Equilibrium Bidding without the Independence Axiom: A Graphical Analysis[1]

by

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1 Introduction

In the framework of expected utility and independent private values the ascending-bid and the second-price sealed-bid auction and the descending-bid and the first-price sealed-bid auction are equivalent and hence lead to the same outcome. However, experimental work on decision making under risk revealed that individuals tend to systematically violate the independence axiom of expected utility theory. This has motivated researchers to develop non-expected utility theories (NEU), which rest on weakened variants of the independence axiom.[3] Many experimental studies of auctions did also not confirm the results of the expected utility framework, but showed significantly different bidding strategies in the two first-price auctions.[4] These experimental results gave rise to a small literature that tried to explain the dynamically inconsistent behavior in open auctions employing NEU models. If bidders behave dynamically inconsistent, open and sealed-bid auctions are no longer equivalent. For example, in a descending-bid auction the bidder, as soon as his valuation is reached, chooses between a certain and an uncertain but a little higher payoff each time the seller lowers the actual price. Many authors argue that this situation may induce bidders to accept a lower but certain payoff or, conversely, to risk the ex ante utility maximizing for a little higher payoff depending on the profits that could be realized and their corresponding attitude towards risk.[5] Others argue that different presentations of the same decision problem may place different dimensions of the problem into the foreground, such as *profit* or *price* and therefore induce different results.[6]

In this paper we consider bidders with general risk preferences and examine their behavior in first- and second-price auctions. Note that in the case of deterministic private valuations of the auctioned object, the equivalence of the ascending-bid and the second-price sealed-bid auction can

[3] Cf. Schmidt (1998) for a review.

[4] Cf. Cox et al. (1982), Coppinger et al. (1982), and Kagel et al. (1987). Because only degenerated lotteries were being auctioned, the outcomes of the two second-price auctions had to remain equivalent.

[5] Cf. Weber (1982).

[6] Cf. Kagel (1995).

be derived with a stochastic dominance argument. Therefore, this equivalence remains valid for all NEU models, which are consistent with first-order stochastic dominance. However, it has already been observed that this result is not true if the auctioned object is a lottery. Auctioning a risky prospect with given probabilities allows for maintaining the independent private values assumption but does not longer imply value revealing behavior of the bidders in second-price auctions. The lottery may be regarded as a common value, where it is reasonable to assume that the differences in the subjective probability distributions are negligible.[7]

When analyzing first-price auctions, we relax the assumption that the auctioned object is a lottery since deterministic private valuations are sufficient to obtain different bids in the open- and sealed-bid auctions if the bidders' preferences violate the independence axiom.

Although a small amount of literature already exists on this topic our paper stands out by the very simple graphical analysis that avoids complex mathematical proofs of the various publications and summarizes and generalizes their results. We show that these results are also true without the usual smoothness assumption. Moreover, we present new results concerning the comparison of the bidding strategies in open and sealed-bid auctions. In particular, we show that the fanning out and the fanning in hypotheses allow for ranking the bids in the two second-price auctions, as well as in the two first-price auctions.

2 The Framework

Let $X = [\underline{v}, \overline{v}]$ be a compact interval of the real line and P the set of all probability measures over X. A degenerate lottery, which assigns the entire probability mass to some $x \in X$ is denoted by δ_x. Furthermore, for any $q \in P$ and $y \in X$ the lottery $[q - y] \in P$ is defined by

$$[q - y](x) = q(x + y) \qquad \forall x \in X. \tag{1}$$

Assume that every bidder $i \in I = \{1, \dots, n\}$ has a complete, transitive, and continuous preference relation $\succsim_i$ on P. Suppose that p is be-

[7]Cf. Karni (1986), p. 17.

ing auctioned and let $v_i \in X$ be bidder i's certainty equivalent for p, i.e. $[p - v_i] \sim_i \delta_0$. The v_i, $i = 1, \dots, n$, represent independent private values due to the given probabilities of the consequences. For a given p the identical distribution functions over the v_i are given by $F : [\underline{v}, \overline{v}] \to [0, 1]$ and the density function $f(v_i) > 0$. Assuming symmetry of the bidders allows for representing the joint distribution function as the product of the individual distribution functions,

$$\mathcal{F}(v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n) = F(v)^{n-1}. \tag{2}$$

Moreover, let $F_y(v)^{n^+-1}$ be the joint distribution function in an open auction assuming that the actual price is y and n^+ is the number of bidders who are still active. Thus, in an ascending-bid auction the support of $F_y(v)^{n^+-1}$ is $[y, \overline{v}]$ and $n^+ = \sharp\{i \in I | b_i \geq y\}$ where b_a denotes the price at which the bidder plans to quit the auction. In contrast in a descending-bid auction $n^+ = n$ and the support of $F_y(v)^{n-1}$ is $[\underline{v}, y]$.

In the following we analyze the performance of the four standard auctions with non-linear preferences of the bidders employing the well-known triangle diagram.

If we draw attention to only three possible consequences $x_1 \succ x_2 \succ x_3$, and define $p(x_2) = 1 - p(x_1) - p(x_3)$, the set of all lotteries over these consequences can be represented in the $[p(x_1), p(x_3)]$-plane. Applying expected utility theory yields $U(p) = u(x_1)p(x_1) + u(x_2)p(x_2) + u(x_3)p(x_3)$ and, by considering a fixed utility level $\overline{U}$, an indifference curve is given by

$$p(x_1) = \frac{u(x_2) - u(x_3)}{u(x_1) - u(x_2)} p(x_3) + \frac{\overline{U} - u(x_2)}{u(x_1) - u(x_2)} \tag{3}$$

Since all the utilities are constant, (3) is a linear equation. Note that the slope is positive and independent of the utility level $\overline{U}$. Thus, indifference curves are parallel straight lines and movements in north western direction lead to a higher utility level. By equation (3) a higher degree of risk aversion leads to steeper but still parallel indifference curves. Furthermore, indifference curves remain parallel straight lines when the corners of the triangle are non-degenerate lotteries and not certain consequences.

In contrast, indifference curves for NEU models are either linear and not parallel or non-linear. For instance utility theories with the betweenness property exhibit linear but not necessarily parallel indifference curves. This is because the betweenness property, which is a weakening of the independence axiom, demands indifference between all linear combinations of two indifferent lotteries. Another important concept in NEU theory is the fanning out hypothesis, which was developed by Machina (1982) in order to accommodate the empirically observed failures of the independence axiom. The fanning out hypothesis states that first-order stochastically dominating shifts from an initial distribution lead to higher degree of risk aversion. Since a higher degree of risk aversion corresponds to steeper indifference curves in the triangle diagram fanning out forces the indifference curves to become steeper in north-western direction. If the corners of the triangle are non-degenerate lotteries, this shape of indifference curves is also implied as long the lotteries are ordered by first-order stochastic dominance[8], i.e. if the corners of triangle are given by the lotteries r, q, and s with $r(s)$ being the upper (right) corner, we must have $r >_{SD} q >_{SD} s$, where $>_{SD}$ denotes the order of stochastic dominance.

However, for some experimental evidence in auction theory, individual choices are better accommmodated by the fanning in hypothesis, which is the reverse of fanning out and, thus, implies indifference curves to become flatter in north-western direction.

3 Second-Price Auctions

The main difference between ascending-bid and second-price sealed-bid auctions consists in the fact that in a sealed-bid auction the bidder decides ex ante how much to bid while in an ascending-bid auction he decides at each actual price whether to place a higher bid or to quit the auction. Maximizing expected utility these two procedures lead to the same outcome due to dynamic consistency. Violations of the independence axiom, however, result in dynamically inconsistent behavior of the bidders as shown by Karni (1988).[9] To model the difference between the two decision problems

[8] Cf. Machina (1982), Theorem 5.

[9] Dynamical consistency implies that the utility maximizing bid chosen ex ante does not change during the auction procedure. Cf. Karni (1988).

we consider, for ascending-bid auctions, the decision of a bidder i whose ex ante certainty equivalent of p, v_i, is slightly higher than the actual price. In the following b^* denotes the optimal bid chosen ex ante in a sealed-bid auction while b^a denotes the *actual bid* placed by a bidder quitting an open auction.

Let the actual price in an ascending-bid auction be $v_i - \varepsilon$, where ε is a infinitesimal small number. Facing this situation bidder i decides between getting δ_0 for certain by quitting the auction and getting $[p - v_i]$ with probability $F_{v_i-\varepsilon}(v_i)^{n^+-1}$ or δ_0 with probability $1 - F_{v_i-\varepsilon}(v_i)^{n^+-1}$ by bidding v_i.

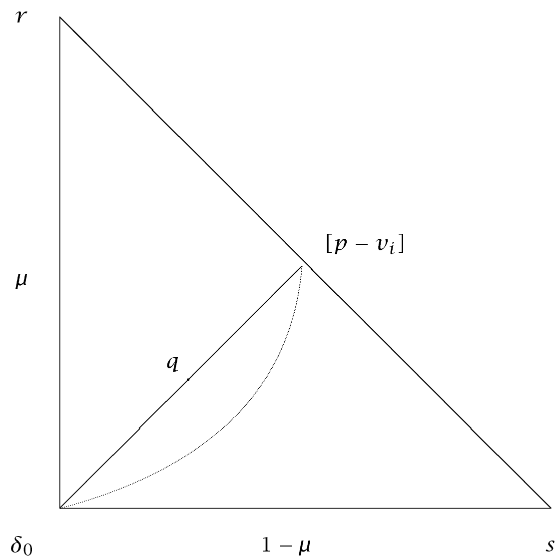

Figure 1: The Decision Problem in Ascending-Bid Auctions

Figure 1 shows the graphical representation of this decision problem in the triangle diagram. The upper (r) and the right corner (s) of this diagram are constructed by dividing the support of $[p - v_i]$ into two parts, i.e. all positive consequences $\{x \in \text{supp}([p - v_i]) : x \geq 0\}$ and all negative consequences $\{x \in \text{supp}([p - v_i]) : x < 0\}$.[10] The lottery r (respectively s) is now obtained if the probabilities of all positive (negative) consequences

[10]The fact that $\text{supp}([p - v_i])$ contains negative consequences follows from consistency with first-order stochastic dominance

of $[p - v_i]$ are devided by $\mu = \sum_{x \geq 0}[p - v_i](x)$ (respectively by $1 - \mu$). Since we have by construction $[p - v_i] = \mu r + (1 - \mu)s$ the lottery $[p - v_i]$ is located on the straight line between r and s. Moreover it is obviously true that $r >_{SD} \delta_0 >_{SD} s$. The decision problem the bidder faces if the actual price is $v_i - \varepsilon$ is to choose between δ_0 and

$$q = [p - v_i]F_{v_i - \varepsilon}(v_i)^{n^+ - 1} + \delta_0(1 - F_{v_i - \varepsilon}(v_i)^{n^+ - 1}), \tag{4}$$

which is located on the straight line between δ_0 and $[p - v_i]$. Note that δ_0 and $[p - v_i]$ lead to the same utility level since v_i is the bidder‘s ex ante certainty equivalent for p.

Figure 1 immediately proves the following result:

Proposition 1 [Karni and Safra (1986)] The ascending-bid auction is demand revealing for all $p \in P$ if and only if the preferences satisfy betweenness. Furthermore, optimal bids exceed (are less than) the bidders' ex ante certainty equivalents, iff preferences are quasiconcave (quasiconvex).

This result can be explained very simply: Since bidder i is indifferent between δ_0 and $[p - v_i]$ and, furthermore, the lottery q, i.e. $[p - v_i]F_{v_i - \varepsilon}(v_i)^{n^+ - 1} + \delta_0(1 - F_{v_i - \varepsilon}(v_i)^{n^+ - 1})$, lies on the straight line between δ_0 and $[p - v_i]$, he is indifferent between δ_0 and q iff his preferences satisfy betweeness. Furthermore, his optimal bid equals v_i iff he is indifferent between δ_0 and q. Thus, the conditional certainty equivalent for p might change during the auction procedure due to dynamic inconsistency but the actual bid equals the bidders ex ante certainty equivalent for p at the moment he is quitting the auction. In contrast, assuming for instance quasiconcave preferences, as depicted in Figure 1, the lottery q is strictly preferred to $[p - v_i]$ and δ_0 and, thus, the optimal bid exceeds the ex ante certainty equivalent v_i at the actual price v_i. Quasiconcave preferences are commonly interpreted as a *preference for randomization*: As soon as the bidder's certainty equivalent is reached he originally planned to quit the auction. But at this point he is confronted with the possibility to get the lottery by placing only slightly higher a bid. Thus, the randomization due to the possibility of winning the auction is strictly preferred to δ_0 and $[p - v_i]$. Note that this argument does not apply if a degenerated lottery is auctioned since the bidder makes a certain loss getting the object for a price higher than v_i. This would constitute a violation of stochastic dominance.

In a second-price sealed-bid auction the bidder lacks the information that other bidders would continue participating at a certain price y. Thus, there is a positive probability of getting the lottery for a price smaller than v_i. This implies that the choice of the optimal bid in a second-price sealed-bid auction has to be derived from a higher indifference curve than in an ascending bid auction at the actual price $v_i - \varepsilon$. In other words, for every bid b smaller than v_i there exists a strictly positive $g \in X$ such that

$$\int_{\underline{v}}^{b} [p - x] dF(x)^{n-1} + [1 - \int_{\underline{v}}^{b} dF(x)^{n-1}]\delta_0 \sim \delta_g \quad . \tag{5}$$

Obviously, the bidder chooses a bid where he is just indifferent between $[p - b]$ and δ_0 conditional on $\int_{\underline{v}}^{b} [p - x] dF(x)^{n-1} + [1 - \int_{\underline{v}}^{b} dF(x)^{n-1}]\delta_0$. Thus, the optimal bid b^* is the solution to

$$\begin{aligned} &[p - b^*] dF(b^*)^{n-1} + \int_{\underline{v}}^{b^*-db} [p - x] dF(x)^{n-1} + [1 - \int_{\underline{v}}^{b^*} dF(x)^{n-1}]\delta_0 \\ &\sim \delta_0 dF(b^*)^{n-1} + \int_{\underline{v}}^{b^*-db} [p - x] dF(x)^{n-1} + [1 - \int_{\underline{v}}^{b^*} dF(x)^{n-1}]\delta_0 \end{aligned} \tag{6}$$

If we define $\lambda = dF(b^*)^{n-1}$ and $t = \frac{1}{1-\lambda}(\int_{\underline{v}}^{b^*-db} [p - x] dF(x)^{n-1} + [1 - \int_{\underline{v}}^{b^*} dF(x)^{n-1}]\delta_0)$ equation (6) reduces to

$$\lambda[p - b^*] + (1 - \lambda)t \sim \lambda\delta_0 + (1 - \lambda)t. \tag{7}$$

Recall that $[p - v_i] \sim \delta_0$. Thus, demand revelation, i.e. we have $b^* = v_i$ for all t, says that the certainty equivalent of p must not change if p is mixed with any t. Since this condition is equivalent to the independence axiom we get:

PROPOSITION 2 [KARNI AND SAFRA (1986)] The second-price sealed-bid auction is demand revealing if and only if the preferences satisfy the independence axiom.

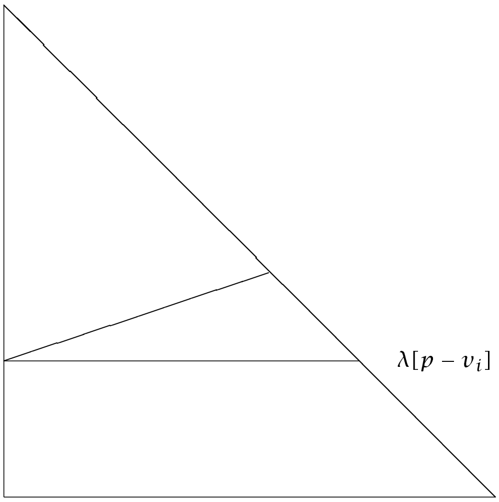

Figure 2: Optimal Bidding in Second-Price Auctions

The validity of Proposition 2 follows also immediately from Figure 2 since the line between $\lambda\delta_0 + (1-\lambda)t$ and $\lambda[p-v_i] + (1-\lambda)t$ is parallel to the line between δ_0 and $[p-v_i]$.

Suppose that the preferences satisfy betweenness and, in contrast to the independence axiom, that the indifference curve beginning in $\lambda\delta_0 + (1-\lambda)t$ is steeper than the indifference curve between δ_0 and $[p-v_i]$. This implies that $\lambda\delta_0 + (1-\lambda)t$ is strictly preferred to $\lambda[p-v_i] + (1-\lambda)t$, which yields, according to (7), $b^* < v_i$. In order to arrive at the following proposition it remains to show that this shape of indifference curves is under betweenness equivalent to the fanning out hypothesis since we have neither $t >_{SD} \delta_0$ nor $\delta_0 >_{SD} [p-v]$. By assumption, a stochastically dominating shift leads to a higher indifference set. Since the degree of risk aversion is in the case of betweenness constant in an indifference set, fanning out, thus, implies that a higher degree of risk aversion is equivalent to an higher indifference set. Since obviously $t \succ \delta_0$ and $t \succ [p-v_i]$ fanning out guarantees the desired shape of indifference curves.

PROPOSITION 3 Suppose that preferences are consistent with betweenness. Then the optimal bid in the second-price sealed-bid auction is lower than (exceeds) the optimal bid in the ascending bid auction if and only if preferences satisfy the fanning out (fanning in) hypothesis.

This result can be explained as follows: The optimal bid in the second-price sealed-bid auction is derived from a higher indifference curve than in the ascending bid auction at the actual price v_i. This yields, according to the fanning out hypothesis, a higher degree of risk aversion, which lowers the conditional certainty equivalent of p and, thus, the optimal bid.

4 First-Price Auctions

In this section we assume that any deterministic object is offered in the auction such that bidder i is indifferent to receiving the object or not at the price v_i. Note that the value v_i cannot change during the auction of a deterministic object.

Since the bidder has no dominant strategy in first-price auctions he chooses his optimal bid presuming given decision rules of the other bidders. Thus, the optimal bid b^a in a descending-bid auction is given by a Nash strategy which is the solution to

$$\begin{aligned} & \delta_{(v_i - b^a(v_i))} F_{b^a}(v_i)^{n-1} + \delta_0 (1 - F_{b^a}(v_i)^{n-1}) \\ \succsim \quad & \delta_{(v_i - b)} F_b(\rho(b))^{n-1} + \delta_0 (1 - F_b(\rho(b))^{n-1}) \qquad \forall b \in X, \end{aligned} \tag{8}$$

where ρ is the inverse of b^a, i.e. $\rho(b)$ is the bidders hypothetical valuation if b was a Nash strategy.

Note that a bidder, whose valuation is already reached, chooses between a certain profit and a lottery determined by the chance to get a little higher profit and the chance to receive nothing if another bidder claims the item. The situation can be characterized as depicted in Figure 3.

In the descending-bid auction the bidder chooses a bid where he is indifferent between getting $v_i - b$ for certain and getting a little higher amount

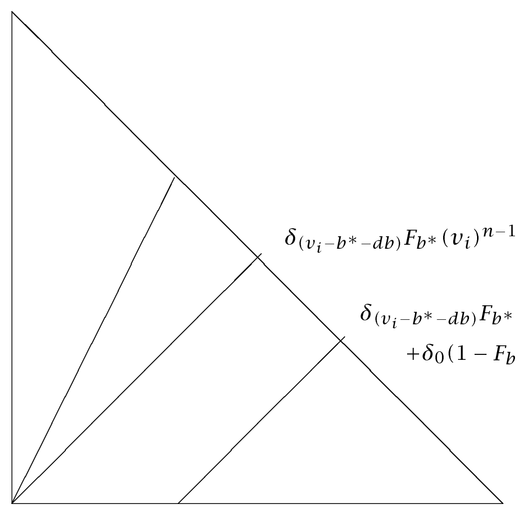

Figure 3: Optimal Bidding in First-Price Auctions

$v_i - b - db$ with probability $F_b(v_i)^{n-1}$ or δ_0 otherwise. Deriving the optimal bid b^* in the first-price sealed-bid auction the bidder i faces the chance of being outbidden at any price lower than v_i. We can describe the decision problem in the sealed-bid auction as compound lotteries, which yield $\delta_{(v_i-b)}$ or $\delta_{(v_i-b-db)}F_b(v_i)^{n-1} + \delta_0(1 - F_b(v_i)^{n-1})$ with probability $F(v_i)^{n-1}$, and δ_0 otherwise. Considering the actual price level b^* allows for comparing the selling price in the descending-bid auction with the selling price in the sealed-bid auction, which is b^* by definition. Note that $F(v_i)^{n-1}$ is the ex ante probability that no valuation among the bidders is higher than v_i. Since the ex ante probability of getting δ_0, $1 - F(v_i)^{n-1}$, is strictly positive and higher than the probability of δ_0 faced by a bidder at the moment he decides in a descending-bid auction, the decision problem in the sealed-bid auction is located on a lower indifference curve as depicted in Figure 4.

Suppose that the optimal bids in the open and the sealed-bid auction coincide, i.e. $b^* = b^a$. Figure 3 shows that this is always true iff the bidder's indifference curves are parallel straight lines. Hence, we can conclude

PROPOSITION 4 [KARNI 1988] The optimal bid in the descending bid auction and the first-price sealed-bid auction coincide for every auctioned object if and only if the preferences of every bidder satisfy the independence axiom.

Consider now the case in which the two indifference curves in Figure 4 are not parallel. Since $v_i - b^* - db > v_i - b^* > 0$ fanning out implies that the slope of the indifference curve for the open auction is in every point higher than its slope for the sealed-bid auction. Hence, the sure profit $v_i - b^*$ is strictly preferred to a slightly higher bid, which means that the optimal bid in the open auction is in this case strictly higher than b^*. Since all lotteries are on the border of the triangle this argument does not rely on betweenness.

PROPOSITION 5 The actual bid in the descending-bid auction is always higher (lower) than the optimal bid in the first-price sealed-bid auction if and only if the bidder's preferences satisfy fanning out (fanning in).

This result can be interpreted as follows. When deriving the optimal bid b^a in the open auction the bidder knows that his bid is the highest bid, and thus, the decision problem is located on a higher indifference curve. Due to the increased degree of risk aversion the certainty equivalent of the lottery $z = \delta_{(v_i - b^* - db)} F_{b^*}(v_i)^{n-1} + \delta_0(1 - F_{b^*}(v_i)^{n-1})$ is lower than the conditional certainty equivalent of z if z is mixed with a dominated lottery (δ_0) as in the sealed-bid auction.

The same problem as in Proposition 5 was already studied by Weber (1982) with weighted utility, which is a special variant of the utility theories with the betweenness property. Weber (1982) derives a condition on the weighting function, which is sufficient (though not necessary) for the optimal bid in the open auction being higher than in the sealed-bid auction.

5 Conclusion

We can conclude that fanning out implies that the optimal bids in the open auctions are always higher than in the sealed-bid auctions. This is not obviously clear since for first-price auctions the decision problem in the open auction is located on the higher indifference curve than in the sealed-bid auction, while the converse is true for second-price auctions. However, a higher degree of risk aversion corresponds to a higher bid in first-price auctions and, again, the converse is true in second-price auctions. This becomes quite clear, if we realize the different situations in the two open auctions . In the ascending-bid auction the bidder decides between getting nothing and playing the lottery at each actual price so that a lower degree of risk aversion leads to a higher certainty equivalent and thus, a higher bid, in the open auction. In the descending-bid auction, conversely, the bidder chooses between a certain amount and the lottery at each actual price lower than v_i. Thus, a higher degree of risk aversion induces the bidder to place a higher bid by claiming the object earlier.

Most experimental results concerning first-price auctions, however, are according to proposition 5 not consistent with fanning out, but with the fanning in hypothesis. This is commonly explained by the low expected payoffs that represent low opportunity costs of feeling the excitement of the game by staying active a little bit longer than planned ex ante in descending-bid auctions. However, it seems plausible to suspect that fanning in will weaken, and finally reverse, as the payoff scale is increased, since high profits represent high opportunity costs.

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