The Stochastic Equation $P_{t+1} = A_t P_t + B_t$ with Non-Stationary Coefficients

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Abstract

In this paper we consider the stochastic sequence $\{P_t\}_{t\in\mathbb{N}}$ defined recursively by the linear relation $P_{t+1} = A_t P_t + B_t$ in a random environment which is described by the non-stationary process $\psi = \{(A_t, B_t)\}_{t\in\mathbb{N}}$. We formulate sufficient conditions on ψ which ensure that the finite-dimensional distributions of $\{P_t\}_{t\in\mathbb{N}}$ converge weakly to the finite-dimensional distribution of a unique stationary process. If the driving sequence ψ has a "nice" tail behaviour, then we can establish a global convergence result. This extends results of Brandt (1986) and Borovkov (1998) from the stationary to the non-stationary case.

Key Words: Stochastic difference equation, stochastic stability, ergodicity

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1 Introduction

In this paper we consider the stochastic sequence $\{P_t\}_{t\in\mathbb{N}}$ defined recursively by the linear relation

$$P_{t+1} = A_t P_t + B_t \quad (t \geq 0) \tag{1}$$

in a random environment which is described by the stochastic process $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We formulate sufficient conditions on $\psi := \{(A_t, B_t)\}_{t\in\mathbb{N}}$ which guarantee asymptotic stability of the solution of (1) in the sense that the sequence $\{P_t\}_{t\in\mathbb{N}}$ converges in distribution as $t \to \infty$.

Dynamics of the form (1) have been extensively investigated under a contraction condition and under the assumption that the environment ψ is stationary. For example, Vervaat (1979) considers the case where the environment consists of i.i.d. random variables. Brandt (1986) and Borovkov (1998) assume that the environment ψ is stationary and ergodic under the measure $\mathbb{P}$.

Our purpose is to replace this stationarity assumption by an asymptotic stability condition on the driving sequence ψ. This is essential for applications where a stationarity condition on ψ seems too restrictive. For example, the process $\{P_t\}_{t\in\mathbb{N}}$ could be a sequence of temporary equilibrium prices of a risky asset generated by the interaction of different types of economic agents; see, e.g., Horst (1999a) or Horst (1999b) for models in this direction. In such a model the sequence ψ describes the evolution of the empirical distribution of individual agents' characteristics, i.e., the "mood" of the market. In that setting, it seems natural to investigate the asymptotic dynamics of the price process $\{P_t\}_{t\in\mathbb{N}}$ under the assumption that the environment ψ is is out of equilibrium, i.e., a non-stationary sequence. However, given that the "mood" of the market settles down in the long run, it is desirable to have sufficient conditions which ensure that the "mood" drives the price process into equilibrium.

For a non-stationary process ψ, ergodic theorems based on the existence of stationary renovation events for $\{P_t\}_{t\in\mathbb{N}}$ can be proved, if the driving sequence $\{(A_t, B_t)\}_{t\in\mathbb{N}}$ converges in the sense of a strong coupling to a stationary process; see Borovkov (1998), Chapter 3. However, for many applications, for example in the model analysed in Horst (1999b), this condition is too strong or too hard to verify.

In this paper we formulate conditions on the non-stationary environment ψ which guarantee that the solution of (1) settles down to a unique equilibrium in the long run. In a first step we will analyse the dynamics of the process $\{P_t\}_{t\in\mathbb{N}}$ governed by (1) under the assumption that ψ has a "nice" tail structure. More precisely, we assume that there exists a probability measure $\mathbb{P}^*$ such that the driving sequence ψ is ergodic under $\mathbb{P}^*$ and which coincides with the original measure $\mathbb{P}$ on $\mathcal{T}$, the tail-field generated by ψ. This assumption is satisfied if, for example, the environment is driven by some underlying Markov process which converges in the total variation norm to a unique invariant distribution. In this case we establish a "global" convergence result under the measure $\mathbb{P}$, namely convergence in law of the sequence $\{P_t\}_{t\in\mathbb{N}}$ governed by (1) to the uniquely determined stationary solution of (1) under the measure $\mathbb{P}^*$. The results of Brandt (1986) correspond to the case $\mathbb{P} = \mathbb{P}^*$, and in this case one can prove almost sure convergence.

In a second step we will weaken the regularity condition that the measures $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $\mathcal{T}$. Instead, we shall assume that the environment ψ can be approximated in law by a sequence of "nice" processes. In this more general case we establish weak convergence of the finite-dimensional distributions of the process $\{P_t\}_{t\in\mathbb{N}}$ under $\mathbb{P}$ to the finite-dimensional distributions of the uniquely determined stationary solution of (1) under $\mathbb{P}^*$, i.e., a "local" version of the convergence result.

The paper is organised as follows. In Section 2 we formulate our main results. Section 3 analyses solutions of (1) driven by non-stationary but "nice" sequences ψ. In Section 4 we study the case where ψ itself does not satisfy this regularity condition but can be approximated in law by "nice" processes. Section 5 is devoted to a Markovian case study, where the assumption that the environment ψ can be approximated by "nice" processes can indeed be verified.

2 Assumptions and the Main Results

Let $\psi := \{\psi_t\}_{t\in\mathbb{N}} = \{(A_t, B_t)\}_{t\in\mathbb{N}}$ be a sequence of $\mathbb{R}^2$-valued random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_\phi)$ which satisfy $\mathbb{P}_\phi - a.s.$ the relation $(A_0, B_0) = \phi$.

In this section we formulate conditions which guarantee that the finite-dimensional distributions of the sequence $\{P_t\}_{t\in\mathbb{N}}$ governed by (1) and driven by the "input" ψ converge to the finite-dimensional distributions of a uniquely determined stationary process as $t \to \infty$.

Assumption 1 *There exists a probability measure μ^* on $\mathbb{R}^2$ such that the environment ψ is stationary and ergodic under the law*

$$\mathbb{P}^*(\cdot) := \int \mathbb{P}_\phi(\cdot)\mu^*(d\phi).$$

This assumption is satisfied if, for example, the "input" ψ is driven by some underlying Markov process M which admits a unique invariant measure, c.f. Examples 3.7 and 3.8 below.

Throughout this paper we denote by $\mathbb{E}_\phi$ the expectation with respect to the measure $\mathbb{P}_\phi$ and by $\mathbb{E}^*$ the expectation with respect to the measure $\mathbb{P}^*$. We impose the following integrability conditions with respect to the measure $\mathbb{P}^*$.

Assumption 2 *We assume* $\mathbb{E}^*(\ln B_0)^+ < \infty$ *and*

$$-\infty < \mathbb{E}^* \ln |A_0| < 0. \tag{2}$$

Here, (2) may be viewed as a mean contraction condition for the dynamics defined by (1). Such a contraction condition has also been imposed by, e.g., Vervaat (1979).

Remark 2.1 *If the driving process ψ is already in equilibrium, i.e., if we work under the measure $\mathbb{P}^*$, then we find ourselves in the setting analysed in Brandt (1986). In this case Assumption 2 coincides with condition (0.4) in Brandt (1986), and there exists a unique stationary solution $\{P_t^*\}_{t\in\mathbb{N}}$ of (1) (c.f. Theorem 3.5 below).*

Let us introduce the following σ-fields. For $t, l \in \mathbb{N}$ we put

$$\hat{\mathcal{F}}_{t,l} := \sigma\left(\{\psi_s\}_{t\le s\le l}\right), \quad \hat{\mathcal{F}}_t := \sigma\left(\{\psi_s\}_{s\ge t}\right). \tag{3}$$

Furthermore, we denote by $\mathrm{Law}(Y, \mathbb{P})$ the distribution of a random variable Y under the measure $\mathbb{P}$. For example, by $\mathrm{Law}(\psi, \mathbb{P}_\phi)$ we mean the distribution of the process ψ on the path space $(\mathbb{R}^2)^{\mathbb{N}}$ induced by the measure $\mathbb{P}_\phi$.

In a first step we extend the results of Brandt (1986) from the stationary to the non-stationary setting under the assumption that the driving sequence ψ is "nice" in the following sense.

Definition 2.2 *A driving sequence ψ is called "nice" if the following assumption about its asymptotic behaviour is satisfied. For any initial condition $\phi \in \mathbb{R}^2$, we have that*

$$\mathrm{Law}(\psi, \mathbb{P}_\phi) = \mathrm{Law}(\psi, \mathbb{P}^*) \;\; on \;\mathcal{T}, \tag{4}$$

where $\mathcal{T} := \bigcap_{t\in\mathbb{N}} \hat{\mathcal{F}}_t$ denotes the tail-σ-field generated by the sequence ψ.

We introduce some additional notation. For a given driving sequence ψ we put

$$p_t(p, \psi) := \sum_{j=0}^{t-1} \left(\prod_{i=t-j}^{t-1} A_i \right) B_{t-j-1} + \left(\prod_{i=0}^{t-1} A_i \right) p \tag{5}$$

and denote by $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ the governed by (1) with initial value p. Thus, $p_t(p,\psi)$ can be interpreted as the state at time t of a system governed by (1) and by the "input" ψ if it starts at the (random) state p.

Theorem 2.3 *Suppose that the driving sequence ψ satisfies Assumptions 1 and 2 and that ψ is "nice" in the sense of Definition 2.2. Then for any initial condition $\phi \in \mathbb{R}^2$ of ψ and for any $p \in \mathbb{R}$, the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ converges in law to the unique stationary solution $\{P_t^*\}_{t\in\mathbb{N}}$ of (1) defined on $(\Omega, \mathcal{F}, \mathbb{P}^*)$. More precisely, we have that*

$$\mathrm{Law}(\{p_{t+T}(p,\psi)\}_{t\in\mathbb{N}}, \mathbb{P}_\phi) \xrightarrow{w} \mathrm{Law}(\{P_t^*\}_{t\in\mathbb{N}}, \mathbb{P}^*) \;\; on \;\mathbb{R}^{\mathbb{N}} \quad (T \to \infty). \tag{6}$$

Here, by $\mathrm{Law}(\{p_{t+T}(p,\psi)\}_{t\in\mathbb{N}}, \mathbb{P}_\phi)$ we mean the distribution of the "shifted" sequence $\{p_{t+T}(p,\psi)\}_{t\in\mathbb{N}}$ on the path space $\mathbb{R}^{\mathbb{N}}$ under the measure $\mathbb{P}_\phi$.

Section 3 is devoted to the proof of this theorem.

Let us now consider the case where the environment itself does not have a "nice" tail structure. Theorem 2.5 below states that the finite-dimensional distributions of the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ under the measure $\mathbb{P}_\phi$ converge weakly to the finite-dimensional distributions of the unique stationary solution of (1) under $\mathbb{P}^*$ as soon as the driving sequence ψ can be approximated in law by a suitable sequence of processes $\{\psi^n\}_{n\in\mathbb{N}}$ which are "nice" in the sense defined above. In particular, the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ converges in law to to a unique stationary measure ν_∞.

The approximating sequence $\{\psi^n\}_{n\in\mathbb{N}}$, $\psi^n = \{(A_t^n, B_t^n)\}_{t\in\mathbb{N}}$, is defined as follows. Let $\{\epsilon_t\}_{t\in\mathbb{N}}$ and $\{\eta_t\}_{t\in\mathbb{N}}$ be sequences of random variables defined on $(\Omega,\mathcal{F})$ which are independent, which satisfy $\text{Law}((\epsilon_0,\eta_0),\mathbb{P}_\phi) = \text{Law}((\epsilon_0,\eta_0),\mathbb{P}'_\phi)$ for all $\phi,\phi'\in\mathbb{R}^2$ and independent of all (A_t,B_t) $(t\in\mathbb{N})$. We put

$$A_t^n := A_t + \sigma_n\epsilon_t \quad \text{and} \quad B_t^n := B_t + \sigma_n\eta_t$$

for a sequence $\sigma_n \downarrow 0$. Thus, the sequences $\psi^n := \{(A_t^n, B_t^n)\}_{t\in\mathbb{N}}$ $(n\in\mathbb{N})$ are stationary and ergodic under $\mathbb{P}^*$, and the process $\{\psi^n\}_{n\in\mathbb{N}}$ converges in distribution to ψ both in the stationary and in the non-stationary setting. Furthermore, for each $t\in\mathbb{N}$ we set

$$\widetilde{\psi}_t := (|A_t| + \sigma_1|\epsilon_t|, |B_t| + \sigma_1|\eta_t|) \quad \text{and} \quad \widetilde{\psi} := \{\widetilde{\psi}_t\}_{t\in\mathbb{N}}.$$

Assumption 3 *The sequences $\widetilde{\psi},\psi^1,\psi^2,\ldots$ are nice in the sense of Definition 2.2 and satisfy the following regularity conditions:*

$$\mathbb{E}^*(\ln|A_0^n|)^+ \to \mathbb{E}^*(\ln|A_0|)^+, \quad \mathbb{E}^*(\ln|B_0^n|)^+ \to \mathbb{E}^*(\ln|B_0|)^+ \quad (n\to\infty). \tag{7}$$

$$\mathbb{E}^*(\ln|A_0^n|) \to \mathbb{E}^*(\ln|A_0|), \quad \mathbb{E}^*(\ln|B_0^n|) \to \mathbb{E}^*(\ln|B_0|) \quad (n\to\infty). \tag{8}$$

Moreover, we impose the following integrability condition:

$$\mathbb{E}^*(\ln(|A_0| + \sigma_1|\epsilon_0|) < \infty, \quad \mathbb{E}^*(\ln(|B_0| + \sigma_1|\eta_0|) < \infty. \tag{9}$$

In Section 5 we shall verify this assumption if the environment ψ is driven by a certain class of underlying Markov processes.

Remark 2.4 *Under the measure $\mathbb{P}^*$, i.e., if all driving sequences are already in equilibrium, (7) and (8) coiincide with conditions (0.11) and (0.10) respectively in Brandt (1986).*

Let us now state our main result.

Theorem 2.5 *Suppose that Assumptions 1-3 are satisfied. Then the finite dimensional distributions of the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ under $\mathbb{P}_\phi$ converge weakly to the finite dimensional distributions of the unique stationary solution of (1) under $\mathbb{P}^*$. More precisely, for any $m\in\mathbb{N}$ and for all $t_1 < \cdots < t_m$ we have that*

$$\text{Law}((p_{t_1+T}(p,\psi),\ldots,p_{t_m+T}(p,\psi)),\mathbb{P}_\phi) \xrightarrow{w} \text{Law}((P(t_1),\ldots,P(t_m)),\mathbb{P}^*)$$

as $T\to\infty$. In particular, there exists a unique probability measure ν_∞ on $\mathbb{R}$ such that

$$\text{Law}(p_t(p,\psi),\mathbb{P}_\phi) \xrightarrow{w} \nu_\infty \quad (t\to\infty).$$

The proof of this theorem is given in Section 4.

Our results should be compared to Theorem 1 in Brandt (1986), see also Theorem 3.5 below. There, the author shows that the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ governed by the input ψ becomes stationary in the long run assuming that the driving sequence $\{(A_t,B_t)\}_{t\in\mathbb{N}}$ is stationary and ergodic. More precisely, it is verified that

$$\mathbb{P}^*[\lim_{t\to\infty}|p_t(p,\psi)-P_t^*|=0]=1, \tag{10}$$

where $\{P_t^*\}_{t\in\mathbb{N}}$ denotes the unique stationary solution of (1) under $\mathbb{P}^*$. Theorem 2.3 above shows that in case of non-stationary but "nice" driving sequences, we can still establish a "global" result, namely convergence in distribution of the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ to the unique stationary solution of (1) under the measure $\mathbb{P}^*$ instead of almost sure convergence. Furthermore, if the governing sequence ψ can be approximated in law by "nice" processes, then we can prove a "local" version of the convergence result, i.e., convergence of the finite-dimensional distributions of the sequence $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ under $\mathbb{P}_\phi$.

3 Linear Stochastic Sequences Driven by Nice Processes

This section is devoted to the proof of Theorem 2.3. Thus, throughout this section we assume that the driving sequence ψ is "nice" in the sense of Definition 2.2. In a first step we shall analyse the sequence $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ given a stationary "input" ψ, i.e., we will study the long run dynamics of this process under the measure $\mathbb{P}^*$. In a second step we will demonstrate that for any initial value $\phi=(A_0,B_0)$ of ψ, the asymptotic distribution for $t\to\infty$ of the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ is the same under $\mathbb{P}_\phi$ and $\mathbb{P}^*$. More precisely, we shall verify that

$$\mathrm{Law}(p_t(p,\psi),\mathbb{P}_\phi)\stackrel{w}{\longrightarrow}\mathrm{Law}(P_0^*,\mathbb{P}^*)\quad(t\to\infty).$$

In a final step we establish the "global" convergence result stated in Theorem 2.3.

Let us start with some preliminaries. First, we shall establish a condition on the driving sequence which turns out to be equivalent to (4). Below, we will verify this assumption for a certain class of Markovian models.

Remark 3.1 *As the σ-fields $\{\hat{\mathcal{F}}_t\}_{t\in\mathbb{N}}$ decrease to $\mathcal{T}$, we have that*

$$\lim_{t\to\infty}||\mathrm{Law}(\psi,\mathbb{P}_\phi)-\mathrm{Law}(\psi,\mathbb{P}^*)||_{\hat{\mathcal{F}}_t}=||\mathrm{Law}(\psi,\mathbb{P}_\phi)-\mathrm{Law}(\psi,\mathbb{P}^*)||_{\mathcal{T}}=0, \tag{11}$$

where $||\cdot||_{\mathcal{E}}$ denotes the total variation of a signed measure on $\mathcal{E}$. A simple martingale proof can be found in, e.g., Föllmer (1979), Remark 2.1.

Assumption 4 *There exists a sequence* $c_t \to 0$ *such that, for any* $l \in \mathbb{N}$*, the following holds true:*

$$||\mathrm{Law}(\psi, \mathbb{P}_\phi) - \mathrm{Law}(\psi, \mathbb{P}^*)||_{\hat{\mathcal{F}}_{t,l}} \leq c_t. \tag{12}$$

Proposition 3.2 *A driving sequence* ψ *is "nice" in the sense of Definition 2.2 if and only if it satisfies Assumption 4.*

Proof: Due to (11), a "nice" sequence ψ satisfies Assumption 4. Thus, we just have to verify the "only if" part. By Remark 3.1 it suffices to show that Assumption 4 implies

$$\lim_{t\to\infty} ||\mathrm{Law}(\psi, \mathbb{P}_\phi) - \mathrm{Law}(\psi, \mathbb{P}^*)||_{\hat{\mathcal{F}}_t} = 0.$$

This, however, can be verified using a monotone class argument. To this end, let $A \in \hat{\mathcal{F}}_{t,l}$ $(t \leq l)$. We have that

$$\left| \mathbb{P}_\phi \left[\psi \in \times_{i=0}^{t-1} \mathbb{R}^2 \times A \times (\mathbb{R}^2)^{\mathbb{N}} \right] - \int \mathbb{P}_\phi \left[\psi \in \times_{i=0}^{t-1} \mathbb{R}^2 \times A \times (\mathbb{R}^2)^{\mathbb{N}} \right] \mu^*(d\phi) \right| \leq c_t \tag{13}$$

by Assumption 4. Thus (13) holds true for any $A \in \mathcal{G}_t := \bigcup_{l \geq t} \hat{\mathcal{F}}_{t,l}$.

Let us denote by $\mathcal{G}$ the system of all sets $A \in \mathcal{B}((\mathbb{R}^2)^{\mathbb{N}})$ which satisfy (13). It is easily seen that $\mathcal{G}$ is a monotone class including $\mathcal{G}_t$. As $\mathcal{G}_t$ generates the $\sigma-$algebra $\hat{\mathcal{F}}_t$ and because $\mathcal{G}_t \subset \mathcal{G}$, our assertion follows immediately from the Monotone Class Theorem. □

Next, observe that for any measurable function $F : \mathbb{R}^2 \to \mathbb{R}$ which satisfies $\mathbb{E}^*|F(\psi_1)| < \infty$, the event

$$\left\{ \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} F(\psi_i) = \mathbb{E}^* F(\psi_1) \right\}$$

is a tail event. Therefore, the following proposition follows immediately from Assumption 1 and Assumption 3.

Proposition 3.3 *For any initial value* $\phi \in \mathbb{R}^2$*, the sequence* ψ *satisfies the strong law of large numbers under the measure* $\mathbb{P}_\phi$*.*

Remark 3.4 *Whenever we are working under the measure* $\mathbb{P}^*$*, we may without loss of generality assume that the stationary sequence* $\psi = \{(A_t, B_t)\}_{t\in\mathbb{N}}$ *is defined for all integers* $t \in \mathbb{Z}$ *due to Kolmogorov's extension theorem.*

We are going to use the following result which appears as Theorem 1 in Brandt (1986).

Theorem 3.5 *In the stationary situation, i.e., under the measure* $\mathbb{P}^*$*, the following holds true as soon as Assumption 2 is satisfied.*

1. *There exists a unique stationary solution of (1) for the input* ψ, *i.e., there exists a unique stationary process* $\{P_t^*\}_{t\in\mathbb{N}}$ *which obeys the relation* $P^*t+1 = A_t P_t^* + B_t$. *Furthermore,* P_t^* *takes the form* $P_t^* = \sum_{j=0}^{\infty}\left(\prod_{i=-j}^{t-1} A_i\right) B_{t-j-1}$ $(t \in \mathbb{N})$.

2. *The process* $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ *converges almost surely to the stationary solution. More precisely,* $\lim_{t\to\infty} |p_t(p,\psi) - P_t^*| = 0$ $\mathbb{P}^* - a.s.$

3. *In particular, under the measure* $\mathbb{P}^*$ *the sequence* $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ *converges in distribution to the almost surely finite random variable*

$$P_0^* = \sum_{j=0}^{\infty}\left(\prod_{i=-j}^{-1} A_i\right) B_{-j-1}$$

as $t \to \infty$.

Let us now establish convergence in distribution for the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ driven by a non-stationary but "nice" sequence ψ. As a first step we shall prove convergence of the one-dimensional distributions.

Theorem 3.6 *We have that*

$$\mathrm{Law}(p_t(p,\psi), \mathbb{P}_\phi) \xrightarrow{w} \mathrm{Law}(P_0^*, \mathbb{P}^*) \quad (t \to \infty).$$

Proof: In analogy to Brandt (1986) we define for each $l \in \mathbb{N}$ the following random variable:

$$p_t^l(p,\psi) := \sum_{j=0}^{t-1-l}\left(\prod_{i=t-j}^{t-1} A_i\right) B_{t-j-1} + \left(\prod_{i=l}^{t-1} A_i\right) p. \tag{14}$$

Obviously, $p_t^l(p,\psi)$ can be interpreted as the state at time t of a system governed by (1) and by the "input" ψ if it starts at time l at the (random) state p. In particular, $p_t^l(p,\psi)$ does not depend on the random variables $\{A_1, B_1, \ldots, A_{l-1}, B_{l-1}\}$. Therefore, we have that

$$\begin{aligned}
|p_t(p,\psi) - p_t^l(p,\psi)| &\leq \left|\sum_{j=t-l}^{t-1}\left(\prod_{i=t-j}^{t-1} A_i\right) B_{t-j-1} + p\prod_{i=0}^{t-1} A_i - p\prod_{i=l}^{t-1} A_i\right| \\
&\leq \exp\left(\frac{1}{t-1}\sum_{i=1}^{t-1}\ln|A_i| + \frac{\ln|B_0|}{t-1}\right)^{t-1} \\
&\quad + \exp\left(\frac{1}{t-2}\sum_{i=1}^{t-2}\ln|A_{i+1}| + \frac{\ln|B_1|}{t-2}\right)^{t-2} \\
&\quad + \ldots \\
&\quad + \exp\left(\frac{1}{t-l-1}\sum_{i=1}^{t-l-1}\ln|A_{i+l}| + \frac{\ln|B_l|}{t-l-1}\right)^{t-l-1}
\end{aligned}$$

$$+\left(\exp\left(\frac{1}{t}\sum_{i=0}^{t-1}\ln|A_i|\right)\right)^t |p|$$
$$+\left(\exp\left(\frac{1}{t-l}\sum_{i=l}^{t-1}\ln|A_i|\right)\right)^{t-l} |p|.$$

Observe that the strong law of large numbers – see Proposition 3.3 above – yields

$$\lim_{t\to\infty}|p_t(p,\psi)-p_t^l(p,\psi)|=0 \quad \mathbb{P}_\phi- \text{ and } \mathbb{P}^*-a.s. \tag{15}$$

Indeed, for any $j \in \mathbb{N}$ we have that

$$\frac{1}{t-j}\sum_{i=0}^{t-j-1}\ln|A_{i+j}| \to \mathbb{E}^*|\ln A_0|<0 \quad \mathbb{P}_\phi- \text{ and } \mathbb{P}^*-a.s. \quad (t\to\infty)$$

by Assumption 2, and therefore

$$\exp\left(\frac{1}{t-j}\sum_{i=0}^{t-j-1}\ln|A_{i+j}|+\frac{\ln|B_j|}{t-j}\right)^{t-j} \to 0 \quad \mathbb{P}_\phi- \text{ and } \mathbb{P}^*-a.s. \quad (t\to\infty)$$

which yields (15).

Let $\epsilon>0$ and F, a bounded continuous real valued function with compact support, be given. According to Theorem 3.5 there exists $T_1=T_1(\epsilon)\in\mathbb{N}$ such that

$$\left|\int F(p_t(p,\psi))d\mathbb{P}^*-\nu_\infty(F)\right|<\epsilon/4 \quad \text{for all } t\geq T_1, \tag{16}$$

where $\nu_\infty := \text{Law}(P_0^*,\mathbb{P}^*)$.

Observe now that the event $\{p_t^l(p,\psi)\in A\}$, $A\in\mathcal{B}(\mathbb{R})$, belongs to the $\sigma-$algebra $\hat{\mathcal{F}}_l$ defined in (3). As the sequence ψ is "nice" we deduce from (11) that there exist a sequence $c_l\to 0$ such that

$$\sup_{A,t\geq l}|\mathbb{P}_\phi[p_t^l(p,\psi)\in A]-\mathbb{P}^*[p_t^l(p,\psi)\in A]|\leq c_l. \tag{17}$$

In particular, as $\mathbb{P}_\phi-\mathbb{P}^*$ is a signed measure on $\mathcal{F}$, this yields the following inequality.

$$\begin{aligned}
&\sup_{t\geq l}\left|\int F(p_t^l(p,\psi))\,(d\mathbb{P}_\phi-d\mathbb{P}^*)\right| \\
&\leq |F|_\infty \sup_{A,t\geq l}|\mathbb{P}_\phi[p_t^l(p,\psi)\in A]-\mathbb{P}^*[p_t^l(p,\psi)\in A]| \\
&\leq |F|_\infty c_l \\
&\leq \frac{\epsilon}{4}
\end{aligned} \tag{18}$$

for l sufficiently large by (17) and because ψ is "nice".

For the rest of the proof we fix $l \in \mathbb{N}$ such that (18) holds true. Using (15) we can easily deduce that there exists a constant $T_2 = T_2(l) \in \mathbb{N}$ such that

$$\left| \int \left\{ F(p_t^l(p,\psi)) - F(p_t(p,\psi)) \right\} d\mathbb{P}_\phi \right| \leq \frac{\epsilon}{4} \quad \text{for} \quad t \geq T_2 \tag{19}$$

and that

$$\left| \int \left\{ F(p_t^l(p,\psi)) - F(p_t(p,\psi)) \right\} d\mathbb{P}^* \right| \leq \frac{\epsilon}{4} \quad \text{for} \quad t \geq T_2. \tag{20}$$

Therefore, for $t \geq \max\{T_1, T_2\}$ we obtain the following inequality:

$$\begin{aligned}
&\left| \int F(p_t(p,\psi)) d\mathbb{P}_\phi - \nu_\infty(F) \right| \\
\leq & \left| \int \left\{ F(p_t(p,\psi)) - F(p_t^l(p,\psi)) \right\} d\mathbb{P}_\phi \right| \\
&+ \left| \int F(p_t^l(p,\psi)) \left\{ d\mathbb{P}_\phi - d\mathbb{P}^* \right\} \right| \\
&+ \left| \int \left\{ F(p_t^l(p,\psi)) - F(p_t(p,\psi)) \right\} d\mathbb{P}^* \right| \\
&+ \left| \int F(p_t(p,\psi)) d\mathbb{P}^* - \mu_\infty(F) \right| \\
\leq & \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

by (19), (18), (20) and by (16). As $\epsilon > 0$ is arbitrary, this proves our assertion. □

Now we are able to establish the "global" convergence result stated in Theorem 2.3.

Proof of Theorem 2.3:

From the proof of Theorem 3.6 we can easily deduce that the finite dimensional distributions of the shifted process $\{p_{t+T}(p,\psi)\}_{t\in\mathbb{N}}$ under the measure $\mathbb{P}_\phi$ converge weakly to the finite dimensional distributions of the unique stationary solution $\{P_t^*\}_{t\in\mathbb{N}}$ of (1) under the measure $\mathbb{P}^*$ as $T \to \infty$. Indeed, let $m \in \mathbb{N}$ and $t_1 < t_2 < \cdots < t_m \in \mathbb{N}$ be given. For $t_1 \geq l$ we put

$$p_{t_1,\cdots,t_m}(p,\psi) := (p_{t_1}(p,\psi), \ldots, p_{t_m}(p,\psi)),$$

and

$$p^l_{t_1,\cdots,t_m}(p,\psi) := (p^l_{t_1}(p,\psi), \ldots, p^l_{t_m}(p,\psi)).$$

As in the proof of Theorem 3.6 we have that

$$\lim_{t_1\to\infty} ||p_{t_1,\cdots,t_m}(p,\psi) - p^l_{t_1,\cdots,t_m}(p,\psi)|| = 0 \quad \mathbb{P}_\phi - \text{ and } \mathbb{P}^* - a.s.,$$

where $||\cdot||$ denotes the Euclidian distance. Now let $A \in \mathcal{B}(\mathbb{R}^m)$ be given. The event $\{p^l_{t_1,\cdots,t_m}(p,\psi) \in A\}$ belongs to the σ-algebra $\hat{\mathcal{F}}_l$. Therefore, we can use exactly the same arguments as in the proof of Theorem 3.6 to verify weak convergence of the finite-dimensional distributions.

Thus, in order to prove our assertion it suffices to show that the family of random variables $(\{p_{t+T}(p,\psi)\}_{t\in\mathbb{N}})_{T\in\mathbb{N}}$ – viewed as random variables taking values in the space $\mathbb{R}^{\mathbb{N}}$ – is tight. For this, we have to verify that for any $\epsilon > 0$ and for all $t \in \mathbb{N}$ there exists a compact set $K_t \subset \mathbb{R}$ such that

$$\sup_T \mathbb{P}_\phi[p_{t+T}(p,\psi) \in K_t] \geq 1 - \epsilon,$$

see, e.g., Ethier and Kurtz (1986), Theorem 3.7.2. To this end, let $\epsilon > 0$ and $t \in \mathbb{N}$ be given. As the driving sequence ψ is "nice" we can choose $l \in \mathbb{N}$ such that

$$\sup_{A,T\geq l} |\mathbb{P}_\phi[p^l_{t+T}(p,\psi) \in A] - \mathbb{P}^*[p^l_{t+T}(p,\psi) \in A]| < \frac{\epsilon}{4}$$

due to (11). Furthermore, we can choose a constant $k \in \mathbb{R}$ such that

$$\mathbb{P}^*[|P^*_t| \leq k] = \mathbb{P}^*[|P^*_0| \leq k] > 1 - \frac{\epsilon}{4} \quad (T \in \mathbb{N})$$

because the random variable P^*_0 is $\mathbb{P}^*$-a.s. finite. Due to (15) there exists a constant $T_0 = T_0(\epsilon, k)$ which satisfies

$$|\mathbb{P}_\phi[|p_{t+T}(p,\psi)| \leq k + 2\epsilon] - \mathbb{P}_\phi[|p^l_{t+T}(p,\psi)| \leq k + \epsilon]| < \frac{\epsilon}{4}$$

for all $T \geq T_0$ and for a given $\epsilon > 0$. Now recall that in the stationary setting, i.e., under the measure $\mathbb{P}^*$, the process $\{p_{t+T}(p,\psi)\}_{T\in\mathbb{N}}$ — and therefore, by (15), also the process $\{p^l_{t+T}(p,\psi)\}_{T\in\mathbb{N}}$ — converges $\mathbb{P}^*$-a.s. to the stationary process $\{P^*_{t+T}\}_{T\in\mathbb{N}}$. In particular, we deduce that

$$\mathbb{P}^*[|p^l_{t+T}(p,\psi)| \leq k + \epsilon] \geq \mathbb{P}^*[|P^*_t| \leq k] - \frac{\epsilon}{4}$$

for all T sufficiently large, say for $T \geq T_0(\epsilon, K)$. Altogether, we obtain that

$$\begin{aligned}
\mathbb{P}_\phi[|p_{t+T}(p,\psi)| \leq k + 2\epsilon] &\geq \mathbb{P}_\phi[|p^l_{t+T}(p,\psi)| \leq k + \epsilon] - \frac{\epsilon}{4} \\
&\geq \mathbb{P}^*[|p^l_{t+T}(p,\psi)| \leq k + \epsilon] - \frac{\epsilon}{2} \\
&\geq \mathbb{P}^*[|P^*_t| \leq k] - \frac{3\epsilon}{4} \\
&= \mathbb{P}^*[|P^*_0| \leq k] - \frac{3\epsilon}{4} \geq 1 - \epsilon
\end{aligned}$$

for all $T \geq T_0(\epsilon, K)$. Now we can obviously choose a compact set $\tilde{K} = \tilde{K}(\epsilon) \subset \mathbb{R}$ such that

$$\sup_T \mathbb{P}_\phi[p_{t+T}(p,\psi) \in \tilde{K}] \geq 1 - 2\epsilon.$$

This proves the theorem. □

Let us now consider two case studies, where we can verify the assumption that the environment ψ has a "nice" asymptotic behaviour. We assume that ψ is driven by some underlying Markov process $M = \{m_t\}_{t\in\mathbb{N}}$ with transition operator U and with state space $X \subset \mathbb{R}$. The sequence M is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P}_m)$. We assume that for any initial distribution μ_0 of m the distributions μ_t of m_t converge weakly to a unique stationary measure μ^* as $t \to \infty$. Uniqueness of the stationary probability measure implies that the Markov chain M is ergodic under the measure $\mathbb{P}^*(\cdot) := \int_X \mathbb{P}_m(\cdot)\mu^*(dm)$. Furthermore, we suppose that

$$A_t := f(m_t) \quad \text{and} \quad B_t := g(m_t)$$

for some bounded measurable real-valued functions f and g on $\mathbb{R}$. It follows that the sequence $\psi = \{(f(m_t), g(m_t))\}_{t\in\mathbb{N}}$ is also stationary and ergodic by Propositions 4.1 and 4.3 in Krengel (1988).

Example 3.7 *Suppose that the distribution μ_t of m_t converge to μ^* in the norm of total variation. This assumption is satisfied if, for example, M satisfies Doeblin's condition.*[1] *In this case we have that*

$$||U^t - \mu^*||_\infty = \sup_{F\in B(X),|F|_\infty\leq 1} |U^tF - \mu^*(F)|_\infty \to 0 \quad (t\to\infty). \tag{21}$$

Here, U^t denotes the t-fold iteration of the operator U. Thus, Assumption 4 holds true as the mapping $m \mapsto \mathbb{P}_m[\{\psi_1,\ldots,\psi_l\} \in A^l]$ $(l\in\mathbb{N},\ A^l \in \mathcal{B}(\mathbb{R}^{2l}))$ is bounded and measurable. Indeed, for any $l \in \mathbb{N}$ we have that

$$\begin{aligned} & \left| U^{t+1}\mathbb{P}_m[\{\psi_1,\ldots,\psi_l\}\in A^l] - \int \mathbb{P}_m[\{\psi_1,\ldots,\psi_l\}\in A^l]\mu^*(dm)\right|_\infty \\ \leq\ & ||\mathrm{Law}(\psi,\mathbb{P}_m) - \mathrm{Law}(\psi,\mathbb{P}^*)||_{\hat{\mathcal{F}}_{t+1,l}} \\ \leq\ & ||U^t - \mu^*||_\infty \to 0 \quad (t\to\infty) \end{aligned}$$

by (21). Hence the environment $\psi = \{(f(m_t), g(m_t))\}_{t\in\mathbb{N}}$ has a "nice" tail structure.

Example 3.8 *Let f and g be continuous and suppose that the transition operator U of M maps $C(X)$ into itself. Let the sequence $\{\mu_t\}_{t\in\mathbb{N}}$ be uniformly weakly convergent to μ^* in the sense that*

$$\sup_{F\in C(X),|F|_\infty\leq 1} |U^tF - \mu^*(F)|_\infty \to 0 \quad (t\to\infty).$$

Suppose furthermore, that the transition kernel P of the Markov chain M is strongly continuous, i.e., for any event $A \in \mathcal{B}(\mathbb{R})$, the mapping $m \mapsto P(m, A)$ is continuous. Using the same arguments as above, it is easily verified that Assumption 4 is satisfied and that the sequence ψ is "nice" in the sense of Definition 2.2.

[1] For a discussion of uniformly ergodic Markov chains and Markov chains satisfying Doeblin's condition we refer the reader to Meyn and Tweedie (1993), Chapter 16.

4 The General Case

This section is devoted to the proof of Theorem 2.4. Let us first sketch the basic idea. Recall that we introduced a sequence of "perturbed" random environments ψ^n $(n \in \mathbb{N})$ defined on a common probability space with the sequence ψ, such that $\text{Law}(\psi^n, \mathbb{P}_\phi) \xrightarrow{w} \text{Law}(\psi, \mathbb{P}_\phi)$ as $n \to \infty$. Each of the sequences ψ^n is stationary and ergodic under $\mathbb{P}^*$ and "nice" by Assumption 3. The following simple result implies for any $n \in \mathbb{N}$ the existence of a unique stationary solution $\{P_t^n\}_{t\in\mathbb{N}}$ of (1) under the measure $\mathbb{P}^*$ driven by the process ψ^n.

Remark 4.1 *Due to (8) we may assume that there exists a negative constant β, such that*

$$\sup_n \mathbb{E}^* \ln |A_0^n| < \beta < 0. \tag{22}$$

Moreover we have that $\sup_n \mathbb{E}^*(\ln |B_0^n|)^+ < \infty$. Thus, each sequence ψ^n $(n \in \mathbb{N})$ satisfies Assumption 2.

As each of the sequences ψ^n $(n \in \mathbb{N})$ satisfies Assumptions 1 and 2 we know by Theorem 2.3 that for any $n \in \mathbb{N}$ the following holds true:

$$\text{Law}(p_t(p, \psi^n), \mathbb{P}_\phi) \xrightarrow{w} \text{Law}(P_0^n, \mathbb{P}^*) := \nu_\infty^n \quad (t \to \infty).$$

Using an approximation result provided in Brandt (1986) we will deduce that the sequence $\{p_t(p, \psi)\}_{t\in\mathbb{N}}$, driven by the non-stationary input ψ, converges in law to the unique weak limit ν_∞ of the sequence $\{\nu_\infty^n\}_{n\in\mathbb{N}}$. Establishing weak convergence of the finite-dimensional distributions will then be an easy task.

The following result, which is an immediate consequence of Theorem 2 in Brandt (1986), shows that the asymptotic distribution of the processes $\{p_t(p, \psi^n)\}_{t\in\mathbb{N}}$ converge in distribution to $\text{Law}(P_0^*, \mathbb{P}^*)$ as $n \to \infty$.

Remark 4.2 *Suppose that the Assumptions of Theorem 2.5 are satisfied. In the stationary setting we have the following "global" stability result:*

$$\text{Law}(\{P_t^n\}_{t\in\mathbb{N}}, \mathbb{P}^*) \xrightarrow{w} \text{Law}(\{P_t^*\}_{t\in\mathbb{N}}, \mathbb{P}^*) \quad (n \to \infty). \tag{23}$$

In particular,

$$\text{Law}(P_0^n, \mathbb{P}^*) \xrightarrow{w} \text{Law}(P_0^*, \mathbb{P}^*) \quad (n \to \infty).$$

Furthermore, we can easily verify another "global" approximation result.

Remark 4.3 *The following "global" approximation result holds true:*

$$\text{Law}(\{p_t(p, \psi^n)\}_{t\in\mathbb{N}}, \mathbb{P}_\phi) \xrightarrow{w} \text{Law}(\{p_t(p, \psi)\}_{t\in\mathbb{N}}, \mathbb{P}_\phi) \text{ on } \mathbb{R}^\mathbb{N} \quad (n \to \infty)$$

Proof: The assertion follows immediately from the continuous mapping theorem because $\text{Law}(\psi^n, \mathbb{P}_\phi) \xrightarrow{w} \text{Law}(\psi, \mathbb{P}_\phi)$ and since the mapping $\psi \mapsto \{p(t,\psi)\}_{t\in\mathbb{N}}$ from $\mathbb{R}^{2\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$ is continuous in the product topology. □

The rest of this section is devoted to the analysis of the asymptotic distribution of the random variable $p_t(p,\psi)$ under the measure $\mathbb{P}_\phi$. The next proposition yields a uniform approximation result, which will turn out to be the key for the proof of Theorem 2.5.

Proposition 4.4 *For any Lipschitz continuous function F with compact support we have*

$$\lim_{n\to\infty} \sup_t \mathbb{E}_\phi |F(p_t(p,\psi)) - F(p_t(p,\psi^n))| = 0.$$

Proof: Let F be a Lipschitz continuous function on $\mathbb{R}$ with compact support and with Lipschitz constant $m(F) = 1$. Without loss of generality we may assume that diam(supp F)=1. In this case $|F(x) - F(y)| \le G(|x-y|)$, where $G(x) := |x| \wedge 1$, and it suffices to show that

$$\overline{\lim}_{n\to\infty} \sup_t \mathbb{E}_\phi G(|p_t(p,\psi) - p_t(p,\psi^n)|) = 0.$$

To this end, we can proceed as follows. Without loss of generality we choose $\sigma_n = \frac{\eta}{n}$ for some constant $\eta > 0$. Thus we can write

$$\begin{aligned} p_t(p,\psi^n) &= \sum_{j=0}^{t-1} \left\{ \sum_{\mathcal{I}\subset\mathcal{I}_j} \left(\prod_{i\in\mathcal{I}} A_i \prod_{i\in\mathcal{I}_j\setminus\mathcal{I}} \frac{\eta}{n}\epsilon_i \right) \right\} \left(B_{t-1-j} + \frac{\eta}{n}\eta_{t-1-j} \right) \\ &\quad + \sum_{\mathcal{I}\subset\mathcal{I}_t} \left(\prod_{i\in\mathcal{I}} A_i \prod_{i\in\mathcal{I}_t\setminus\mathcal{I}} \frac{\eta}{n}\epsilon_i \right) \end{aligned}$$

where $\mathcal{I}_j := \{t-j,\dots,t-1\}$ and $\mathcal{I}_0 := \emptyset$. This gives us

$$\begin{aligned} &|p_t(p,\psi) - p_t(p,\psi^n)| \\ &\le \sum_{j=0}^{t-1} \sum_{\mathcal{I} \subsetneq \mathcal{I}_j} \left(\prod_{i\in\mathcal{I}} |A_i| \prod_{i\in\mathcal{I}\setminus\mathcal{I}_j} \frac{\eta}{n}|\epsilon_i| \right) |B_{t-j-1}| \\ &\quad + \sum_{j=0}^{t-1} \sum_{\mathcal{I}\subset\mathcal{I}_j} \left(\prod_{i\in\mathcal{I}} |A_i| \prod_{i\in\mathcal{I}\setminus\mathcal{I}_j} \frac{\eta}{n}|\epsilon_i| \right) \frac{\eta}{n}|\eta_{t-j-1}| \\ &\quad + \sum_{\mathcal{I}\subsetneq\mathcal{I}_t} \prod_{i\in\mathcal{I}} |A_i| \prod_{i\in\mathcal{I}_t\setminus\mathcal{I}} \frac{\eta}{n}|\epsilon_i| \\ &\le \frac{1}{n} \left\{ \sum_{j=0}^{t-1} \sum_{\mathcal{I}\subset\mathcal{I}_j} \left(\prod_{i\in\mathcal{I}} |A_i| \prod_{i\in\mathcal{I}\setminus\mathcal{I}_j} \eta|\epsilon_i| \right) (|B_{t-j-1}| + \eta|\eta_{t-j-1}|) \right. \end{aligned}$$

$$
\begin{aligned}
& + \sum_{\mathcal{I} \subset \mathcal{I}_t} \left(\prod_{i \in \mathcal{I}} |A_i| \prod_{i \in \mathcal{I}_t \setminus \mathcal{I}} \eta |\epsilon_i| \right) \Bigg\} \\
= \;& \frac{1}{n} p_t(p, \widetilde{\psi}).
\end{aligned}
$$

Thus, it suffices to show that

$$
\overline{\lim}_{n \to \infty} \sup_t \mathbb{E}_\phi \left[G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) \right] = 0.
$$

To this end, we proceed in several steps.

Step 1: Suppose that under some measure Q the sequence $\{p_t(p, \widetilde{\psi})\}_{t \in \mathbb{N}}$ is stationary and that $p_1(p, \widetilde{\psi})$ is $Q - a.s.$ finite. In this case we have

$$
\lim_{n \to \infty} \sup_t \mathbb{E}_Q \left[G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) \right] = \lim_{n \to \infty} \mathbb{E}_Q \left[G\left(\frac{1}{n} p_1(p, \widetilde{\psi}) \right) \right] = G(0) = 0
$$

by Lebesque's theorem.

Step 2: Suppose now that the process $\{p_t(p, \widetilde{\psi})\}_{t \in \mathbb{N}}$ itself is not stationary under the measure Q. Instead, suppose that this sequence converges with probability one to an almost surely finite process $\{\widetilde{P}_t\}_{t \in \mathbb{N}}$ which is stationary under Q, i.e., let

$$
\lim_{t \to \infty} |p_t(p, \widetilde{\psi}) - \widetilde{P}_t| = 0 \quad Q - a.s.
$$

hold true. In this case we know by Step 1 above that

$$
\begin{aligned}
& \overline{\lim}_{n \to \infty} \sup_t \mathbb{E}_Q G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) \\
\leq \;& \overline{\lim}_{n \to \infty} \sup_t \mathbb{E}_Q \left| G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) - G\left(\frac{1}{n} \widetilde{P}_t \right) \right| + \lim_{n \to \infty} \sup_t \mathbb{E}_Q G\left(\frac{1}{n} \widetilde{P}_t \right) \\
= \;& \overline{\lim}_{n \to \infty} \sup_t \mathbb{E}_Q \left| G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) - G\left(\frac{1}{n} \widetilde{P}_t \right) \right|.
\end{aligned}
$$

Now let $\epsilon > 0$ be given. As G is bounded an Lipschitz continuous we can choose $T = T(\epsilon) \in \mathbb{N}$ such that

$$
\mathbb{E}_Q \left| G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) - G\left(\frac{1}{n} \widetilde{P}_t \right) \right| \leq 2\epsilon \quad \text{for all } t \geq T \text{ and for all } n \in \mathbb{N}.
$$

As $G(0) = 0$, and since the random variables $p_t(p, \widetilde{\psi})$ $(t \in \mathbb{N})$ are $Q - a.s.$ finite we have that

$$
\overline{\lim}_{n \to \infty} \sup_t \mathbb{E}_Q \left| G\left(\frac{1}{n} p_t(p, \widetilde{\psi}) \right) - G\left(\frac{1}{n} \widetilde{P}_t \right) \right|
$$

$$
\begin{aligned}
&\leq \overline{\lim}_{n\to\infty} \sup_{t\geq T} \mathbb{E}_Q \left| G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) - G\left(\frac{1}{n}\widetilde{P}_t\right)\right| \\
&\quad + \overline{\lim}_{n\to\infty} \sum_{t=1}^{T-1} \left\{ \mathbb{E}_Q G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) + \mathbb{E}_Q G\left(\frac{1}{n}\widetilde{P}_t\right)\right\} \\
&\leq 2\epsilon
\end{aligned}
$$

by Lebesque's Theorem. As $\epsilon > 0$ is arbitrary we conclude that

$$
\overline{\lim}_{n\to\infty} \sup_t \mathbb{E}_Q G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) = 0.
$$

Step 3: For a given $\epsilon > 0$ we can choose $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ the following holds true:

$$
\sup_{t\geq l} \mathbb{E}_\phi G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right) \leq \sup_{t\geq l} \mathbb{E}^* G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right) + \epsilon \tag{24}
$$

where $p_t^l(p,\widetilde{\psi})$ is defined as in (14). Indeed, as $\mathbb{P}_\phi - \mathbb{P}^*$ is a signed measure on $\mathcal{F}$ we have that

$$
\begin{aligned}
&\int G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right)(d\mathbb{P}_\phi - d\mathbb{P}^*) \\
&\leq \sup_{A\in\mathcal{B}(\mathbb{R}), t\geq l} |\mathbb{P}_\phi[p_t^l(p,\widetilde{\psi}) \in A] - \mathbb{P}^*[p_t^l(p,\widetilde{\psi}) \in A]|.
\end{aligned}
$$

Now observe that the event $\{p_t^l(p,\widetilde{\psi}) \in A\}$, $A \in \mathcal{B}(\mathbb{R})$, belongs to the $\sigma-$field $\hat{\mathcal{G}}_l = \sigma(\{\widetilde{\psi}_s\}_{s\geq l})$. Thus, as the sequence $\widetilde{\psi}$ has a "nice" tail structure, we deduce from Proposition 3.2 that there exists a sequence $c_l \to 0$ such that

$$
\sup_{t\geq l} \mathbb{E}_\phi G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right) \leq \sup_{t\geq l} \mathbb{E}^* G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right) + c_l
$$

which yields (24) for l sufficiently large.

Our aim is now to control the left hand side of inequality (24), and to this end, we apply Step 2 with $Q = \mathbb{P}^*$ in order to control the right hand side of (24) for l sufficiently large. Note that we can apply Theorem 3.5 to the sequence $\{p_t(p,\widetilde{\psi})\}_{t\in\mathbb{N}}$ as soon as

$$
\mathbb{E}^* \ln[|A_0| + \eta|\epsilon_0|] < 0.
$$

This, however, follows immediately from a monotone convergence argument as soon as η is sufficiently small due to (9). Thus, there exists a process $\{\widetilde{P}_t\}_{t\in\mathbb{N}}$ which is stationary under the measure $\mathbb{P}^*$ such that

$$
\lim_{t\to\infty} |p_t(p,\widetilde{\psi}) - \widetilde{P}_t| = 0 \quad \mathbb{P}^* - a.s.
$$

Using Proposition 3.3 we can proceed as in the proof of Theorem 3.6 to show that

$$
\lim_{t\to\infty} |p_t^l(p,\widetilde{\psi}) - p_t(p,\widetilde{\psi})| = 0 \quad \mathbb{P}_\phi - \text{ and } \mathbb{P}^* - a.s. \quad (l \in \mathbb{N}) \tag{25}
$$

for η sufficiently small. In particular, we have, for any $l \in \mathbb{N}$, that

$$\lim_{t\to\infty} |p_t^l(p,\widetilde{\psi}) - \widetilde{P}_t| = 0 \quad \mathbb{P}^* - a.s.$$

Therefore, we can apply Step 2 in order to control the right hand side of (24). We obtain that

$$\overline{\lim}_{n\to\infty} \sup_{t\geq l} \mathbb{E}_\phi G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right) \;\leq\; \epsilon + \overline{\lim}_{n\to\infty} \sup_{t\geq l} \mathbb{E}^* G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right) = \epsilon \quad (26)$$

for l sufficiently large.

Step 4: Let $\epsilon > 0$ be given. We choose $l \in \mathbb{N}$ such that (26) holds true and deduce from Step 3 and from $G(0) = 0$ that

$$\begin{aligned}
&\overline{\lim}_{n\to\infty} \sup_{t} \mathbb{E}_\phi G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right)\\
=\;& \overline{\lim}_{n\to\infty} \sup_{t\geq l} \mathbb{E}_\phi G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right)\\
\leq\;& \overline{\lim}_{n\to\infty} \sup_{t\geq l} \left|\mathbb{E}_\phi \left\{G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) - G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right)\right\}\right| + \epsilon.
\end{aligned}$$

Now, by (25) and as G is bounded and Lipschitz continuous, we can choose $T = T(\epsilon, l)$ such that $T \geq l$ and such that

$$\sup_{t\geq T} \left|\mathbb{E}_\phi \left\{G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) - G\left(\frac{1}{n} p_t^l(p,\widetilde{\psi})\right)\right\}\right| < \epsilon.$$

As $G(0) = 0$ we can now proceed as in Step 2 to conclude that

$$\overline{\lim}_{n\to\infty} \sup_{t} \mathbb{E}_\phi G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) < 2\epsilon.$$

As $\epsilon > 0$ is arbitrary we deduce that $\overline{\lim}_{n\to\infty} \sup_t \mathbb{E}_\phi G\left(\frac{1}{n} p_t(p,\widetilde{\psi})\right) = 0$. This proves our assertion. □

Corollary 4.5 *Let $m \in \mathbb{N}$ be given. For any Lipschitz continuous function F on $\mathbb{R}^m$ with compact support we have that*

$$\lim_{n\to\infty} \sup_{t_1,\ldots,t_m} \mathbb{E}_\phi |F(p_{t_1}(p,\psi^n),\ldots,p_{t_m}(p,\psi^n)) - F(p_{t_1}(p,\psi),\ldots,p_{t_m}(p,\psi))| = 0.$$

Proof: Without loss of generality we may again assume that $m(F) = 1$ and that diam(supp F)=1. Thus, Jensen's inequality yields

$$\mathbb{E}_\phi |F(p_{t_1}(p,\psi^n),\ldots,p_{t_m}(p,\psi^n)) - F(p_{t_1}(p,\psi),\ldots,p_{t_m}(p,\psi))|$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{m} \mathbb{E}_\phi^{1/2}[|p_{t_i}(p,\psi^n) - p_{t_i}(p,\psi)|^2 \wedge 1] \\
&\leq \sum_{i=1}^{m} \mathbb{E}_\phi^{1/2}[|p_{t_i}(p,\psi^n) - p_{t_i}(p,\psi)| \wedge 1]^2 \\
&\leq m \sup_t \mathbb{E}_\phi^{1/2} G^2(|p_t(p,\psi^n) - p_t(p,\psi)|).
\end{aligned}
$$

Therefore, the assertion follows by the same arguments as in the proof of Proposition 4.4. □

As an easy consequence of Proposition 4.4 and Theorem 3.6, we can now establish weak convergence of the finite dimensional distributions of the process $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ under $\mathbb{P}_\phi$ to the finite-dimensional distributions of the unique stationary solution of (1) under the measure $\mathbb{P}^*$.

Proof of Theorem 2.5:

Let us first establish convergence of the one-dimensional distributions, i.e., let us show that

$$
\mathrm{Law}(p_t(p,\psi), \mathbb{P}_\phi) \xrightarrow{w} \nu_\infty := \mathrm{Law}(P_0^*, \mathbb{P}^*) \quad (t \to \infty).
$$

To this end, let F be a real valued Lipschitz continuous function on $\mathbb{R}$ with compact support. According to Proposition 4.4 we know that

$$
\lim_{n\to\infty} \sup_t \mathbb{E}_\phi |F(p_t(p,\psi)) - F(p_t(p,\psi^n))| = 0 \tag{27}
$$

for a suitable sequence $\sigma_n \downarrow 0$ $(n \to \infty)$. As each of the sequences ψ^n $(n \in \mathbb{N})$ is "nice" in the sense of Definition 2.2 we can proceed as follows. We have that

$$
\begin{aligned}
& |\mathbb{E}_\phi F(p_t(p,\psi)) - \nu_\infty(F)| \\
\leq\; & |\mathbb{E}_\phi F(p_t(p,\psi)) - \mathbb{E}_\phi F(p_t(p,\psi^n))| && (28) \\
& + |\mathbb{E}_\phi F(p_t(p,\psi^n)) - \nu_\infty^n(F)| && (29) \\
& + |\nu_\infty^n(F) - \nu_\infty(F)| && (30)
\end{aligned}
$$

where we put $\nu_\infty^n = \mathrm{Law}(P_0^n, \mathbb{P}^*)$. Obviously, (28) can be estimated by

$$
(28) \leq \sup_t \mathbb{E}_\phi[|F(p_t(p,\psi)) - F(p_t(p,\psi^n))|]. \tag{31}
$$

Now let $\epsilon > 0$ be given. By (27) there exists $N \in \mathbb{N}$ such that

$$
(28) \leq \frac{\epsilon}{3} \quad \text{for all} \quad n \geq N \quad \text{and for all} \quad t \in \mathbb{N}.
$$

By Remark 4.2 we also have that $(30) \leq \epsilon/3$ for $n \geq N$. Now let $n \geq N$ be fixed. By Theorem 2.3 there exists a constant $T(n)$ such that

$$
(29) = |\nu_t^n(F) - \nu_\infty^n(F)| < \frac{\epsilon}{3} \quad \text{for all} \quad t \geq T(n)
$$

where $\nu_t^n := \text{Law}(p_t(p, \psi^n), \mathbb{P}_\phi)$. In particular we get

$$|\mathbb{E}_\phi F(p_t(p, \psi)) - \nu_\infty(F)| \leq \epsilon$$

for all $t \geq T(n)$. This shows vage convergence of the sequence $\{\text{Law}(p_t(p, \psi), \mathbb{P}_\phi)\}_{t\in\mathbb{N}}$ to the probability measure ν_∞, and therefore we have

$$\text{Law}(p_t(p, \psi), \mathbb{P}_\phi) \xrightarrow{w} \nu_\infty \quad (t \to \infty).$$

Now convergence of the finite-dimensional distributions follows immediately from Corollary 4.5 and from Remark 4.2 in analogy to the proof of Theorem 2.3. □

5 A Markovian Case Study

Let us now study a class of Markovian models where our Assumption 3 that the driving process ψ can be approximated in law by sequence of "nice" processes $\{\psi^n\}_{n\in\mathbb{N}}$ can indeed be verified.

We assume that the random variables ϵ_t, η_t $(t \in \mathbb{N})$ are N(0,1) distributed and that $M = \{m_t\}_{t\in\mathbb{N}}$ is a Markov process associated to a random system with complete connections[2] defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_m)$. More precisely, we suppose that there exists a measurable space $(E, \mathcal{E})$, a compact set $X \subset \mathbb{R}$, a measurable function $u : X \times E \to X$ and a stochastic kernel K on $(X, \mathcal{E})$ such that the random variables $\{m_t\}_{t\in\mathbb{N}}$ obey the relation

$$m_{t+1} = u(m_t, e_t) \quad \text{where} \quad e_t \sim K(m_t; \cdot). \tag{32}$$

In particular, the transition operator U of the process M acts on the set of all bounded measurable functions F on X according to the equation

$$UF(m) = \int_E F(u(m,e))K(m, de).$$

We are going to consider the driving sequence

$$\psi = \{(f(m_t), g(m_t))\}_{t\in\mathbb{N}},$$

where f and g are Lipschitz continuous functions which are assumed to be bounded away from zero.

Assumption 5 *The kernel K satisfies the Lipschitz condition*

$$\sup_{A\in\mathcal{E}} \sup_{m\neq m'} \frac{|K(m,A) - K(m',A)|}{|m - m'|} \leq C \quad (C < \infty),$$

[2]For a detailed discussion of random systems with complete connections we refer the reader to the books of Iosefescu and Theodorescu (1968) and Iosefescu and Gregorescu (1993).

and the mapping u has the mean contraction property, i.e.,

$$\left| \int_E (u(m,e) - u(m',e)K(m,de) \right| \leq \theta |m - m'|$$

for some constant $\theta < 1$.

In Norman (1963), Chapters 3 and 4, several criteria are given, which ensure that the process M governed by (32) converges in distribution to a unique equilibrium measure μ^*, and that for any Lipschitz continuous function F on X we have that

$$|U^t F - \mu^*(F)|_\infty \leq c(F)\alpha^t \tag{33}$$

for some $\alpha < 1$. Here, the constant $c(F)$ depends on F only through its Lipschitz constant and through its global maximum $|F|_\infty$. For the rest of this section we shall assume that (33) holds true.

Let us now verify Assumption 3. First, we will show that the sequences $\widetilde{\psi}$ and ψ^n ($n \in \mathbb{N}$) have a "nice" tail structure.

Lemma 5.1 *Let $f, g : X \to \mathbb{R}$ be Lipschitz continuous and suppose that Assumption 5 is satisfied. For any fixed $n \in \mathbb{N}$ there exists a constant $C(n) < \infty$ such that*

$$\sup_{A^l \in \mathcal{B}(\mathbb{R}^{2l})} \sup_{m \neq m'} \frac{|Q^{n,l}(m, A^l) - Q^{n,l}(m', A^l)|}{|m - m'|} \leq C(n),$$

and

$$\sup_{A^l \in \mathcal{B}(\mathbb{R}^{2l}_+)} \sup_{m \neq m'} \frac{|\widetilde{Q}^{1,l}(m, A^l) - \widetilde{Q}^{1,l}(m', A^l)|}{|m - m'|} \leq C(1)$$

uniformly in $l \in \mathbb{N}$. Here, we defined $Q^{n,l}(m, A^l) := \mathbb{P}_m[\{\psi^n_1, \ldots, \psi^n_l\} \in A^l]$ and $\widetilde{Q}^{1,l}(m, A^l) := \mathbb{P}_m[\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_l\} \in A^l]$.

Proof: Without loss of generality we may assume that $n = 1$. Let us first consider the case $l = 1$. For $m, m' \in X$ and for $A \in \mathcal{B}(\mathbb{R}^2)$ we have that

$$\begin{aligned} & |Q^{1,1}(m, A) - Q^{1,1}(m', A)| \\ = \; & \tfrac{1}{\sqrt{2\pi}\sigma_1} \Big| \textstyle\int_A \exp\left(-\tfrac{1}{2\sigma_1^2}\{(x_1 - f(m)) + (x_2 - g(m))\}\right) dx_1 dx_2 \\ & - \textstyle\int_A \exp\left(-\tfrac{1}{2\sigma_1^2}\{(x_1 - f(m')) + (x_2 - g(m'))\}\right) dx_1 dx_2 \Big| . \end{aligned}$$

For $m \in X$ we put

$$\mathcal{M}^+(m) := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq |f(m)|, x_2 \geq |g(m)|\},$$

$$\mathcal{M}^-(m) := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq -|f(m)|, x_2 \leq -|g(m)|\}.$$

For $A \in \mathcal{B}(\mathbb{R}_+^2)$ we get

$$\begin{aligned}
& |\widetilde{Q}^{1,1}(m,A) - \widetilde{Q}^{1,1}(m',A)| \\
\leq \quad & \tfrac{1}{\sqrt{2\pi}} \Big| \int_A \exp\left(-\tfrac{1}{2\sigma_1^2}\{(x_1 - |f(m)|) + (x_2 - |g(m)|)\}\right) \mathbf{1}_{\mathcal{M}^+(m)}(x_1,x_2)dx_1dx_2 \\
& - \int_A \exp\left(-\{(x_1 - |f(m')|) + (x_2 - |g(m')|)\}\right) \mathbf{1}_{\mathcal{M}^+(m')}(x_1,x_2)dx_1dx_2 \Big| \\
& + \tfrac{1}{\sqrt{2\pi}} \Big| \int_{-A} \exp\left(-\tfrac{1}{2\sigma_1^2}\{(x_1 + |f(m)|) + (x_2 + |g(m)|)\}\right) \mathbf{1}_{\mathcal{M}^-(m)}(x_1,x_2)dx_1dx_2 \\
& - \int_{-A} \exp\left(-\tfrac{1}{2\sigma_1^2}\{(x_1 + |f(m')|) + (x_2 + |g(m')|)\}\right) \mathbf{1}_{\mathcal{M}^-(m')}(x_1,x_2)dx_1dx_2 \Big|
\end{aligned}$$

An easy calculation shows that the assertion follows from the Lipschitz continuity of the functions f and g. Thus for any $n \in \mathbb{N}$ there exists a constant $c(n)$ such that

$$\sup_{A^1 \in \mathcal{B}(\mathbb{R})} \frac{|Q^{n,1}(m,A^1) - Q^{n,1}(m',A^1)|}{|m-m'|} \leq c(n)$$

and such that

$$\sup_{A^1 \in \mathcal{B}(\mathbb{R})} \frac{|\widetilde{Q}^{1,1}(m,A^1) - \widetilde{Q}^{1,1}(m',A^1)|}{|m-m'|} \leq c(1).$$

Now we can proceed as in the proof of Lemma 2.1.63 in Iosefescu and Theodorescu (1968) to verify that the constant $C(n)$ defined by $C(n) := \frac{C+c(n)}{1-\theta}$ yields the desired result. Indeed, let us put

$$\alpha_l^n := \sup_{A^l \in \mathcal{B}(\mathbb{R}^{2l})} \sup_{m \neq m'} \frac{|Q^{n,l}(m;A^l) - Q^{n,l}(m';A^l)|}{|m-m'|}.$$

In particular, $\alpha_1^n = c(n)$. We shall now demonstrate that $\alpha_l^n \leq \frac{C+c(n)}{1-\theta}$ uniformly in $l \in \mathbb{N}$. To this end, we can proceed as follows. Let $A^{l+1} \in \mathcal{B}(\mathbb{R}^{2(l+1)})$ be given and note that

$$\begin{aligned}
& |Q^{n,l+1}(m,A^{l+1}) - Q^{n,l+1}(m',A^{l+1})| \\
= \quad & \Big| \int_{A^{l+1}} \int_E \big\{ Q^{n,1}(m;d\eta) \otimes Q^{n,l}(u(m,e);d\zeta) \otimes K(m;de) \\
& - Q^{n,1}(m';d\eta) \otimes Q^{n,l}(u(m',e);d\zeta) \otimes K(m';de) \big\} \Big| \\
\leq \quad & \sup_{A^l \in \mathcal{B}(\mathbb{R}^{2l})} \left\{ \int_E \{ Q^{n,l}(u(m,e);A^l) \otimes K(m;de) - Q^{n,l}(u(m',e);A^l) \otimes K(m';de) \} \right\} \\
& + c(n)|m-m'| \\
\leq \quad & (c(n)+C)|m-m'| + \sup_{A^l} \left| \int_E \{ Q^{n,l}(u(m,e);A^l) - Q^{n,l}(u(m',e);A^l) \} K(m;de) \right| \\
\leq \quad & (c(n) + C + \theta\alpha_l^n)|m-m'|,
\end{aligned}$$

where the last inequality follows from the mean contraction property of the mapping u and from the definition of α_l^n. In particular, we have that

$$\alpha_{l+1}^n \leq c(n) + C + \theta\alpha_l^n,$$

and from this it is easily seen that

$$\alpha_l^n \leq (c(n) + C) \sum_{i \in \mathbb{N}} \theta^i \leq \frac{c(n) + C}{1 - \theta}.$$

The same arguments remain valid if we replace $Q^{n,l}$ by $\widetilde{Q}^{1.l}$ in the above calculations. This proves our assertion. □

Corollary 5.2 *Under the assumptions of Lemma 5.1 the approximating sequences ψ^n $(n \in \mathbb{N})$ and $\widetilde{\psi}$ are "nice".*

Proof: For each $n \in \mathbb{N}$ the mappings

$$m \mapsto \mathbb{P}_m[\{\psi_1^n, \ldots, \psi_l^n\} \in A^l], \quad m \mapsto \mathbb{P}_m[\{\widetilde{\psi}_1, \ldots, \widetilde{\psi}_l\} \in A^l]$$

are Lipschitz continuous uniformly in $l \in \mathbb{N}$ and in $A^l \in \mathcal{B}(\mathbb{R}^{2l})$ by Lemma 5.1. Thus, from (33) we deduce that there exists a constant $\alpha < 1$ such that

$$\begin{aligned} & \sup_l ||\mathrm{Law}(\psi^n, \mathbb{P}_\phi) - \mathrm{Law}(\psi^n, \mathbb{P}^*)||_{\hat{\mathcal{F}}_{t,l}} \\ \leq & \sup_l \sup_{A^l \in \mathcal{B}(\mathbb{R}^{2l})} \Big| U^t \mathbb{P}_m[\{\psi_0^n, \ldots, \psi_{l-1}^n\} \in A^l] \\ & - \int_X \mathbb{P}_m[\{\psi_0^n, \ldots, \psi_{l-1}^n\} \in A^l] \mu^*(dm) \Big|_\infty \\ \leq & \widetilde{C}(n) \alpha^t \end{aligned}$$

for some constant $\widetilde{C}(n)$ depending only on $C(n)$. Of course, the same arguments remain valid if we replace ψ^n by $\widetilde{\psi}$ in the above calculations. Thus we verified Assumption 4 and deduce that the sequences $\widetilde{\psi}$ and ψ^n $(n \in \mathbb{N})$ are "nice" in the sense of Definition 2.2. □

Let us now consider the integrability conditions (7), (8) and (9).

Remark 5.3 *In our present setting (7), (8) and (9) hold true.*

Proof: Observe first, that (9) holds true as the functions f and g are bounded. To verify (7), let $c > 0$ be such that $|f|_\infty \leq e^c$. Observe that the mapping

$$(m, z) \mapsto (\ln |f(m) + \sigma_n z|)^+ \wedge c$$

is bounded and continuous. Thus,

$$\mathbb{E}^* \left[(\ln |f(m_1) + \sigma_n \epsilon_1|)^+ \wedge c \right] \to \mathbb{E}^* \left[(\ln |f(m_1)|)^+ \right] \quad (n \to \infty).$$

As

$$\{(\ln |f(m_1) + \sigma_n \epsilon_1|)^+ \geq c\} \subset \left\{ \epsilon_1 \geq \frac{e^c - |f|_\infty}{\sigma_n} \right\}$$

we have that

$$
\begin{aligned}
& |\mathbb{E}^* \left[(\ln|f(m_1)+\sigma_n\epsilon_1|)^+ \wedge c\right] - \mathbb{E}^* \left[(\ln|f(m_1)+\sigma_n\epsilon_1|)^+\right]| \\
\leq & \int_{c'/\sigma_n} (x+|f|_\infty) dN(0,1) \\
\to & \ 0 \quad (n\to\infty).
\end{aligned}
$$

Here, we put $c' = \frac{e^c - |f|_\infty}{\sigma_n}$. Of course, the same arguments remain valid, if we replace f by g in the above calculations.

As the functions f and g bounded away from zero, (8) follows from an uniform integrability argument. □

In our present Markovian setting Theorem 2.5 yields the following result.

Theorem 5.4 *Suppose that the Markov chain $M = \{m_t\}_{t\in\mathbb{N}}$ admits a unique invariant measure μ^* and that Assumptions 2 and 5 are satisfied. Let f and g be Lipschitz continuous functions on X which are bounded away from zero and consider the input*

$$\psi = \{(f(m_t), g(m_t))\}_{t\in\mathbb{N}}.$$

In this case the finite dimensional distributions of the sequence $\{p_t(p,\psi)\}_{t\in\mathbb{N}}$ under the measure $\mathbb{P}_\phi$ ($\phi \in \mathbb{R}^2$) converge weakly to the finite dimensional distributions of the unique stationary solution of (1) in the sense described in Theorem 2.5.

Proof: The assertion follows immediately from Theorem 2.5, from Corollary 5.2 and from our above considerations. □

Remark 5.5
1. *Suppose that the functions f and g are not bounded away from zero. In this case our result remains valid as soon as the convergence condition (8) is satisfied, i.e., as soon as the invariant probability μ^* of M is "sufficiently regular".*

2. *The random variables ϵ_t and η_t ($t \in \mathbb{N}$) in our approximation scheme do not need to be normally distributed. This was only used for convenience in order to obtain an easy proof of Lemma 5.1. Furthermore, some straightforward modifications show that the random variables m_t may take values in any compact metric space.*

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