Auctions

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Abstract

This paper surveys the literature on auctions. To a large part it is concerned with revenue and efficiency considerations. It addresses the problems of finding the auction that yields the greatest expected revenues to the auctioneer and that allocates the good(s) to the bidders which value them most highly. Models with private value, common value and general value will be considered. Open (English, Dutch) and sealed-bid (discriminatory, uniform-price, first-price, Vickrey) auctions are analysed. The analysis focuses on the strategic aspect of the bidding process, i.e. the game-theoretic auction literature will be heavily relied on. The employed equilibrium concept is Bayes-Nash. The theoretical analysis is supplemented by reviewing empirical work resulting from real-life auctions or controlled laboratory experiments. The importance of auctions in designing trading institutions is documented for financial markets.

1 Introduction

The use of auctions to allocate goods is quite common and has been known for centuries. Numerous and quite different goods are sold by auctions, i.e. antiques, fish, flowers, real estate, gold, securities and oil exploration rights. Cassedy (1967) gives a complete list of goods sold by auction. The literature on auctions is enormous, and it is increasing rapidly. It is mainly concerned with revenue and efficiency considerations, namely to find the auction that yields the greatest expected revenues to the auctioneer and that allocates the good(s) to the bidders which value them most highly. Further issues, like the robustness of auctions, transaction costs, bid preparation costs and the vulnerability to cheating are rarely addressed.

2 Definitions

According to Cassedy (1967), auctioning is

> "a unique system of allocating [...] property based on price making by competition of buyers for the right to purchase."
>
> Cassedy (1967), p.8

More formally, Myerson (1981) defines an auction mechanism by (i) a pair of outcome functions, which give the allocation of the good(s) and the bidders' fees in dependence of the bidders' strategies, and (ii) a description of bidders' strategic plans. According to those definitions, there is a plethora of different auctions. To fix ideas, assume that there are N, $N \geq 1$, units of a homogenous good for sale and that each bidder is restricted to one unit of the good. First of all, open and sealed-bid auctions can be distinguished. An auction is called *open*, if the bids are submitted openly, i.e. they become known to all participants of the auction. In contrast to this, an auction is called *sealed-bid*, if bidders submit their bids privately. Thus bidders have no knowledge of the bids of their competitors. The most well-known open auctions are the English and the Dutch auction (cp. Milgrom/Weber (1982)). In the *English auction* the auctioneer raises the price up to the point, where N bidders remain, with the understanding that the N units of the good are sold to the remaining bidders at the last called-out price. In the *Dutch auction* the auctioneer lowers the price interrupted only by bidders accepting the last called-out price for one unit of the good. The auction ends, when all units are sold or a lower price limit is reached. The most well-known sealed-bid auctions are the discriminatory and the uniform-price (competitive) auction (cp. Smith (1966), Harris/Raviv (1981)). In the *discriminatory auction* the N units are allocated to the bidders with the highest price bids in descending order. Successful bidders are charged an amount equal to their bids. Thus, for more than one unit for sale, different bidders pay different unit-prices in a discriminatory auction. If there is only one unit for sale, the discriminatory auction is also called *first-price auction*. The rules of the *uniform-price auction* are the same as those of the discriminatory auction, except for the prices paid by successful bidders. In the *uniform-price auction* all successful bidders are charged the same price, which equals the highest unsuccessful bid. If there is only one unit for sale, the uniform-price auction is also called *second-price auction* or *Vickrey auction*. Vickrey (1961) introduced this type of auction.

Allowing for more than one seller, a double auction results. In a *double auction* sellers submit bids to sell and buyers submit bids to buy. The sellers' (buyers') bids are ranked in ascending (descending) order. The price in a *non-discriminatory double auction* results from the intersection of the obtained supply and demand schedule. This auction as well as other types may be performed as a *discrete-time* auction at predetermined times during a trading period, or as a *continuous-time* auction at any moment during a trading period (cp. Friedman (1991)).

The goods sold by auction can be classified with respect to the monetary value that bidders attach to them. Milgrom/Weber (1982) distinguish between goods with private value, goods with common value and goods with general value. A good is called to be of the *private value* type (or of *preference uncertainty* (cp. Myerson (1981)), if its value to a bidder is a purely personal matter. A painting bought for purely personal reasons provides an example. A good is called to be of the *common value* type (or *of quality uncertainty* (cp. Myerson (1981)), if the monetary value bidders attach to it is the same for all bidders. This value, however, will generally be unknown at the time the auction takes place. Thus it is considered a random variable. Typically, securities have a future monetary value which is considered uncertain. Thus they serve as an example for goods with common value. A good is called to be of the *general value* type, if its value to a bidder is a combination of private and common values. A painting bought for both personal reasons and later resale could be considered being of the general value type.

3 Analysing auctions

The auction literature can be divided into three groups: *decision-theoretic auction models*, *game-theoretic auction models* and *empirical studies*. Decision-theoretic auction models deal with the bidder's decision problem in a way which ignores his strategic response to the bidding behavior of his competitors. The game-theoretic approach, on the contrary, is based on this strategic aspect of the bidding process. Empirical studies make use of either the decision-theoretic or the game-theoretic approach when examing data sets resulting from real-life auctions or controlled laboratory experiments.

The earliest auction models are of the decision-theoretic type. They are motivated by real-world auction design problems concerning the sale of oil exploration rights (Hughart (1975)) and US treasury bills (Smith (1966), Scott/Wolf (1979)). Formal in nature, they assume that bidders are expected utility maximizers, where the expectation is with respect to a subjective probability density function for the lowest accepted bid. (Cp., for example, Smith (1966).) In particular, it is assumed that a bidder's action does not affect the realization of the lowest successful bid. (Cp., for example, Scott/Wolf (1979).) Despite obvious modelling deficiencies, those early models already succeed in identifying certain effects, which later are proved rigorously within the game-theoretic approach to auction theory.

The earliest game-theoretic auction studies are primarily concerned with the determination of Bayes-Nash equilibrium bidding strategies. The models are quite simple in that they assume symmetric bidders with independent value estimates. Those results are generalized in different directions in order to account for bidders with correlated value estimates, for goods of the general value type (Milgrom/Weber(1982)), and for goods with a resale market (Bikhchandani/Huang (1989)).

The empirical studies are concerned with the analysis of real-life auction data sets or controlled laboratory experiments. Examples of studies in the area of real-life auctions are Cammack (1991), who considers the US treasury bill auction, and Hendricks/Porter (1988), who consider auctions for Federal leases on the Outer Continental Shelf.

There is an increasing number of studies, which provide tests of auction market behavior by means of experimental methods. Quite often, they are concerned with the independent private values model, which predictions are being tested (cp. Smith (1991) and its citations). Fundamental to this area is the study of Smith (1967), who examines individual bidding behavior in discriminatory and uniform-price auctions for a common value set-up. His results are consistent with the predictions of auction theory. He establishes that bids in the discriminatory auction are lower than those in the uniform-price auction and that the same is true for the auctioneer's revenues. Kagel/Levin (1986) perform a test of the Nash equilibrium bidding theory. They test the hypotheses of efficiency, departure from Nash equilibrium and auctioneer's average revenues rise under public information disclosure for the first-price auction. Experimental studies of common value auctions are of great importance in the double auction set-up, where a satisfactory model is not yet available (cp. Friedman/Rust (1993)). Furthermore, specific issues such as the influence of insider information can be addressed by experiments (cp. Kagel/Levin (1999)).

4 A game-theoretic auction model

For ease of exposition only single-unit auctions will be considered, i.e. there is one unit of a good to be sold in an auction. The *auctioneer*, who for simplicity is assumed to be the owner of the good, sets the auction rules and makes them known to all participants. The auction rules are binding even though it might be in the auctioneer's interest to change them after the bids have been submitted. The auctioneer is risk-neutral, i.e. he or she is interested in maximizing his or her expected revenues resulting from the sale of the good. There are n, $n \geq 2$, bidders competing with each other in the auction. Each bidder i, $i = 1, ..., n$, has some private information $x_i \in \mathbf{R}_+$. This information is considered a realization of a random variable X_i, *bidder i's information variable*. Bidders are distinguished solely by their private information; otherwise they are identical. When preparing their bids, bidders do not cooperate. Furthermore, bidders are risk-neutral, i.e. bidder i, $i = 1, ..., n$, is interested in maximizing his or her expected gains from the auction. It is assumed that the number of bidders and the joint distribution of $(X_1, ..., X_n)$ is common knowledge.

In an auction a rational bidder takes the bidding behavior of his competitors into account and behaves strategically. However, a bidder has only incomplete information about the relevant features of the bidding situation. Thus an auction is to be considered a *non-cooperative game with incomplete information* among the bidders. To analyse this game the Bayes-Nash equilibrium concept due to Harsanyi (1967/68) is employed. For that purpose, a *bidder's strategy* is defined as a real-valued function mapping possible realizations of his information variable into bids. A *Bayes-Nash equilibrium* is defined as a n-tuple $(B_1, ..., B_n)$ of bidders' strategies with the following properties:

1. B_i, $i = 1, ..., n$, maximizes bidder i's expected gains assuming bidding strategies B_1,..., B_{i-1}, B_{i+1},..., B_n of his competitors.

2. The assumed bidding behavior of his competitors is correct.

4.1 The private value model

The private value of the good to bidder i is given by x_i, the realization of bidder i's information variable X_i. If p_i denotes the price bidder i pays, if he is successful, and $1_{\{i \text{ successful}\}}$ denotes the indicator variable, which is 1 if i is successful and 0 otherwise, the expected gains of bidder i are given by

$$E[(x_i - p_i)1_{\{i \text{ successful}\}} \mid x_i].$$

Thus bidder i's expected gains can be written in the form

$$x_i \times \text{probability of winning} - \text{expected payments}. \qquad (1)$$

The optimal bid maximizes (1). Note that lowering bidder i's bid has two opposing effects on his expected gains. First, the probability of winning decreases and second, the gains from the bid increases. Intuitively, the optimal bid will be such that it balances those two effects.

4.2 The common value model

The common value of the good is unknown at the time the auction takes place. This common value is considered a realization of a random variable V, the *value variable.* As to the common value V, the auctioneer has some private information, which is summarized by a positive number x_0. x_0 is considered to be the realization of a random variable X_0, the *auctioneer's information* *variable*. The joint distribution of $(V, X_0, X_1, ..., X_n)$ is common knowledge. If p_i denotes the price bidder i pays, if he is successful, the expected gains of bidder i are given by

$$E[(V - p_i)1_{\{i \text{ successful}\}} \mid x_i].$$

The major assumption, which drives the model, is the one of affiliated value and information variables introduced by Milgrom/Weber (1982). Roughly speaking, random variables are affiliated, if large values of one variable are more likely, when the other variables are large as well. In particular, independent random variables are affiliated, and affiliated random variables are non-negatively correlated. Note that affiliation generalizes the concept of non-negative correlation to arbitrary vectors of random variables. In the following, the cases of independence and strict affiliation (positive correlation) will be dealt with separately.

4.3 The general value model

Suppose $(S_1, ..., S_l)$ represents a vector of value variables, which are non-observable. The general value is characterized by a function h such that the value of the good to bidder i is given by

$$V_i = h(X_i, X_1, ..., X_{i-1}, X_{i+1}, ...X_n, S_1, ..., S_l).$$

h is symmetric in the arguments $2, 3, ..., n$. The private value model results for $V_i = X_i$, and the common value model for $V_i = S_1$.

5 Results

Due to the symmetry assumption, the analysis of the auction game centers around symmetric Bayes-Nash equilibria. Thus bidding behavior can be described by a common bidding strategy B. Different bids are the result of different information only. Effects apart from that are not modelled.

5.1 The private value model

5.1.1 Independent private values

In the independent private values set-up the individual information variables $X_i, i = 1, ..., n$, are assumed to be independent and identically distributed (i.i.d.). Furthermore, for technical reasons, a continuous and positive (on its support $[0, a]$) density f exists with distribution function F. Under those assumptions, the Bayes-Nash bidding behavior can be easily described in the standard sealed-bid auctions.

Bidding strategies (Vickrey (1961), Harris/Raviv (1981))

(a) *In the second-price (Vickrey) auction the symmetric bidding strategy is given by*

$$B(x) = x.$$

(b) *In the first-price auction the symmetric bidding strategy is given by*

$$B(x) = x - \frac{\int_0^x (F(y))^{n-1} dy}{(F(x))^{n-1}}. \quad (2)$$

Thus a bidder with information x submits a lower bid in the first-price auction than in the second-price auction. This difference in bids is represented by the second term of (2).

Uniqueness of symmetric equilibria can be established (cp. Milgrom/Weber (1982)). Furthermore, efficiency is guaranteed in both types of auction. The bidder with the highest information is the one who wins the good both in the first-price and in the second-price auction. From the described bidding behavior, bidders' expected gains and the auctioneer's expected revenues can be derived. Both coincide in the case of first-price and second-price auction. This is a special case of the revenue equivalence theorem due to Vickrey (1961), Harris/Raviv (1981), Myerson (1981) and Riley/Samuelson (1981).

Revenue Equivalence

The auctioneer's expected revenues are the same in the single-unit English, single-unit Dutch, first-price and second-price auction.

Similarly, a bidder's expected gains coincide in the abovementioned auction types.

5.1.2 Strictly affiliated private values

Suppose that in the private value set-up bidders information variables are strictly affiliated. Milgrom/Weber (1982) established the following ranking of auctions.

Auction Ranking

The auctioneer's expected revenues in the first-price auction are never higher than those in the second-price auction. The auctioneer's expected revenues in the second-price auction are never higher than those in the single-unit English auction. The single-unit Dutch auction yields the same expected revenues as the first-price auction.

Similarly, in the second-price auction a bidder's expected gains are smaller or equal to those in the first-price auction.

5.2 The common value model

5.2.1 Winner' s curse

Common value models suffer from an effect, which is called the *winner's curse*

> "learning that others have bid less than [his own bid] ... is bad news about the value of the item being acquired."
>
> Milgrom (1987), p. 18

Unsophisticated bidding leads to the effect that the successful bidder tends to be the one who overvalued the good. Thus rational bidders take account of the winner's curse effect, when preparing their bids. The "winner's curse", was the subject of a lot of controversy. (Cp., for example, Kagel/Levin (1986)). Milgrom (1987) formalizes the winner's curse effect in the following way. Consider the decision problem of bidder 1 with information X_1 in a sealed-bid common value auction for one unit of the good, where the highest bid wins the auction. Before the auction takes place, bidder 1's expected unit valuation of the good is given by $E[V \mid X_1]$. Suppose all other bidders bid according to $B(X_j)$, $j \neq 1$, where B is a strictly increasing function and X_j is bidder j's information variable. Then bidder 1's expected unit valuation of the good, when he has won one unit with his bid b amounts to

$$E[V \mid X_1, \max(B(X_2), ..., B(X_n)) < b].$$

The winner's curse effect is based on a comparison of the expected unit valuation of the good by the successful bidder before and after the auction. Thus it can be measured by

$$E[V \mid X_1] - E[V \mid X_1, \max(B(X_2), ..., B(X_n)) < b]. \tag{3}$$

By the affiliation property, (3) is positive (cp. Milgrom (1987)). For a symmetric Bayes-Nash equilibrium in strictly increasing strategies, the winner' s curse effect reduces to

$$E[V \mid X_1] - E[V \mid X_1, \max(X_2, ..., X_n) < X_1].$$

Any information on the unknown common value will weaken the winner's curse effect. In particular, linking the price in an auction to affiliated information of the common value reduces the winner's curse. This is the essence of the linkage principle (Milgrom/Weber (1982)). Reducing the winner's curse effect, however, leads to more aggressive bidding and higher auction prices. It is interesting to note a fact about the influence of increased competition on the winner's curse effect. The winner's curse effect gets stronger with an increasing number of bidders.

5.2.2 Independent information variables

If the information variables are independent, the results for the common value model resemble those of the independent private value model. In particular, the auctioneer's expected revenues in the first-price and second-price auction coincide. Similarly, the same is true for a bidder's expected gains in both auctions.

5.2.3 Strictly affiliated information variables

The bidding behavior in the common value set up can be described by means of

$$v^*(x) = E[V \mid X_1 = x, \max(X_2, ..., X_n) = x].$$

In the following $g(. \mid r)$ and $G(. \mid r)$ denote the density and the distribution function, respectively, of $\max(X_2, ..., X_n)$ conditional on $X_1 = r$.

Bidding strategies (Milgrom/Weber(1982))

(a) *In the second-price (Vickrey) auction the symmetric bidding strategy is given by*

$$B(x) = v^*(x).$$

(b) *In the first-price auction the symmetric bidding strategy is given by*

$$B(x) = v^*(x) - \int_0^x (v^*)'(s) \exp\left(-\int_s^x \frac{g(r \mid r)}{G(r \mid r)} dr\right) ds.$$

Uniqueness of symmetric equilibria can be established (cp. Milgrom/Weber (1982)).

The central results of the common value auction literature concern the ranking of auctions, the auctioneer's information policy and the information aggregation property of auctions. All those results rely on the comparison of symmetric equilibria of the respective auctions, i.e. Bayes-Nash equilibria of the form $B_1 = ... = B_n$. Taking this into account, the basic results are as follows.

Auction Ranking (Milgrom/Weber (1982))

The auctioneer's expected revenues in the first-price auction are never higher than those in the second-price auction. The auctioneer's expected revenues in the second-price auction are never higher than those in the single-unit English auction.

If the auctioneer has private information about the unknown common value, he may use this fact to influence the outcome of the auction. He may commit himself to supply information publicly in order to increase his revenues. There is a whole range of possible information policies. Milgrom/Weber (1982) mention complete revelation (fully reporting of all information), censoring (reporting of favorable information only), randomizing (reporting after having added noise to the information) and concealment (no reporting of private information).

Information Revelation (Milgrom/Weber (1982))

The auctioneer's policy of revealing any affiliated information publicly and completely raises his expected revenues in the first-price auction, the second-price auction and the single-unit English auction.

There has always been the question of whether the price of a good of unknown value reflects (aggregates) the dispersed information of market participants. Under certain assumptions this could be answered in the affirmative in the case of auctions.

Information Aggregation (Wilson (1977))
The winning bid in a first-price auction converges almost surely to the common value V as the number of bidders n goes to infinity.

5.3 The general value model

The general value model gives results in much the same way as the common value model. In particular, the auction ranking and the information revelation results hold true. However, the information aggregation property relies on the simple structure of the pure common value model and thus does not generalize.

6 Double auctions

For more than a century, to a large extent trade was conducted by means of double auctions. Nowadays, double auctions are important tools for computerized trade. They also serve well for the organization of markets for natural gas as well as for electric power networks. The theoretical analysis concerning double auctions is much more involved than that of one-sided auctions. Determination of Bayes-Nash equilibrium is complicated, if at all feasible. Often the analysis gives rise to a multiplicity problem. For that reason, the theoretical analysis centers around the independent private values model for a single indivisible unit. Satterthwaite/Williams (1993) give results for particular sealed-bid non-discriminatory double auctions. For m risk-neutral sellers and m risk-neutral buyers symmetric Bayes-Nash equilibria are determined. Convergence of equilibria to ex-post efficiency as both the number of sellers and buyers increase is shown to hold.

However, due to the abovementioned technical problems, the analysis of double auctions is done almost always by means of empirical or experimental studies (cp., for example, section III in Friedman/Rust (1993)).

7 Application to financial markets

The interest into the market institutions that govern the trade of financial securities grew rapidly during the last decades. Along with the successful introduction of new exchanges, the trading rules at existing exchanges were examined more closely. Common to all those activities is an interest into the price formation process under alternative trading rules. An important example of this approach is the discussion on the design of the US Treasury bill auction, one of the most important auctions of financial securities. Every Monday the US Treasury auctions a previously announced quantity of Treasury bills maturing in 91 and 182 days, respectively.

Since the late fifties there has been quite a controversy concerning the design of this auction. This controversy started with a proposal of Milton Friedman to change the Treasury bill auction from discriminatory to uniform-pricing. The reasons given in favor of this proposal were quite different in nature. They ranged from revelation of bidders' true demand curves and an increase in the number of bidders to discouragement of collusion under uniform-pricing. They did not, however, include an increase in the Treasury's average revenues from a sale under uniform-pricing. In fact, the early opponents to this proposal argued by a reduction in revenues through a change from discriminatory to uniform-pricing. Several years later, models of the decision- and the game-theoretic type lent additional support to the Friedman proposal. In that respect, the results of Smith (1966) and Milgrom/Weber (1982) on the superiority of uniform- over discriminatory pricing in terms of auctioneer's expected revenues were especially successful. Despite the fact that a completely satisfactory theoretical model of the Treasury bill auction is not available yet, those results made the Treasury reconsider the Friedman proposal. Those considerations led to experimentation with a uniform-price auction in the 1970's. However, the Treasury retained the discriminatory rules. After the Salomon Brothers Inc admission of violating the US Treasury auction rules, the experiments were taken up again in September 1992 for the two-year and five-year note auctions (cp. Bikhchandani/Huang (1993)). Experimentation ended in a decision effective with the 11/2/98 auction to auctioning all bills using the uniform-pricing rules.

8 Conclusion

The basic game-theoretic auction model gives rise to a number of strong results, which have been subject to empirical and/or experimental tests. Naturally, the question arises as to whether those results are robust with respect to a relaxation of assumptions. Unfortunately, there are several assumptions which, when relaxed, give rise to different results. There are a number of factors which have to be dealt with. First, risk neutrality matters. When allowing for risk aversion on the bidders' part, the expected revenue comparison will yield different results. For example, in models with both risk-aversion and strictly affiliated information variables, the established ranking of first- and second-price auctions with respect to expected revenues fails to hold. Only in special cases (e.g. i.i.d. information variables) general results can be established (cp. Milgrom/Weber (1982)). Second, multi-unit demand on the bidders' part changes results dramatically. Endogenizing quantity, for example, as in the case of a share auction, might result in a multiplicity of equilibria and a substantial lower expected sale price than in comparable unit-auctions (cp. Wilson (1979)). Third, in auctions of financial securities bidders usually meet more than once. The introduction of this phenomenon produces all effects that apply to repeated non-cooperative games. Finally, a number of features cannot be solved for in the outlined theoretical auction model due to mathematical complexities. It is in those cases that one has to rely on controlled laboratory experiments in order to draw conclusions.

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