A GENERALIZED FRACTIONAL TIME SERIES MODEL

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ABSTRACT

We propose in this article a general time series model, whose components are modelled in terms of fractionally integrated processes. This specification allows us to consider the trend, the seasonal and the cyclical components as stochastic processes, including the unit root models as particular cases. A very general version of the tests of Robinson (1994) is used to test the order of integration of each component. Finite-sample critical values of the tests are evaluated and, an empirical application, is also carried out at the end of the article.

Key words: Time series model; Long memory; Fractional integration.

JEL Classification: C22

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1. Introduction

We propose in this article a general time series model whose components are separated into a trend, a seasonal and a cyclical component. The first two components may be formulated in terms of explanatory variables consisting of a time trend and a set of seasonal dummies. However, for many time series, this may be inadequate. The necessary flexibility may then be achieved by letting the regression coefficients change over time. A similar treatment may be accorded to the cyclical component, which may be expressed in terms of a stochastic process instead of a deterministic model. Traditionally, the simplest way of modelling the trend component is a random walk, i.e.,

$$(1 - L)T_t = \varepsilon_{1t}, \tag{1}$$

with white noise ε_{1t}, and similarly, for the seasonal component,

$$(1 - L^s)S_t = \varepsilon_{2t}, \tag{2}$$

where s indicates the number of periods per year. Finally, the cyclical component may be specified as

$$(1 - 2\mu L + L^2)C_t = \varepsilon_{3t}, \tag{3}$$

where the periodicity is implicit in μ. (See, eg. Harvey, 1985 and Ahtola and Tiao, 1987). In this article we generalize the above specifications by allowing each of the components to be fractionally integrated. Thus, the trend component will be specified as

$$(1 - L)^{d_1} T_t = \varepsilon_{1t}, \tag{4}$$

where d_1 is a given real value. Note that the polynomial in (4) may be expressed in terms of its Binomial expansion such that

$$(1 - L)^{d_1} = \sum_{j=0}^{\infty} \binom{d_1}{j} (-1)^j L^j .$$

for any real d_1. Clearly, if ε_{1t} is I(0), defined for the purpose of the present paper as a covariance stationary process with spectral density that is positive and finite at any frequency, T_t is (fractionally) integrated of order d_1, and if $d_1 > 0$, T_t is said to be long memory, so-named because of the strong association between observations widely separated in time. This type of processes were proposed by Granger and Joyeux (1980) and Hosking (1981, 1984), and were theoretically justified by Robinson (1978) and Granger (1980), showing that they can arise from aggregation of individual processes. Examples of empirical applications based on fractional models like (4) are Diebold and Rudebusch (1989, 1991) and more recently, Gil-Alana and Robinson (1997) and a good survey can be found in Baillie (1996).

Similarly, the seasonal unit root model (2) can be generalized to permit long memory and thus, we can consider

$$(1 - L^s)^{d_2} S_t = \varepsilon_{2t}, \tag{5}$$

where $(1 - L^s)^{d_2}$ can be expanded as

$$(1 - L^s)^{d_2} = \sum_{j=0}^{\infty} \binom{d_2}{j} (-L^s)^j ,$$

for any real d_2. This model was studied in Porter-Hudak (1990) and Ray (1991), and another empirical application is Gil-Alana and Robinson (2000).

The cyclical component can also be extended for long memory. Gray et al. (1989, 1994) showed that the cyclical stochastic model (3) can be generalized as

$$(1 - 2\mu L + L^2)^{d_3} C_t = \varepsilon_{3t}, \tag{6}$$

where $(1 - 2\mu L + L^2)^{d_3}$ can be expressed in terms of the polynomial $C_{j,d_3}(\mu)$, such that for all $d_3 \neq 0$,

$$(1 - 2\mu L + L^2)^{-d_3} = \sum_{j=0}^{[j/2]} C_{j,d_3}(\mu) L^j ,$$

where

$$C_{j,d_3}(\mu) = \sum_{k=0}^{\infty} \frac{(-1)^k (d_3)_{j-k} (2\mu)^{j-2k}}{k!(j-2k)!}; \qquad (d_3)_j = \frac{\Gamma(d_3 + j)}{\Gamma(d_3)},$$

and Γ(x) means the Gamma function. Simulated realizations based on fractionally cyclic models like (6) can be found in Gray et al. (1989) and an empirical application is Gil-Alana (2000).

All these components can be jointly considered in the following model,

$$(1 - L)^{d_1} (1 - L^s)^{d_2} (1 - 2\mu L + L^2)^{d_3} Y_t = U_t, \qquad (7)$$

where U_t is I(0) which may include possible weakly autocorrelated processes. We propose in this article the use of a very general version of the tests of Robinson (1994) for testing the order of integration of each of the components in (7). The outline of the paper is as follows: Section 2 briefly describes the tests of Robinson (1994) for testing the orders of integration of the trend, the seasonal and the cyclical components. Section 3 evaluates finite-sample critical values of the tests. Section 4 contains an empirical application and finally, Section 5 contains some concluding comments.

2. The tests of Robinson (1994)

Robinson (1994) considers the following regression model,

$$Y_t = \beta' Z_t + X_t, \qquad (8)$$

where Y_t is the time series we observe; $\beta = (\beta_1, \ldots, \beta_k)'$ is a (kx1) vector of unknown parameters; Z_t is a (kx1) vector of deterministic regressors, and the regression errors X_t are such that

$$\rho(L;\theta) X_t = U_t, \qquad (9)$$

where U_t is I(0) and the function ρ is specified as

$$\rho(L;\theta) = (1 - L)^{d_1 + \theta_1} (1 + L)^{d_2 + \theta_2} \prod_{j=3}^{p} (1 - 2\cos w_r L + L^2)^{d_j + \theta_j}, \qquad (10)$$

for given real numbers $d_1, d_2, \ldots, d_p$ and w_r.

Based on (8), (9) and (10), Robinson (1994) proposes a Lagrange Multiplier (LM) tests of

$$H_o: \theta = (\theta_1, \theta_2, \ldots, \theta_p)' = 0, \qquad (11)$$

against the alternative H_a: $\theta \neq 0$. Specifically, the test statistic is given by

$$\hat{R} = \frac{T}{\hat{\sigma}^4} \hat{a}' \hat{A}^{-1} \hat{a}, \qquad (12)$$

where T is the sample size and

$$\hat{a} = \frac{-2\pi}{T} \sum_j^* \psi(\lambda_j) g(\lambda_j; \hat{\tau})^{-1} I_{\hat{u}}(\lambda_j); \qquad \psi(\lambda_j) = \operatorname{Re}\left[\frac{\partial}{\partial \theta}\left(\log \rho(e^{i\lambda_j}; \theta)\right)\right]_{\theta=0};$$

$$\hat{A} = \frac{2}{T}\left(\sum_j^* \psi(\lambda_j)\psi(\lambda_j)' - \sum_j^* \psi(\lambda_j)\hat{\varepsilon}(\lambda_j)' \, x \left(\sum_j^* \hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)'\right)^{-1} x \sum_j^* \hat{\varepsilon}(\lambda_j)\psi(\lambda_j)'\right);$$

$$\hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial \tau} \log g(\lambda_j; \hat{\tau}); \qquad \hat{\sigma}^2 = \frac{2\pi}{T} \sum_j^* g(\lambda_j; \hat{\tau})^{-1} I_{\hat{u}}(\lambda_j); \qquad \lambda_j = \frac{2\pi j}{T},$$

$g(\lambda; \tau)$ is the function appearing in the spectral density of U_t: $f(\lambda; \tau) = (\sigma^2/2\pi)\, g(\lambda; \tau)$, evaluated at $\hat{\tau} = \arg\min \sigma^2(\tau)$, and $I(\lambda)$ is the periodogram of $\hat{U}_t$ defined as:

$$I(\lambda) = \frac{1}{2\pi T}\left|\sum_{j=1}^{T} \hat{U}_t e^{i\lambda t}\right|^2;$$

$$\hat{U}_t = \rho(L) Y_t - \hat{\beta}' W_t; \quad W_t = \rho(L) Z_t; \quad \hat{\beta} = \left(\sum_{t=1}^{T} W_t W_t'\right)^{-1} \sum_{t=1}^{T} W_t \rho(L) Y_t,$$

where $\rho(L) = \rho(L; \theta = 0)$ and the summation on * in the above expressions are over $\lambda \in M$ where $M = \{\lambda: -\pi < \lambda < \pi, \lambda \notin (\rho_l - \lambda_1, \rho_l + \lambda_1), l=1,2,\ldots s\}$, such that ρ_l, $l = 1,2,\ldots s < \infty$ are the distinct poles of $\psi(\lambda)$ on $(-\pi, \pi]$.

Robinson (1994) showed that, under very general conditions, the above test statistic has an asymptotic distribution given by

$$\hat{R} \to \mathrm{X}_p^2, \qquad as \quad T \to \infty, \qquad (13)$$

and the same limit distribution holds whether or not deterministic regressors are included in (8). Furthermore, he shows that the test is efficient in the Pitman sense, i.e., that against local alternatives of form: H_a: $\theta = \delta\, T^{-1/2}$, for $\delta \neq 0$, the limit distribution is $\chi_p^2(\upsilon)$ with a non-centrality parameter, υ, which is optimal under Gaussianity of U_t.

We can now particularize Robinson's (1994) model, and consider $\beta = 0$ a priori in (8) and

$$\rho(L;\theta) \; = \; (1 - L)^{d_1+\theta_1} \; (1 \; - L^s)^{d_2+\theta_2} \; (1 \; - \; 2\cos w_r L \; + \; L^2)^{d_3+\theta_3}. \qquad (14)$$

Thus, under the null hypothesis (11), (9) and (14) imply the fractional model (7) described in Section 1, and testing (11) against H_a: $\theta \neq 0$ will be a test of the orders of integration of the trend, the seasonal and the cyclical components of the series. Furthermore, the limit distribution will be standard and will be given by a χ_3^2 distribution.

3. Finite-sample critical values

In this section we evaluate finite-sample critical values of the tests of Robinson (1994) by means of Montecarlo simulations. We consider a model given by (9) and (14) with s = 4 and white noise U_t. In this context of white noise disturbances, with s = 4 and p = 3, the test statistic greatly simplifies and becomes

$$\tilde{R} \; = \; \frac{T}{\tilde{\sigma}^4} \, \tilde{a}' \, \tilde{A}^{-1} \, \tilde{a}, \qquad (15)$$

where

$$\tilde{a} \; = \; \frac{-2\pi}{T} \sum_j^* \psi(\lambda_j) I(\lambda_j); \qquad \tilde{A} \; = \; \frac{2}{T} \sum_j^* \psi(\lambda_j) \psi(\lambda_j)'; \qquad \tilde{\sigma}^2 \; = \; \frac{2\pi}{T} \sum_j^* I(\lambda_j)$$

and $\psi(\lambda_j) \; = \; [\psi_1(\lambda_j), \psi_2(\lambda_j), \psi_3(\lambda_j)]'$, with

$$\psi_1(\lambda_j) \; = \; \log \left| 2\sin\frac{\lambda_j}{2} \right|, \qquad (16)$$

$$\psi_2(\lambda_j) \; = \; \log \left| 2\sin\frac{\lambda_j}{2} \right| \; + \; \log\left(2\cos\frac{\lambda_j}{2} \right) \; + \; \log \left| 2\, \cos\lambda_j \right|, \qquad (17)$$

$$\psi_3(\lambda_j) \; = \; \log \left| 2\, (\cos\lambda_j \; - \cos w_r) \right|. \qquad (18)$$

We first compute the empirical distributions of $\tilde{R}$ in (15) for sample sizes of 48, 96 and 144 observations, generating Gaussian series obtained by the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Vetterling (1986) with 50,000 replications in each case. Note that the empirical distributions are not affected by the orders of integration d_1, d_2 and d_3, given that the test statistic is based on the null differenced model. However, for the cyclical component, we need to impose values of $w_r = 2\pi r/T$, i.e., the number of periods per cycle. We do it for r = T/4, T/6, T/12 and T/24, i.e., we consider cycles occurring every 4, 6, 12 and 24 periods. This choice is not completely arbitrary: The results obtained in several empirical applications based on unit and fractional root cycles (eg., Gil-Alana, 2000) suggest that most of the cycles occur at approximately six years. Thus, it has interest to consider r = T/6 (if the data are annual) and r = T/24 (it they are quarterly); r = T/4 and r = T/12 have also interest in case of quarterly and monthly data.

(Table 1 about here)

The results are given in Table 1. We see that in practically all the cases, the finite-sample critical values are higher than those given by the χ_3^2 distribution. Thus, when testing H_o (11), the tests based on the asymptotic results will reject the null more often than those based on the finite-sample critical values. We also see that these values increase with r, i.e., with the number of periods per cycle, indicating that, for a given value of the test statistic, shorter the cycles are, more often the null will be rejected. Finally, and as we should expect, increasing the sample size, the values approximate to the χ_3^2 distribution.

(Tables 2 and 3 about here)

Tables 2 and 3 report the sizes of the tests based on both, the asymptotic and the finite-sample critical values, with nominal sizes respectively of 5% and 1%. In both tables we see that using the finite-sample critical values, the sizes are very close to the nominal values for all sample sizes. However, using the asymptotic critical values given by the χ_3^2 distribution, the sizes are in all the cases too large, though they improve considerably with T. The power of the tests was also examined against different fractional alternatives and, though we do not report the results here, higher rejection frequencies were obtained with the asymptotic tests which should be clearly associated with the higher sizes of the tests when the asymptotic critical values are used.

4. An empirical application

In this section, the version of the tests of Robinson (1994) described in Section 3 is applied to the UK consumption and income. The time series data are the logs of the UK consumption expenditure on non-durables and the personal disposable income, quarterly, from 1955q1 to 1984q4. These two series were also analysed in Hylleberg et al. (1990) and in Gil-Alana and Robinson (2000), studying respectively the cases of seasonal integration and seasonal fractional integration.

(Figure 1 about here)

Plots of the two series are given in Figure 1. We see that both may include a trend, a seasonal and a cyclical component, with a possible changing pattern across time. Tables 4 and 5 report values of $\tilde{R}$ in (15), testing H_o (11) in a model given by (9) and (14) with s = 4 and r = T/4, T/6, T/12 and T/24. We employ throughout the null model,

$$(1 - L)^{d_1} (1 - L^4)^{d_2} (1 - 2\cos w_r L + L^2)^{d_3} X_t = \varepsilon_t ,$$

with white noise ε_t and values of d_1, d_2 and d_3 equal to 0, 0.50 and 1. Thus, we test for a simple random walk ($d_1 = 1$, and $d_2 = d_3 = 0$); for a seasonal unit root model ($d_2 = 1$ and $d_1 = d_3 = 0$); for cyclic I(1) models ($d_3 = 1$ and $d_1 = d_2 = 0$); and also for fractional alternatives involving all of these components. The FORTRAN code used in this application is available from the author upon request.

Table 4 gives the results for consumption. We see that, imposing $d_3 = 0$, (i.e., without a cyclical component), H_o (11) always result rejected except when $d_2 = 0$ and $d_1 = 1$. In other words, a random walk model appears as a plausible way of modelling this series. The fact that H_o (11) cannot be rejected in this case for any value of r is consistent with the fact that the cyclical component is not required in this context. Supposing however that $d_3 = 0.50$, H_o (11) cannot be rejected if r = T/6 and T/24, whether the values of d_1 and d_2 are 0, 0.50 or 1. This suggests that, when modelling the cyclical component in terms of a fractionally integrated process, (with d_3 = 0.50), the cycles appear important, and they seem to occur every 6 or 24 periods. Note that, in general, we observe lower statistics when r = T/24 than when r = T/6, suggesting that the cycles may occur approximately each six years. Finally, imposing a unit root cycle (i.e., d_3 = 1), all the models result rejected.

(Tables 4 and 5 about here)

A very similar picture is observed in Table 5 when modelling the UK personal disposable income. The null hypothesis (11) cannot be rejected when d_1 = 1 and d_2 = d_3 = 0, implying that the random walk model also appears as a plausible alternative for this series. Surprisingly, all the non-rejection values occur at exactly the same (d, r) combination as in Table 4. Thus, apart from the random walk model, all the remaining non-rejection values take place when d_3 = 0.50, independently of the values of d_1 and d_2, with cycles occurring every 6 or 24 periods.

We finally observe that in both tables, the lowest statistics are obtained when $d_1 = d_2 = 1$ and d_3 = 0.50, suggesting that a plausible model for both series would be

$$(1 - L)\ (1 - L^4)\ (1 - 2\cos w_4 L + L^2)^{0.50}\ X_t = \varepsilon_t,$$

implying nonstationarities for the three components of the series. Hylleberg et al. (1990) and Gil-Alana and Robinson (2000) also studied these series. They concentrated on the seasonal component and did not take into account the trend and the cyclical components, finding evidence of four seasonal unit roots. In this article, we also find evidence of four seasonal unit roots, along with another (unit) root corresponding to the trend component and with a fractional process for the cyclical term.

5. Concluding comments

We have proposed in this article the use of a version of the tests of Robinson (1994) for testing the order of integration of each of the components in a given raw time series. Modelling the trend, the seasonal and the cyclical component in terms of fractionally integrated processes allows us to consider a richer structure in the dynamics of each of these components, and permit us to consider the unit root models as particular cases of interest. The tests have standard null and local limit distributions given by a χ^2 distribution. However, we also computed finite-sample critical values, observing that they were in practically all cases higher than those given by the asymptotic distribution.

The tests of Robinson (1994) were applied to the UK consumption and income series, and the results indicate that, though a random walk model could be appropriate, fractional cyclical models may also be adequate, with cycles occurring approximately each six years. Thus, a model with unit roots for the trend and the seasonal components, along with a cyclic I(0.5) component cannot be rejected in this context. Extensions based on autoregressive disturbances could also have been performed. However, and as a preliminary step, the results in this paper suggest that fractionally cyclical models may be adequate when modelling macroeconomic time series.

It would be worthwhile proceeding to get point estimates of the orders of integration of each of the components of the series. However, not only would this be computationally more expensive, but it is then in any case confidence intervals rather than point estimates which should be stressed. The approach used in this article simply generates computed diagnostic for departures from real orders of integration and thus, it is not surprising that different models may result non-rejected. Ooms (1997) suggests Wald tests based on Robinson's (1994) model in (8) – (10), using for the estimation a modified periodogram regression procedure of Hassler (1994), whose distribution is evaluated under simulation. Similar methods based on this and other semi-parametric estimation procedures of the fractional differencing parameters (eg. Robinson, 1995a, b) can also be implemented in these and other macroeconomic time series.

References

Ahtola, J. and G.C. Tiao, 1987, Distributions of least squares estimators of autoregressive parameters for a process with complex roots on the unit circle, Journal of Time Series Analysis 8, 1 - 14.

Baillie, R.T., 1996, Long memory processes and fractional integration in econometrics, Journal of Econometrics 73, 5 - 59.

Diebold, F.X. and G. Rudebusch, 1989, Long memory and persistence in aggregate output, Journal of Monetary Economics 24, 189 - 209.

Diebold, F.X. and G. Rudebusch, 1991, Is consumption too smooth? Long memory and the Deaton paradox, Review of Economics and Statistics 74, 1 - 9.

Gil-Alana, L.A., 2000, Testing stochastic cycles in macroeconomic time series, Preprint.

Gil-Alana, L.A. and P.M. Robinson, 1997, Testing of unit roots and other nonstationary hypotheses in macroeconomic time series, Journal of Econometrics 80, 241 - 268.

Gil-Alana, L.A. and P.M. Robinson, 2000, Testing of seasonal fractional integration in UK and Japanese consumption and income, forthcoming in Journal of Applied Econometrics.

Granger, C.W.J., 1980 Long memory relationships and the aggregation of dynamic models, Journal of Econometrics 14, 227 - 238.

Granger, C.W.J. and. R. Joyeux, 1980, An introduction to long memory time series models and fractional differencing, Journal of Time Series Analysis 1, 15 - 30.

Gray, H.L., N. Zhang. and W.A. Woodward, 1989, On generalized fractional processes, Journal of Time Series Analysis 10, 233 - 257.

Gray, H.L., N. Zhang. and W.A. Woodward, 1994, On generalized fractional processes. A correction, Journal of Time Series Analysis 15, 561 - 562.

Harvey, A.C., 1985, Trends and cycles in macroeconomic time series, Journal of Business Economics and Statistics, 3, 216 - 227.

Hassler, U., 1994, Misspecification of long memory seasonal time series, Journal of Time Series Analysis, 15, 19 - 30.

Hylleberg, S., R.F. Engle, C.W.J. Granger and B.S. Yoo, 1990, Seasonal integration and cointegration, Journal of Econometrics 44, 215 - 238.

Hosking, J.R.M., 1981, Fractional differencing, Biometrika 68, 165 - 176.

Hosking, J.R.M., 1984, Modelling persistence in hydrological time series using fractional differencing, Water Resources Research 20, 1898 - 1908.

Ooms, M., 1997, Flexible seasonal long memory and economic time series, Preprint.

Porter-Hudak, S., 1990, An application of the seasonal fractional differenced model to the monetary aggregates, Journal of the American Statistical Association 85, 338 - 344.

Press, W.H., B.P. Flannery, S.A. Teukolsky and W.T. Wetterling, 1986, Numerical recipes. The art of scientific computing, Cambridge University Press, Cambridge.

Ray, B.K., 1991, Fractionally differenced ARMA process: Seasonality and forecasting issues, Ph.D. thesis, Graduate School of Arts and Sciences, Columbia University, U.S.A.

Robinson, P.M., 1978, Statistical inference for a random coefficient autoregressive model, Scandinavian Journal of Statistics 5, 163 - 168.

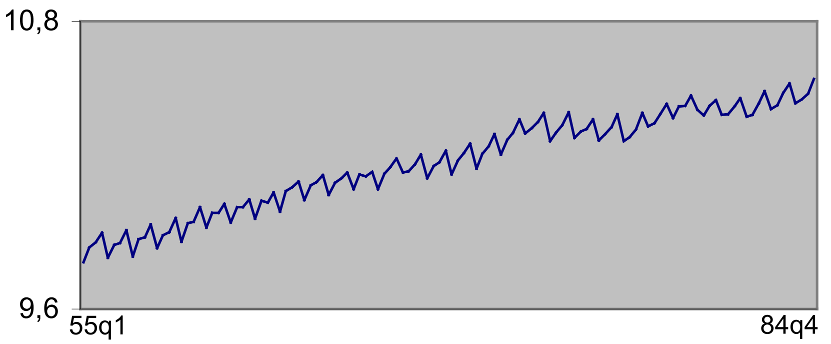
Robinson, P.M., 1994, Efficient tests of nonstationary hypotheses, Journal of the American Statistical Association 89, 1420 -1437.

Robinson, P.M., 1995a, Log-periodogram regression of time series with long range dependence, Annals of Statistics 23, 1048 - 1072.

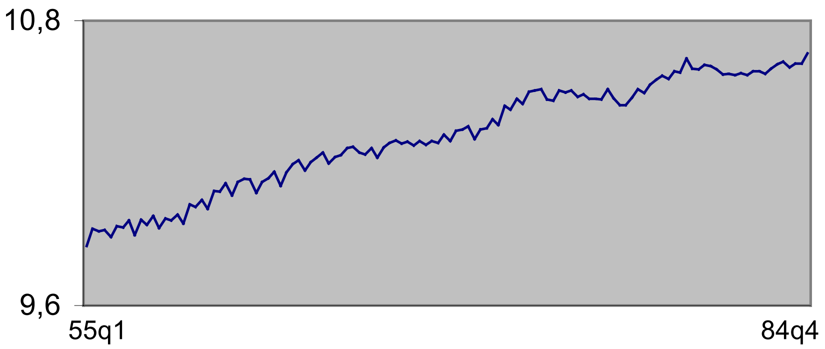
Robinson, P.M., 1995b, Gaussian semi-parametric estimation of long-range dependence, Annals of Statistics 23, 1630 - 1661.

FIGURE 1

Log of the UK consumption and income



Log of the UK consumption expenditure on non-durables



Log of the UK personal disposable income

TABLE 1					
Finite-sample critical values of $\widetilde{R}$ in (15) in a model given by (9) and (14) with s = 4					
T = 48	Values of r				
Percentiles	T/4	T/6	T/12	T/24	χ^2_3
10%	8.99	9.64	10.03	10.24	6.25
5%	10.56	11.34	11.81	12.24	7.81
2.5%	12.10	13.02	13.51	14.34	9.35
2%	12.56	13.54	14.03	15.13	11.30
1%	14.21	15.21	15.93	17.56	12.80
0.5%	16.12	17.12	18.16	20.34	16.30
0.1%	19.73	20.70	24.22	29.78	17.60
T = 96	Values of r				
Percentiles	T/4	T/6	T/12	T/24	χ^2_3
10%	7.50	8.39	8.65	8.64	6.25
5%	8.90	9.96	10.34	10.46	7.81
2.5%	10.41	11.61	12.08	12.44	9.35
2%	10.95	12.04	12.60	12.92	11.30
1%	12.45	13.55	14.47	14.73	12.80
0.5%	14.14	15.09	16.08	17.18	16.30
0.1%	18.19	18.89	21.03	24.08	17.60
T = 144	Values of r				
Percentiles	T/4	T/6	T/12	T/24	χ^2_3
10%	6.87	7.81	8.04	7.88	6.25
5%	8.31	9.34	9.82	9.67	7.81
2.5%	9.78	11.06	11.63	11.51	9.35
2%	10.36	11.62	12.20	12.14	11.30
1%	11.91	13.09	14.08	14.36	12.80
0.5%	13.83	14.68	15.89	16.06	16.30
0.1%	17.10	19.06	20.39	21.71	17.60

TABLE 2						
Sizes of $\widetilde{R}$ in (15 using both, the asymptotic and the finite-sample critical values and a nominal size of 5%.						
Sample size	Critical values		Values of r			
			T/4	T/6	T/12	T/24
T = 48	Finite-sample		**0.051**	**0.051**	**0.050**	**0.051**
	Asymptotic		0.163	0.210	0.234	0.234
T = 96	Finite-sample		**0.051**	**0.050**	**0.050**	**0.051**
	Asymptotic		0.085	0.126	0.140	0.139
T = 144	Finite-sample		**0.051**	**0.050**	**0.050**	**0.050**
	Asymptotic		0.065	0.100	0.110	0.103

TABLE 3						
Sizes of $\widetilde{R}$ in (15) using both, the asymptotic and the finite-sample critical values and a nominal size of 1%.						
Sample size	Critical values		Values of r			
			T/4	T/6	T/12	T/24
T = 48	Finite-sample		**0.011**	**0.010**	**0.011**	**0.011**
	Asymptotic		0.018	0.027	0.034	0.041
T = 96	Finite-sample		**0.010**	**0.010**	**0.011**	**0.010**
	Asymptotic		0.008	0.014	0.019	0.021
T = 144	Finite-sample		**0.010**	**0.010**	**0.010**	**0.010**
	Asymptotic		0.008	0.011	0.015	0.015

TABLE 4

$\widetilde{R}$ in (15) in a model given by (9) and (14) with s = 4 for the log of U.K. consumption expenditure on non-durables*

Integration orders			Values of r			
d_3	d_2	d_1	T/4	T/6	T/12	T/24
0.00	0.00	0.00	66.00	56.99	217.91	1193.19
0.00	0.00	0.50	44.06	40.44	58.25	281.27
0.00	0.00	1.00	**1.29'**	**1.41'**	**1.49'**	**1.48'**
0.00	0.50	0.00	66.20	60.54	85.28	403.46
0.00	0.50	0.50	35.65	35.46	36.49	36.32
0.00	0.50	1.00	11.26	11.65	11.52	11.33
0.00	1.00	0.00	91.81	91.61	92.40	93.64
0.00	1.00	0.50	24.70	28.03	35.42	27.02
0.00	1.00	1.00	18.38	19.52	19.35	18.42
0.50	0.00	0.00	12.09	**2.12'**	19.03	**2.06'**
0.50	0.00	0.50	15.74	**3.60'**	21.58	**0.76'**
0.50	0.00	1.00	14.21	**4.39'**	28.86	**0.24'**
0.50	0.50	0.00	11.96	**2.84'**	17.92	**2.37'**
0.50	0.50	0.50	15.36	**3.38'**	20.08	**0.74'**
0.50	0.50	1.00	13.41	**4.14'**	26.87	**0.23'**
0.50	1.00	0.00	11.81	**1.68'**	16.88	**2.37'**
0.50	1.00	0.50	14.96	**3.18'**	18.65	**0.73'**
0.50	1.00	1.00	12.72	**3.96'**	24.91	**0.21'**
1.00	0.00	0.00	65.42	56.16	15.43	63.54
1.00	0.00	0.50	63.74	28.15	41.52	38.87
1.00	0.00	1.00	28.28	63.50	56.20	45.24
1.00	0.50	0.00	70.08	51.52	27.13	40.41
1.00	0.50	0.50	80.30	41.12	62.82	51.82
1.00	0.50	1.00	25.17	74.70	20.41	57.83
1.00	1.00	0.00	13.53	62.38	57.84	50.25
1.00	1.00	0.50	44.89	54.42	72.07	59.23
1.00	1.00	1.00	27.29	80.58	78.09	64.89

*: Non-rejection values of the null hypothesis at the 95% significance level, using the finite-sample critical values obtained in Table 1.

TABLE 5						
$\widetilde{R}$ in (15) in a model given by (9) and (14) with s = 4 for the log of U.K. personal disposable income*						
Integration orders			Values of r			
d_3	d_2	d_1	T/4	T/6	T/12	T/24
0.00	0.00	0.00	65.98	56.97	217.81	1192.81
0.00	0.00	0.50	43.67	40.03	58.59	285.53
0.00	0.00	1.00	**1.32'**	**1.44'**	**1.53'**	**1.52'**
0.00	0.50	0.00	65.67	59.92	85.75	409.27
0.00	0.50	0.50	35.81	35.63	36.67	36.50
0.00	0.50	1.00	11.10	11.53	11.38	11.17
0.00	1.00	0.00	91.93	91.73	92.51	93.76
0.00	1.00	0.50	24.82	28.19	35.52	27.12
0.00	1.00	1.00	18.27	19.47	19.28	18.30
0.50	0.00	0.00	12.13	**2.05'**	19.04	**2.37'**
0.50	0.00	0.50	15.79	**3.62'**	21.54	**0.76'**
0.50	0.00	1.00	14.28	**4.42'**	28.77	**0.24'**
0.50	0.50	0.00	12.01	**1.86'**	17.94	**2.37'**
0.50	0.50	0.50	15.42	**3.41'**	20.05	**0.75'**
0.50	0.50	1.00	13.49	**4.16'**	26.78	**0.23'**
0.50	1.00	0.00	11.86	**1.70'**	16.90	**2.38'**
0.50	1.00	0.50	15.20	**3.20'**	18.63	**0.73'**
0.50	1.00	1.00	12.79	**3.98'**	24.83	**0.21'**
1.00	0.00	0.00	65.40	56.15	15.98	64.71
1.00	0.00	0.50	63.32	28.29	41.54	38.92
1.00	0.00	1.00	28.33	63.64	56.40	45.40
1.00	0.50	0.00	69.45	51.14	26.78	40.10
1.00	0.50	0.50	80.58	40.92	62.55	51.59
1.00	0.50	1.00	25.12	74.52	70.23	57.66
1.00	1.00	0.00	13.64	62.74	57.58	50.06
1.00	1.00	0.50	45.14	54.19	71.92	59.10
1.00	1.00	1.00	27.20	80.48	77.99	64.79

*: Non-rejection values of the null hypothesis at the 95% significance level, using the finite-sample critical values obtained in Table 1