Efficient hedging for a complete jump-diffusion model

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Abstract.

This paper is devoted to the problem of hedging contingent claims in the framework of a complete two-factor jump-diffusion model. In this context, it is well understood that every contingent claim can be hedged perfectly if one invests the unique arbitrage-free price. Based on the results of H. Föllmer and P. Leukert [4]-[5] in a general semimartingale setting, we determine the unique hedging strategies which minimize a suitably defined shortfall risk under a given cost constraint. We derive explicit formulas for this so-called efficient or quantile hedging strategy for a European call option. We then compare the performance of the optimal strategy for different degrees of the investor's risk-aversion.

Key words: Efficient hedging, Quantile Hedging, jump-diffusion, martingale measure.

JEL classification: G10, G12, G13, D81.

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1.Introduction.

The problem of hedging a contingent claim with probability one is well understood in a complete financial market. In this situation, every contingent claim can be replicated perfectly by investing the unique arbitrage-free price. However, by eliminating the risk completely the investor also takes away the chance of making a profit. Recent work studies the question what an investor who is short the option can do if he is unwilling to invest the price of the option completely in the hedging strategy. Under this constraint, the investor is not able to eliminate the risk completely. Instead, no matter which admissible strategy the investor chooses, he is faced with a strictly positive probability that he will incur a nontrivial *shortfall* (the difference between the value of his portfolio and the value of the option at time T). Hence, some optimality criterion on the shortfall is needed in order to determine the most efficient hedging strategy. In [4], Föllmer and Leukert studied strategies that successfully hedge the option with *maximal probability* in the class of all self-financing strategies with restricted cost. They called the optimal hedging strategy in this class a *quantile hedging strategy* and showed how the determination of the quantile hedging strategy can be reduced to computing the optimal *success set* by means of the Neyman-Person Lemma

In [5], Föllmer and Leukert developed a more general approach to hedge an option in the most effective way given the capital constraint: They replaced the criterion of

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maximizing the probability of "no shortfall" by the demand to minimize the expected size of the shortfall weighted by some loss function l. The optimal strategy with respect to this demand is then called an *efficient hedging* strategy. They showed that the efficient hedging strategy is given by the perfect hedging strategy for some modified option. In a second step, they demonstrated how this modified option can be computed by means of the unique equivalent martingal measure and the derivative of the loss function. The quantile hedging strategy appears as a special case of an efficient hedging strategy for a suitably defined loss function l. Hence we use the term "efficient" in the sequel with the understanding that the quantile case is also included.

In the present paper, we apply the results of Föllmer and Leukert in [4] and [5] to derive explicit formulas for the efficient hedging strategy in a jump-diffusion model given by equations (1),(2) and (5). We show how the efficient hedge can be derived from an ordinary partial differential equation where the boundary condition is given by the modified option. In the special case of a European call option, we demonstrate how the modified option can be constructed by computing the root of a rather simple equation. We then compare the performance of the optimal strategy for different degrees of the investor's risk-aversion.

2.Description of the Model and Auxiliary results.

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geqslant 0}, \mathrm{P})$ be a standard stochastic basis. We assume there are two risky assets S^1 and S^2 whose price-processes are described by the following stochastic differential equations

$$dS_t^i = S_{t-}^i(\mu^i dt + \sigma^i dW_t - \nu^i d\Pi_t), i = 1, 2\,, \tag{1}$$

where W is a standard Wiener process and Π is a Poisson process with positive intensity λ. Suppose also that W and Π are independent and that the filtration $\mathbf{F}$ is generated by W and Π, $\mu^i \in \mathbb{R}, \sigma^i > 0, \nu^i < 1$.

There is a non-risky asset B which follows the equation

$$dB_t = rB_t dt, B_0 = 1, r \in \mathbb{R}\,. \tag{2}$$

Every predictable process $\pi = (\pi_t)_{t\geqslant 0} = ((\beta_t, \gamma_t^1, \gamma_t^2))_{t\geqslant 0}$ can be regarded as *trading strategy* or portfolio. The value of such a portfolio at time t equals

$$X_t^\pi = \beta_t B_t + \gamma_t^1 S_t^1 + \gamma_t^2 S_t^2. \tag{3}$$

If the discounted value $\frac{X^\pi}{B}$ of a strategy π can be represented in the form

$$\frac{X_t^\pi}{B_t} = \frac{X_0^\pi}{B_0} + \int_0^t \sum_{i=1}^2 \gamma_u^i d\left(\frac{S_u^i}{B_u}\right) (\mathrm{P}-\text{a.s.}), \tag{4}$$

then π is called *self-financing*. A self-financing strategy with non-negative value at any time is called *admissible*.

The market (1)-(2) is *complete* if the following conditions are fulfilled (see, for instance Melnikov and Shiryaev [8], Volkov and Kramkov [6])

$$\sigma^2\nu^1 - \sigma^1\nu^2 \neq 0, \frac{(\mu^1 - r)\sigma^2 - (\mu^2 - r)\sigma^1}{\sigma^2\nu^1 - \sigma^1\nu^2} > 0. \tag{5}$$

Under condition (5), there exists a unique equivalent martingale measure P^* with local density

$$Z_t = \left.\frac{d\mathrm{P}^*}{d\mathrm{P}}\right| \mathcal{F}_t = \exp(\alpha^* W_t - \frac{\alpha^{*2}}{2} t + (\lambda - \lambda^*)t + (\ln \lambda^* - \ln \lambda)\Pi_t), \tag{6}$$

where the pair(α^*, λ^*) is given by the unique solution of the equation

$$\begin{cases} \mu^1 - r = & -\sigma^1 \alpha^* + \nu^1 \lambda^* \\ \mu^2 - r = & -\sigma^2 \alpha^* + \nu^2 \lambda^* \end{cases}, \lambda^* > 0\,. \tag{7}$$

Under the measure P^*, $W_t^* = W_t - \alpha^* t$ is a Wiener process, Π is a Poisson process with intensity $\lambda^* > 0$, and W^* is independent of Π.

A non-negative $\mathcal{F}_T$-measurable function f_T is called contingent claim. For a *perfect* hedge, we have to find a self-financing strategy π that eliminates the risk completely in the sense that

$$\mathrm{P}[X_T^\pi \geqslant f_T] = 1$$

and requires minimal initial capital X_0^π over all such strategies. This minimal initial capital is also called the *fair price* of the option.

We consider classical options of the form $f_T = f(S_T^1)$. We will demonstrate that the efficient hedging strategy for such an option coincides with the perfect hedging strategy for a modified option $\tilde{f}_T \leq f_T$ of the form

$$\tilde{f}_T = g(S_T^1, S_T^2)$$

for a measurable function $g : \mathbb{R}_+^2 \to \mathbb{R}_+$, i.e., the modified option is a basket option. For this reason we first study hedging strategies for basket options. According to the general theory of perfect hedging (see [1], [2], [3], [9] and [10]), the unique fair price of a basket option is given by

$$\mathbb{C}(T, S_0^1, S_0^2) := \mathbf{E}^*[g(S_T^1, S_T^2)e^{-rT}],$$

where $\mathbf{E}^*$ denotes expectation with respect to P^*.

By the Ito-formula we obtain

$$\begin{aligned} S_t^i &= S_0^i \exp\{\sigma^i W_t + (\mu^i - \frac{1}{2}(\sigma^i)^2)t\}(1 - \nu^i)^{\Pi_t} \\ &= S_0^i \exp\{\sigma^i W_t^* + (\mu^i + \sigma^i \alpha^* - \frac{1}{2}(\sigma^i)^2)t\}(1 - \nu^i)^{\Pi_t} \\ &= S_0^i \exp\{\sigma^i W_t^* + (r + \nu^i \lambda^* - \frac{1}{2}(\sigma^i)^2)t\}(1 - \nu^i)^{\Pi_t} \end{aligned} \tag{8}$$

and

$$Y_t^i = \frac{S_t^i}{B_t} = Y_0^i \exp\{\sigma W_t^* + (\nu^i \lambda^* - \frac{1}{2}(\sigma^i)^2)t\}(1 - \nu^i)^{\Pi_t}.$$

The basic observation is that the independence of W^* and Π and (8) imply the general pricing formula for basket options:

$$\mathbb{C}(T, S_0^1, S_0^2) = \sum_{n=0}^{\infty} p_{n,T}\, \mathbf{E}^* \left[g(s_{n,T}^1, s_{n,T}^2)e^{-rT}\right], \tag{9}$$

where

$$p_{n,T} := e^{-\lambda^* T} \frac{(\lambda^* T)^n}{n!}$$

are the weights of the poisson distribution and $s^i_{n,T}$ are lognormally distributed random variables under P^*:

$$\ln(s^i_{n,T}) \sim \mathcal{N}\left(\ln[S^i_0(1-\nu^i)^n] + [r - \frac{1}{2}(\sigma^i)^2 + \nu^i\lambda^*]T, \sigma^i\sqrt{T}\right). \tag{10}$$

In the special case of a European call option with payoff $f(S^1_T) = (S^1_T - K)^+$ for some strike $K > 0$, we obtain from (9) and (10) the unique fair price (which is independent of S^2_0):

$$\mathbb{C}(T, S^1_0) = \sum_{n=0}^{\infty} \mathbb{C}^{BS}(S^1_0 \vartheta_{n,T}, K, T) p_{n,T} \tag{11}$$

where $\mathbb{C}^{BS}$ denotes the *Black-Scholes price* formula

$$\mathbb{C}^{BS}(S_0, K, T) = S_0 \Phi(d_+(S_0, K)) - K e^{-rT} \Phi(d_-(S_0, K)), \tag{12}$$

$$d_\pm(S_0, K) = \frac{\ln(S_0) - \ln(K) + (r \pm \frac{1}{2}(\sigma^1)^2)T}{\sigma^1\sqrt{T}}, \tag{13}$$

$\Phi(x)$ is the standard normal distribution function and the constants $\vartheta_{n,T}$ are defined by means of

$$\vartheta_{n,T} = (1-\nu^1)^n e^{\nu^1 \lambda^* T}, \quad n \in \mathbb{N}. \tag{14}$$

The value of an European option can also be computed by means of the following partial differential equation.

Theorem 1. Let some basket option $g_T = g(S^1_T, S^2_T)$ be given. Then the value of the perfect hedging strategy at time t is given by a function $\mathbb{C}(S^1_t, S^2_t, t)$ of the current asset prices and time that satisfies the partial differential equation

$$\begin{aligned}
&\left[\mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t) - \mathbb{C}(S^1_{t-}, S^2_{t-}, t)\right]\lambda^* + rS^1_{t-}\tfrac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) \\
&+ rS^2_{t-}\tfrac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \tfrac{\partial}{\partial t}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \tfrac{1}{2}(\sigma^1 S^1_{t-})^2 \tfrac{\partial^2}{\partial x^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) \\
&+ \tfrac{1}{2}(\sigma^2 S^2_{t-})^2 \tfrac{\partial^2}{\partial y^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + (\sigma^1\sigma^2 S^1_{t-}S^2_{t-})\tfrac{\partial^2}{\partial x \partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) - r\mathbb{C}(S^1_{t-}, S^2_{t-}, t) \\
&+ \tfrac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\nu^1\lambda^* S^1_{t-} + \tfrac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\nu^2\lambda^* S^2_{t-} = 0
\end{aligned} \tag{15}$$

with boundary condition

$$\mathbb{C}(S^1_T, S^2_T, T) = g(S^1_T, S^2_T).$$

The components $\beta, \gamma^1, \gamma^2$ of the perfect hedging strategy are given by

$$\begin{cases} \gamma^1_t \sigma^1 S^1_{t-} + \gamma^2_t \sigma^2 S^2_{t-} = S^1_{t-}\sigma^1 \frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + S^2_{t-}\sigma^2 \frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) \\ \gamma^1_t \nu^1 S^1_{t-} + \gamma^2_t \nu^2 S^2_{t-} = \mathbb{C}(S^1_{t-}, S^2_{t-}, t) - \mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t), \end{cases} \tag{16}$$

and

$$\beta_t = \frac{\mathbb{C}(S^1_{t-}, S^2_{t-}, t) - \gamma^1_t S^1_{t-} - \gamma^2_t S^2_{t-}}{B_t}. \tag{17}$$

We have assembled the proofs to all theorems in the appendix.

Again, we consider the special case of a European call option. Here, the components γ_t^1, γ_t^2 of the hedging strategy are the roots of

$$\begin{cases} \gamma_t^1 \sigma^1 S_{t-}^1 + \gamma_t^2 \sigma^2 S_{t-}^2 = S_{t-}^1 \sigma^1 \frac{\partial}{\partial x} \mathbb{C}(S_{t-}^1, t) \\ \gamma_t^1 \nu^1 S_{t-}^1 + \gamma_t^2 \nu^2 S_{t-}^2 = \mathbb{C}(S_{t-}^1, t) - \mathbb{C}(S_{t-}^1(1-\nu^1), t). \end{cases}$$

and the first component of the hedge can be determined from (11) and (17). Equation (15) simplifies to

$$\left[\mathbb{C}(S_{t-}^1(1-\nu^1), t) - \mathbb{C}(S_{t-}^1, t)\right] \lambda^* + \frac{\partial}{\partial t}\mathbb{C}(S_{t-}^1, t) + \frac{1}{2}(\sigma^1 S_{t-}^1)^2 \frac{\partial^2}{\partial x^2}\mathbb{C}(S_{t-}^1, t)$$
$$+ r S_{t-}^1 \frac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, t) - r\mathbb{C}(S_{t-}^1, t) + \frac{\partial}{\partial x}\mathbb{C}(S_{t-}^1, t)\nu^1 \lambda^* S_{t-}^1 = 0.$$

So far, we have exposed the standard theory of pricing and hedging in the model under consideration. We now turn to the problem of efficient hedging.

3.Efficient Hedging.

Given an upper bound $X_0 < E^*[e^{-rT} f_T]$ on the initial capital which is available for hedging the option f_T, the efficient hedging strategy π is defined as the solution of the following problem:

$$E[l((f_T - X_T^\pi)^+)] = \min_\pi \tag{18}$$

where the minimum is taken over all admissible strategies that satisfy the cost constraint

$$X_0^\pi \leq X_0. \tag{19}$$

For the definition of the value process X^π cf. equations (3) and (4).

Concerning the solution of problem (18), we are going to consider two important special cases:

(i) *Quantile Hedging*
The loss function is given by $l(x) = I_{(0,\infty)}(x)$. In this case, we obtain

$$E[l((f_T - X_T^\pi)^+)] = E[I_{f_T > X_T^\pi}] = P[f_T > X_T^\pi]$$

i.e. problem (18) is equivalent to *maximizing* the probability of a successful hedge $P[f_T \leq X_T^\pi]$.

(ii) *Lower partial moments*
The loss function is given by $l(x) = x^p$ for some $p > 1$.

Note that the risk-neutral case $l(x) = x$ is not included in our analysis. However, using the results of [5] concerning the risk-neutral case and the methodology presented below easily yields formulas similar to the quantile case.

Concerning the quantile case *(i)*, Föllmer and Leukert showed in [5] that the quantile hedging strategy $\tilde{\pi}$ for a European option of the form $f_T = f(S_T^1)$ is given by the perfect hedging strategy for the knockout-option $\tilde{\phi} f_T$ defined by[1]

$$\tilde{\phi} = I_{\{\frac{dP}{dP^*} > \tilde{a} f_T\}}, \tag{20}$$

[1]Here we have used the fact that the randomization part has zero measure, i.e. $P\left[\frac{dP}{dP^*} = const \cdot f(S_T^1)\right] = 0$, cf. equation (45).

where the critical value $\tilde{a}$ has to be computed by means of the capital constraint

$$\mathbf{E}^* \left[e^{-rT} \tilde{\phi} f_T \right] = X_0. \tag{21}$$

The maximal probability $P[f_T \leq X_T^\pi]$ of a successful hedge that can be achieved by any admissible strategy π satisfying the cost constraint (19) is then given by

$$\begin{aligned} 1 - \varepsilon &:= E[\tilde{\phi}] \\ &= P[f_T \leq X_T^{\tilde{\pi}}]. \end{aligned} \tag{22}$$

In the risk averse case *(ii)*, Föllmer and Leukert showed in [4] that the efficient hedging strategy $\tilde{\pi}$ is given by the perfect hedging strategy for the modified claim

$$\tilde{f}_T := f_T - \tilde{a} Z_T^{\frac{1}{p-1}} \wedge f_T \tag{23}$$

where Z_T was defined in (6) and the constant $\tilde{a}$ has to be determined from the capital constraint

$$E^*[\tilde{f}_T \, e^{-rT}] = X_0. \tag{24}$$

Before we state the results concerning efficient hedging strategies, we have to introduce some new objects. First, we define the constants

$$b = \frac{\lambda^*}{\lambda(1-\nu^1)^{\frac{\alpha^*}{\sigma^1}}} \quad \text{and} \tag{25}$$

$$q = -\frac{\alpha^*}{\sigma^1(p-1)}. \tag{26}$$

We denote by $c^1(a;n) \leq c^2(a;n)$ the roots of the equation

$$x^{-\frac{\alpha^*}{\sigma^1}} = b^n a(x-K). \tag{27}$$

If $-\frac{\alpha^*}{\sigma^1} \leqslant 1$, equation (27) has only one solution which we denote by $c(a;n)$. We introduce auxiliary functions $\tilde{\mathbb{C}}_a$ and $\tilde{\mathbb{C}}_b$ as follows:

$$\begin{aligned} \tilde{\mathbb{C}}_a(S,n,c,T) := & \big[\mathbb{C}^{BS}(S\vartheta_{n,T},K,T) - \mathbb{C}^{BS}(S\vartheta_{n,T},c,T) \\ & - (c-K)e^{-rT} \cdot \Phi[d_-(S\vartheta_{n,T},c)]\big] \end{aligned} \tag{28}$$

and

$$\begin{aligned} \tilde{\mathbb{C}}_b(S,n,c^1,c^2,T) := & \mathbb{C}^{BS}(S\vartheta_{n,T},K,T) - \mathbb{C}^{BS}(S\vartheta_{n,T},c^1,T) \\ & + \mathbb{C}^{BS}(S\vartheta_{n,T},c^2,T) - (c^1-K)e^{-rT} \cdot \Phi[d_-(S\vartheta_{n,T},c^1)] \\ & + (c^2-K)e^{-rT} \cdot \Phi[d_-(S\vartheta_{n,T},c^2)]. \end{aligned}$$

We can express the number Π_t of jumps up to time t in terms of (S_t^1, S_t^2, t): From

$$\begin{aligned} S_t^i &= S_0^i \exp(\sigma^i W_t^* + (r + \nu^i \lambda^* - \frac{1}{2}(\sigma^i)^2)t)(1-\nu^i)^{\Pi_t} \\ W_t^* &= \frac{1}{\sigma^i}\left(\ln \frac{\frac{S_t^i}{S_0^i}}{(1-\nu^i)^{\Pi_t}} - (r + \nu^i \lambda^* - \frac{(\sigma^i)^2}{2})t \right) \end{aligned} \tag{29}$$

we obtain

$$\frac{1}{\sigma^1}\left(\ln\frac{S_t^1}{S_0^1}-\Pi_t\ln(1-\nu^1)-(r+\nu^1\lambda^*-\frac{(\sigma^1)^2}{2})t\right)$$
$$=\frac{1}{\sigma^2}\left(\ln\frac{S_t^2}{S_0^2}-\Pi_t\ln(1-\nu^2)-(r+\nu^2\lambda^*-\frac{(\sigma^2)^2}{2})t\right)$$

which implies the desired formula for Π

$$\Pi_t=\frac{\frac{1}{\sigma^1}\ln\frac{S_t^1}{S_0^1}-(r+\nu^1\lambda^*-\frac{(\sigma^1)^2}{2})\frac{t}{\sigma^1}-\frac{1}{\sigma^2}\ln\frac{S_t^2}{S_0^2}+(r+\nu^2\lambda^*-\frac{(\sigma^2)^2}{2})\frac{t}{\sigma^2}}{\frac{1}{\sigma^1}\ln(1-\nu^1)-\frac{1}{\sigma^2}\ln(1-\nu^2)}=:\Pi(S_t^1,S_t^2,t). \tag{30}$$

For quantile hedging in a jump-diffusion model see also [7].

Theorem 2. Consider a European call option with strike K and maturity T on the stock S^1 and some initial capital $X_0<E^*[e^{-rT}(S_T^1-K)^+]$. Concerning the structure of the quantile hedging strategy $\tilde{\pi}$, we distinguish two cases:
a) $-\frac{\alpha^*}{\sigma^1}\leqslant 1$ and b) $-\frac{\alpha^*}{\sigma^1}>1$

Case a. Let $\tilde{a}$ denote the unique solution of the equation

$$X_0=\sum_{n=0}^{\infty}p_{n,T}\tilde{\mathbb{C}}_a(S_0^1,n,c(a;n),T). \tag{31}$$

(a1) The quantile hedging strategy is given by the perfect hedging strategy for the modified claim

$$\tilde{f}(S_T^1,S_T^2)=(S_T^1-K)^+I_{\left\{S_T^1<c(\tilde{a};\Pi(S_T^1,S_T^2,T))\right\}}. \tag{32}$$

(a2) The probability of equation (22) that the hedge is successful equals

$$1-\varepsilon=1-e^{-\lambda T}\sum_{n=0}^{\infty}\Phi\left[d_-(S_0^1(1-\nu^1)^n,c(\tilde{a};n))+\frac{\mu^1-r}{\sigma^1}\sqrt{T}\right]\frac{(\lambda T)^n}{n!}. \tag{33}$$

(a3) The value of the strategy at time t is given by

$$\mathbb{C}(S_t^1,S_t^2,t)=\sum_{n=0}^{\infty}p_{n,T-t}\tilde{\mathbb{C}}_a(S_t^1,n,c(\tilde{a};n+\Pi(S_t^1,S_t^2,t)),T-t).$$

(a4) The components γ^1, γ^2 and β of the strategy can be computed by means of the last equation, (16) and (17).

Case b.
Let $\tilde{a}$ denote the unique solution of the equation

$$X_0=\sum_{n=0}^{\infty}\left[\tilde{\mathbb{C}}_b(S_0^1,n,c^1(a;n),c^2(a;n),T)\right]p_{n,T}. \tag{34}$$

(b1) The quantile hedging strategy is given by the perfect hedging strategy for the modified claim

$$\tilde{f}(S_T^1,S_T^2)=(S_T^1-K)^+\left[I_{\left\{S_T^1<c^1(\tilde{a};\Pi(S_T^1,S_T^2,T))\right\}}+I_{\left\{S_T^1>c^2(\tilde{a};\Pi(S_T^1,S_T^2,T))\right\}}\right]. \tag{35}$$

(b2) The probability of equation (22) that the hedge is successful equals

$$1-\varepsilon = 1-\sum_{n=0}^{\infty} e^{-\lambda T}\left\{\Phi\left[d_-(S_0^1(1-\nu^1)^n, c^1(\tilde{a};n)) + \frac{\mu^1-r}{\sigma^1}\sqrt{T}\right]\right. \quad (36)$$
$$\left. -\Phi\left[d_-(S_0^1(1-\nu^1)^n, c^2(\tilde{a};n)) + \frac{\mu^1-r}{\sigma^1}\sqrt{T}\right]\right\}\frac{(\lambda T)^n}{n!}.$$

(b3) The value of the strategy at time t is given by

$$\mathbb{C}(S_t^1, S_t^2, t) = \sum_{n=0}^{\infty} p_{n,T-t}\tilde{\mathbb{C}}_b(S_t^1, n, c^1(\tilde{a}; n+\Pi(S_t^1, S_t^2, t)), c^2(\tilde{a}; n+\Pi(S_t^1, S_t^2, t)), T-t). \quad (37)$$

(b4) The components γ^1, γ^2 and β of the strategy can be computed by means of the last equation, (16) and (17).

Instead of prescribing the maximal initial capital available for the hedge, we could as well have fixed a maximal shortfall probability ε and determine the minimal initial capital $\tilde{X}_0(\varepsilon)$ such that there exists a strategy $\tilde{\pi}$ that has a shortfall probability of at most ε. In view of the above theorem, this is easily achieved [2]: Given a maximal shortfall probability ε, compute $\tilde{a}(\varepsilon)$ by means of equation (33). Substitute $a = \tilde{a}(\varepsilon)$ in equation (31) and determine $\tilde{X}_0(\varepsilon)$ as the unique solution of this equation for X_0. Then, the strategy $\tilde{\pi}$ given by (a1), (a3) and (a4) with initial capital $\tilde{X}_0(\varepsilon)$ successfully hedges the option with probability $1-\varepsilon$ and $\tilde{X}_0(\varepsilon)$ is the minimal initial capital that is required to establish such a hedging strategy.

Concerning the efficient hedging strategy in the risk-averse case, let $L(a;n)$ denote the unique root of

$$ab^{\frac{n}{p-1}} L^{-q} = (L-K)^+ \quad (38)$$

and define

$$\tilde{\mathbb{C}}_p(S,n,L,T) = \mathbb{C}^{BS}(S\vartheta_{n,T}, L, T) + e^{-rT}(L-K)\Phi(d_-(\vartheta_{n,T}S, L(a;n))) \quad (39)$$
$$+L^q(L-K)(S\vartheta_{n,T})^{-q}\kappa_T\Phi(d_-(\vartheta_{n,T}S, L) - q\sigma^1\sqrt{T}),$$
$$\kappa_T = e^{\frac{1}{2}(\sigma^1)^2Tq(1-q)}e^{-rT(1+q)}. \quad (40)$$

Theorem 3. Again, consider a European call option with maturity T and strike K on the stock S^1 and some initial capital $X_0 < E^*[e^{-rT}(S_T^1-K)^+]$. The structure of the efficient hedging strategy with respect to the loss function $l(x) = x^p$ for some $p > 1$ is the following: Let $\tilde{a}$ denote the unique root of the equation

$$\sum_{n=0}^{\infty} p_{n,T}\, \tilde{\mathbb{C}}_p(S_0^1, L(\tilde{a};n), T) = X_0. \quad (41)$$

(1) The efficient hedging strategy is given by the perfect hedging strategy for the modified claim

$$\tilde{f}(S_T^1, S_T^2) = \left[(S_T^1-K)^+ - \Big(L(\tilde{a};\Pi(S_T^1,S_T^2,T))S_T^1\Big)^q\right. \quad (42)$$
$$\left.\times\Big(L(\tilde{a};\Pi(S_T^1,S_T^2,T)) - K\Big)\right] I_{\{S_T^1 \geq L(\tilde{a};\Pi(S_T^1,S_T^2,T))\}}.$$

[2]For ease of exposition, we assume that case a holds, otherwise one has to use the respective equations for case b

(2) The value of the strategy at time t is given by

$$\mathbb{C}(S_t^1, S_t^2, t) = \sum_{n=0}^{\infty} p_{n,T-t}\tilde{\mathbb{C}}_p(S_t^1, n, L(\tilde{a}; n + \Pi(S_t^1, S_t^2, t)), T-t). \tag{43}$$

(3) The components γ^1, γ^2 and β of the strategy can be computed by means of the last equation, (16) and (17).

4. Numerical Examples.

Consider the following set of parameters:

$$\begin{array}{llll} \mu^1 = 0.12; & \sigma^1 = 0.3; & \nu^1 = 0.05; & \lambda = 1; \\ \mu^2 = 0.102 & \sigma^2 = 0.3 & \nu^2 = 0.01. & \\ r = 0.03; & T = 1; & S_0^1 = 100; & K = 90. \end{array} \tag{44}$$

For this set of parameters, we obtain $\lambda^* = 0.45$, $-\frac{\alpha^*}{\sigma^1} = 0.75$ and the price of the call is $\mathbb{C}(T, S_0^1) = 18.67$. If there where no jumps, i.e., if $\lambda = 0$, the Black-Scholes price of the call would be 18.61.

Figure 1 shows the success probability of the quantile hedging strategy as a function of the invested capital. As illustrated by the additional lines, the investor can save 6.5 percent of the perfect-hedging price (respectively 26 or 43 percent) if he is willing to accept a shortfall-probability of 1 percent (respectively 5 or 10 percent).

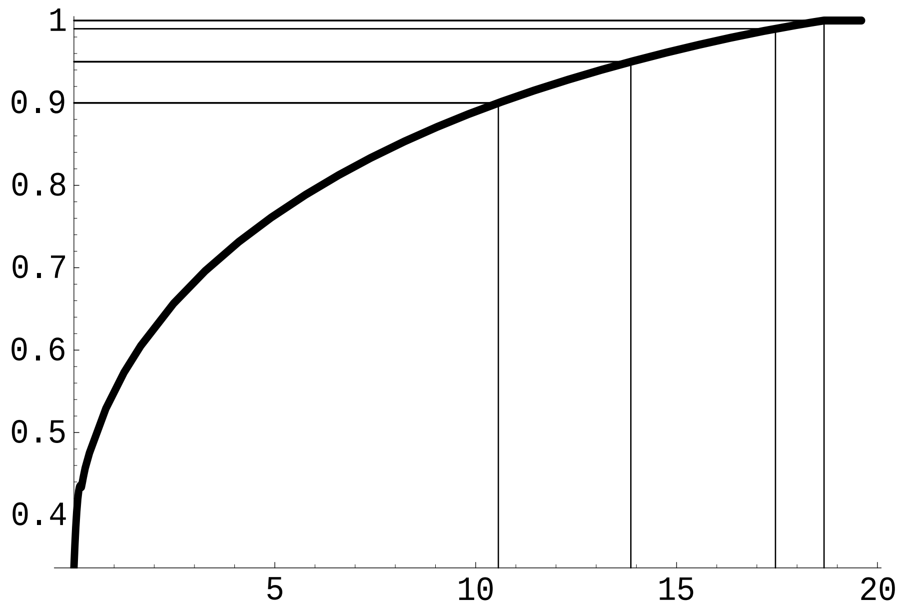

Figure 1: The success probability as a function of the invested capital

Now consider the case where the investor is risk-averse and seeks to establish an efficient hedging strategy for $p = 2$. In this case, we propose to measure risk associated to a nonnegative liability by the L^2-norm which is of the same monetary unit as the actual loss. The bold line in Figure 2 shows the L^2-norm of the shortfall as a function of the invested capital if the investor uses the efficient hedging strategy for

$p = 2$. The dashed line corresponds to the L^2-norm of the shortfall if the investor uses the quantile-hedging strategy of Figure 1. The thin line shows the L^2-norm of the shortfall if the investor establishes the hedging strategy that minimizes the *maximum loss*, i.e. if he hedges a call-option with a higher strike where the strike is chosen such as to satisfy the capital constraint - a strategy popular with traders.

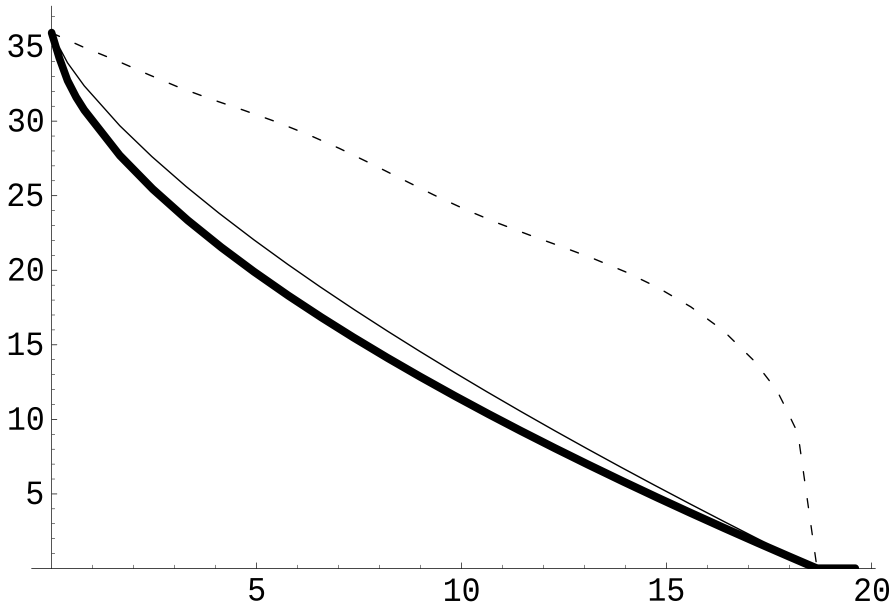

Figure 2: L^2-norm of the shortfall as a function of the invested capital: Efficient (thick) vs. Min-Max-Loss (thin) and Quantile-strategy (dashed)

The comparison of the three lines in Figure 2 demonstrates the increase in performance of the efficient hedging strategy compared to the alternatives quantile- and Min-Max-loss strategy for a moderately risk-averse investor ($p = 2$). The increase in performance of the efficient hedging strategy over the quantile-hedging strategy is more pronounced for larger p. On the other hand, the increase in performance of the efficient hedging strategy over the Min-Max-loss strategy is more pronounced for smaller p. For $p \uparrow \infty$, the distance between the thick and the thin line vanishes whereas the dashed line assumes the value $+\infty$.

References

[1] K. Aase. Contingent claim valuation when the security price is a combination of an ito process and a random point process. *Stoch. Process. Appl.*, 28:185–220, 1988.

[2] J. Bardhan and X. Chao. Pricing options on securities with discontinous returns. *Stoch. Process. Appl*, 48:123–137, 1993.

[3] D.B. Colwell and R. Elliott. Discontinous asset prices and non-attainable contingent claims and corporate policy. *Mathematical Finance*, 3:295–318, 1993.

[4] H. Foellmer and P. Leukert. Quantile hedging. *Finance and Stochastics*, 3:251–273, 1999.

[5] H. Foellmer and P. Leukert. Efficient hedging: Balancing cost versus shortfall risk. *Finance and Stochastics*, 4:117–146, 2000.

[6] D. Kramkov and S.N. Volkov. On methodology of hedging of options. *Survey of Applied and Industrial Math*, 4:18–65, 1997.

[7] A.V. Melnikov, M.L.Nechaev, and S.N.Volkov. *Mathematics of Financial Liabilities*. Publ.House of Higher School of Economics, Moscow, 2001.

[8] A.V. Melnikov and A.N. Shiryaev. Criteria for absence of arbitrage in financial market, 1996.

[9] W. J. Runggaldier and F. Mercurio. Option pricing for jump diffusions: Approximations and their interpretation. *Mathematical Finance*, 3:191–200, 1993.

[10] A.N. Shiryaev. *Essentials of Stochastic Finance*. World Scientificy, 1999.

Appendix.

Proof of Theorem 2:
We know from [4], that the quantile hedging stratey is given by the perfect hedging strategy for the modified claim $\tilde{\phi} f_T$ defined by equations (20) and (21). We first paraphrase the risk neutral density in terms of S^1_T and Π_T:

$$\begin{aligned}
&\tfrac{dP^*}{dP} = \exp(\alpha^* W_T - \tfrac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T + (\ln\lambda^* - \ln\lambda)\Pi_T) \\
&= \left(S^1_0 \exp\left\{\sigma^1 W_T + (\mu^1 - \tfrac{\sigma^{12}}{2})T\right\}(1-\nu^1)^{\Pi_T}\right)^{\frac{\alpha^*}{\sigma^1}} \times \\
&\times \tfrac{1}{S_0^{1\frac{\alpha^*}{\sigma^1}}} \exp\left(-\tfrac{\alpha^*\mu^1}{\sigma^1}T + \tfrac{\sigma^1\alpha^*}{2}T - \tfrac{\alpha^{*2}}{2}T + (\lambda-\lambda^*)T)\right) \times \\
&\times \left(\frac{\lambda^*}{\lambda(1-\nu^1)^{\frac{\alpha^*}{\sigma^1}}}\right)^{\Pi_T} \\
&= g\cdot (S^1_T)^{\frac{\alpha^*}{\sigma^1}} \cdot b^{\Pi_T},
\end{aligned} \tag{45}$$

with b defined in equation (25) and a suitably defined constant g.

Using (45) we can represent $\tilde{\phi}$ in the form

$$\tilde{\phi} = I_{\{\frac{dP}{dP^*} > \tilde{a}_1 f_T\}} = I_{\left\{S_T^{1\,-\frac{\alpha^*}{\sigma^1}} > b^{\Pi_T}\tilde{a}(S^1_T - K)^+\right\}}$$

where we have set $\tilde{a} = \tilde{a}_1 g$. We show how to compute $\tilde{a}$ by means of the capital constraint $\mathbf{E}^*\left[e^{-rT}\phi f\right] = X_0$. It follows from the last equation that

$$\begin{aligned}
\mathbf{E}^*\left[e^{-rT}\phi f_T\right] &= \mathbf{E}^*\left[e^{-rT}(S^1_T - K)^+ I_{\left\{S_T^{1\,-\frac{\alpha^*}{\sigma^1}} > b^{\Pi_T}\tilde{a}(S^1_T-K)^+\right\}}\right] \\
&= \sum_{n=0}^{\infty} \mathbf{E}^*\left[e^{-rT}(s^1_{n,T} - K)^+ I_{\left\{s_{n,T}^{1\,-\frac{\alpha^*}{\sigma^1}} > b^n\tilde{a}(s^1_{n,T}-K)^+\right\}}\right]\cdot p_{n,T}.
\end{aligned} \tag{46}$$

Case a, $-\frac{\alpha^*}{\sigma^1} \leqslant 1$:
In this case, the equation

$$x^{-\frac{\alpha^*}{\sigma^1}} = b^n a(x-K)^+$$

has a unique root $x = c(a; n)$. Thus, for any fixed $n = 1, 2, ...$, the inequality

$$x^{-\frac{\alpha^*}{\sigma^1}} > b^n a(x-K)^+.$$

is equivalent to $x < c(a; n)$. This implies

$$I_{\left\{S_T^{1-\frac{\alpha^*}{\sigma^1}} > b^n a(S_T^1 - K)^+\right\}} = I_{\left\{S_T^1 < c(a;n)\right\}}.$$

We combine the last equation and (46) to obtain

$$\begin{aligned}
\mathbf{E}^*\left[e^{-rT}\phi f_T\right] &= \sum_{n=0}^{\infty} \mathbf{E}^*\left[e^{-rT}(S_T^1 - K)^+ I_{\left\{S_T^1 < c(a;n)\right\}}\right] \cdot p_{n,T} \\
&= \sum_{n=0}^{\infty} \mathbf{E}^*\Big[e^{-rT}\Big((s_{n,T}^1 - K)^+ - (s_{n,T}^1 - c(a;n))^+ \\
&\quad -(c(a;n) - K) I_{\left\{s_{n,T}^1 \geqslant c(a;n)\right\}}\Big)\Big] \cdot p_{n,T} \\
&= \sum_{n=0}^{\infty} p_{n,T}\Big[\mathbb{C}^{BS}(S_0^1\vartheta_{n,T}, K, T) - \mathbb{C}^{BS}(S_0^1\vartheta_{n,T}, c(a;n), T) \\
&\quad -(c(a;n) - K)e^{-rT} \cdot P^*\left[s_{n,T}^1 \geqslant c(a;n)\right]\Big] \\
&= \sum_{n=0}^{\infty} p_{n,T}\Big[\mathbb{C}^{BS}(S_0^1\vartheta_{n,T}, K, T) - \mathbb{C}^{BS}(S_0^1\vartheta_{n,T}, c(a;n), T) \\
&\quad -(c(a;n) - K)e^{-rT} \cdot \Phi[d_-(S_0^1\vartheta_{n,T}, c(a;n))]\Big] \\
&= \sum_{n=0}^{\infty} p_{n,T}\tilde{\mathbb{C}}_a(S_0^1, n, c(a;n), T)
\end{aligned}$$

where $\tilde{\mathbb{C}}_a(S_0^1, n, c, T)$ has been defined in equation (28). Again refering to equations (20) and (21), this proves item (a1) of theorem 2.

Concerning assertion (a2), we proceed as follows: Observe that for a borel-measurable function f,

$$E[f(\Pi_T, S_T^1)] = \sum_{n=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^n}{n!}\,\mathbf{E}\left[f(n, s_{n,T}^1)\right], \tag{47}$$

where $s_{n,T}^1$ is lognormally distributed under P, cf. (8):

$$\ln(s_{n,T}^1) \sim \mathcal{N}\left(\ln[S_0^1(1-\nu^1)^n] + [\mu^1 - \frac{1}{2}(\sigma^i)^2]T, \sigma^i\sqrt{T}\right).$$

Hence, the probability of the hedge beeing succesfull can be computed using the above results:

$$\begin{aligned}
1 - \varepsilon &= 1 - P\left[S_T^{1-\frac{\alpha^*}{\sigma^1}} \leq b^{\Pi_T}\tilde{a}e^{-rT}(S_T^1 - K)^+\right] \\
&= 1 - e^{-\lambda T}\sum_{n=0}^{\infty} P\left[s_{n,T}^1 > c(\tilde{a};n)\right]\frac{(\lambda T)^n}{n!} \\
&= 1 - e^{-\lambda T}\sum_{n=0}^{\infty} \Phi\left[d_-(S_0^1(1-\nu^1)^n, c(\tilde{a};n)) + \frac{\mu^1 - r}{\sigma^1}\sqrt{T}\right]\frac{(\lambda T)^n}{n!},
\end{aligned}$$

i.e. (a2) holds.

We already know that the quantile hedging strategy coincides with the perfect hedge of the modified claim $\tilde{\phi} f_T$. We calculate the value of this strategy:

$$\begin{aligned} X_t^\pi &= \mathbf{E}^*\left[\tilde{\phi} f_T e^{-r(T-t)} | \mathcal{F}_t\right] \\ &= \mathbf{E}^*\left[e^{-r(T-t)}(S_T^1 - K)^+ I_{\left\{ {S_T^1}^{-\frac{\alpha^*}{\sigma^1}} > b^{\Pi_T} \tilde{a}(S_T^1 - K)^+ \right\}} | \mathcal{F}_t\right]. \end{aligned}$$

As above, we can simplify the last equation to

$$\begin{aligned} X_t^\pi &= \mathbf{E}^*\left[e^{-r(T-t)}(S_T^1 - K)^+ I_{\left\{ {S_T^1}^{-\frac{\alpha^*}{\sigma^1}} > g \cdot b^{\Pi_{T-t}} b^{\Pi(S_t^1, S_t^2, t)} \tilde{a}(S_T^1 - K)^+ \right\}} | \mathcal{F}_t\right] \\ &= \sum_{n=0}^{\infty} \mathbf{E}^*\left[e^{-r(T-t)}(S_T^1 - K)^+ I_{\left\{ {S_T^1}^{-\frac{\alpha^*}{\sigma^1}} > b^n b^{\Pi(S_t^1, S_t^2, t)} \tilde{a}(S_T^1 - K)^+ \right\}} | \mathcal{F}_t\right] \frac{(\lambda^*(T-t))^n}{n!} e^{-\lambda^*(T-t)} \\ &= \sum_{n=0}^{\infty} p_{n,T-t} \tilde{\mathbb{C}}_a(S_t^1, n, c(\tilde{a}; n + \Pi(S_t^1, S_t^2, t)), T - t). \end{aligned}$$

This proves (a3) of theorem 2. The last assertion (a4) follows from theorem 1.

Case b, when $-\frac{\alpha^*}{\sigma^1} > 1$.

Now the equation

$$x^{-\frac{\alpha^*}{\sigma^1}} = b^n a(x - K)^+$$

has two roots $c^1(a; n) \leqslant c^2(a; n)$. Thus, the inequality

$$x^{-\frac{\alpha^*}{\sigma^1}} > b^n a(x - K)^+$$

is equivalent to either $x < c^1(a; n)$ or $x > c^2(a; n)$. This implies

$$I_{\left\{ {S_T^1}^{-\frac{\alpha^*}{\sigma^1}} > b^n a(S_T^1 - K)^+ \right\}} = I_{\left\{ S_T^1 < c^1(a;n) \right\}} + I_{\left\{ S_T^1 > c^2(a;n) \right\}}.$$

We can conclude as in case a to obtain

$$\begin{aligned} \mathbf{E}^*\left[e^{-rT} \tilde{\phi} f_T\right] &= \sum_{n=0}^{\infty} \mathbf{E}^*\left[e^{-rT}(s_{n,T}^1 - K)^+ (I_{\left\{ s_{n,T}^1 < c^1(a;n) \right\}} + I_{\left\{ s_{n,T}^1 > c^2(a;n) \right\}}\right] \cdot p_{n,T} \\ &= \sum_{n=0}^{\infty} \mathbf{E}^*\left[e^{-rT}\left((s_{n,T}^1 - K)^+ - (s_{n,T}^1 - c^1(a; n))^+ + (s_{n,T}^1 - c^2(a; n))^+ \right.\right. \\ &\quad \left.\left. -(c^1(a; n) - K) I_{\left\{ s_{n,T}^1 \geqslant c^1(a;n) \right\}} + (c^2(a; n) - K) I_{\left\{ s_{n,T}^1 \geqslant c^2(a;n) \right\}}\right)\right] \cdot p_{n,T} \\ &= \sum_{n=0}^{\infty} \left[\mathbb{C}^{BS}(S_0^1 \vartheta_{n,T}, K, T) - \mathbb{C}^{BS}(S_0^1 \vartheta_{n,T}, c^1(a; n), T) \right. \\ &\quad + \mathbb{C}^{BS}(S_0^1 \vartheta_{n,T}, c^2(a; n), T) - (c^1(a; n) - K) e^{-rT} \cdot \Phi[d_-(S_0^1 \vartheta_{n,T}, c^1(a; n))] \\ &\quad \left. + (c^2(a; n) - K) e^{-rT} \cdot \Phi[d_-(S_0^1 \vartheta_{n,T}, c(a; n))]\right] p_{n,T} \\ &= \sum_{n=0}^{\infty} \left[\tilde{\mathbb{C}}_b(S_0^1, n, c^1(a; n), c^2(a; n), T)\right] p_{n,T}. \end{aligned}$$

This proves item (b1). The assertions (b2) - (b4) can be obtained from (b1) as in case a. □

Proof of Theorem 3:
We know from Equation (23), (26) and (45) that the modified claim is given by

$$\begin{aligned}\tilde{f}_T &= f_T - a_1 Z_T^{\frac{1}{p-1}} \wedge f_T \\ &= (S_T^1 - K)^+ - ab^{\frac{\Pi_T}{p-1}}(S_T^1)^{-q} \wedge (S_T^1 - K)^+\end{aligned}$$

where the constant a has to be determined from the capital constraint. By definition, $L(a;n)$ denotes the unique root of

$$ab^{\frac{n}{p-1}}\, L^{-q} = (L-K)^+.$$

Thus, we have

$$\begin{aligned}\tilde{f}_T &= \Big[(S_T^1 - K)^+ - L(a;\Pi_T)^q (L(a;\Pi_T) - K)(S_T^1)^{-q}\Big] I_{\{S_T^1 \geq L(a;\Pi_T)\}} &(48)\\ &=: g(S_T^1, \Pi_T) &(49)\end{aligned}$$

We can now compute the fair price of $\tilde{f}_T$:

$$\begin{aligned}E^*[e^{-rT}\tilde{f}_T] &= \sum_{n=0}^{\infty} p_{n,T}\,(\Psi_n^1 - \Psi_n^2) &(50)\\ \Psi_n^1 &:= E^*\Big[e^{-rT}(s_{n,T}^1 - K)^+ I_{\{s_{n,T}^1 \geq L(a;n)\}}\Big] \\ \Psi_n^2 &:= E^*\Big[e^{-rT} L(a;n)^q (L(a;n) - K)(s_{n,T}^1)^{-q} I_{\{s_{n,T}^1 \geq L(a;n)\}}\Big]\end{aligned}$$

The quantities Ψ_n^i are easily computed:

$$\begin{aligned}\Psi_n^1 &= \int_{-\infty}^{\infty} e^{-rT}(S_0^1\vartheta_{n,T} e^{\sigma^1\sqrt{T}x + (r-\frac{1}{2}(\sigma^1)^2)T} - K) I_{\{S_0^1\vartheta_{n,T}e^{\sigma^1\sqrt{T}x+(r-\frac{1}{2}(\sigma^1)^2)T} \geq L(a;n)\}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= \int_{x_n}^{\infty} S_0^1 \vartheta_{n,T} e^{-\frac{(x-\sigma^1\sqrt{T})^2}{2}} \frac{dx}{\sqrt{2\pi}} - \int_{x_n}^{\infty} e^{-rT} K e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= \vartheta_{n,T} S_0^1 \phi(d_+(\vartheta_{n,T}S_0^1, L(a;n))) - e^{-rT} K \phi(d_-(\vartheta_{n,T}S_0^1, L(a;n))) \\ &= \mathbb{C}^{BS}(S_0^1\vartheta_{n,T}, L(a,n), T) + e^{-rT}(L-K)\phi(d_-(\vartheta_{n,T}S_0^1, L(a;n)))\end{aligned}$$

where we have set

$$x_n = \frac{\ln L(a;n) - \ln(\vartheta_{n,T}S_0^1) - (r - \frac{1}{2}(\sigma^1)^2)T}{\sigma^1\sqrt{T}}.$$

To simplify the notation regarding the computation of Ψ^2, we denote $y_n = L(a;n)^q(L(a;n) - K)$. Now we compute

$$\begin{aligned}\Psi_n^2 &= y_n \int_{x_n}^{\infty} e^{-rT}(S_0^1\vartheta_{n,T})^{-q} e^{-q(\sigma^1\sqrt{T}x + (r-\frac{1}{2}(\sigma^1)^2)T)} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= y_n e^{-rT}\,(S_0^1\vartheta_{n,T})^{-q} e^{\frac{1}{2}(\sigma^1)^2 Tq(1-q) - rTq} \int_{x_n}^{\infty} e^{-\frac{(x+q\sigma^1\sqrt{T})^2}{2}} \frac{dx}{\sqrt{2\pi}} \\ &= y_n\,(S_0^1\vartheta_{n,T})^{-q}\kappa_T \phi(d_-(\vartheta_{n,T}S_0^1, L(a;n)) - q\sigma^1\sqrt{T}).\end{aligned}$$

where we have defined κ_T in equation (40). Substituting Ψ^i_n in equation (50) yields

$$E^*[e^{-rT}\tilde{f}_T] = \sum_{n=0}^{\infty} p_{n,T}\,\tilde{\mathbb{C}}_p(S^1_0, n, L(a;n), T)$$

where $\tilde{\mathbb{C}}_p$ was defined in (39). Substituting $\Pi_T = \Pi(S^1_T, S^2_T, T)$ in equation (49), yields item (1) of theorem 3.

Now we compute the value of the efficient hedging strategy at time t: From equation (49) we obtain $\tilde{f}_T = g(S^1_T, \Pi_T) = g(S^1_T, \Pi_T - \Pi_t + \Pi_t)$ which yields

$$\begin{aligned} X^\pi_t &= E^*[e^{-r(T-t)}\tilde{f}_T \mid \mathcal{F}_t] \\ &= \sum_{n=0}^{\infty} p_{n,T-t} E^*[e^{-r(T-t)} g(S^1_T, n+\Pi_t) \mid \mathcal{F}_t] \\ &= \sum_{n=0}^{\infty} p_{n,T-t}\tilde{\mathbb{C}}_p(S^1_t, n, L(\tilde{a}; n+\Pi_t), T-t) \end{aligned}$$

where we can replace Π_t by $\Pi(S^1_t, S^2_t, t)$. This proves item (2). The last assertion follows from theorem 1. □

Proof of Theorem 1.
Let some basket option $g_T = g(S^1_T, S^2_T)$ with associated perfect hedging strategy π be given. From (29) and (30) we obtain $(W_t, \Pi_t) = \Gamma_t(S^1_t, S^2_t)$ and $(S^1_t, S^2_t) = \Gamma^{-1}_t(W_t, \Pi_t)$. The markov property of (W, Π) with respect to the filtration $(\mathcal{F}_t)$ implies

$$\begin{aligned} X^\pi_t &= E^*\left[e^{-r(T-t)}\, g(S^1_T, S^2_T)\,|\mathcal{F}_t\right] \\ &= E^*\left[e^{-r(T-t)}\, g\circ\Gamma^{-1}_t(W_t, \Pi_t)\,|\mathcal{F}_t\right] \\ &= E^*\left[e^{-r(T-t)}\, g\circ\Gamma^{-1}_t(W_t, \Pi_t)\,|(W_t, \Pi_t)\right] \\ &=: \hat{\mathbb{C}}_t(W_t, \Pi_t) \\ &= \hat{\mathbb{C}}_t\circ\Gamma_t(S^1_t, S^2_t) \\ &=: \mathbb{C}(S^1_t, S^2_t, t) \end{aligned}$$

i.e. the value of the perfekt hedging strategy is a function $\mathbb{C}$ of the current values of the assets and time. We study the representation

$$\frac{X^\pi_t}{B_t} = \frac{X^\pi_0}{B_0} + \int_0^t \sum_{i=1}^{2} \gamma^i_u d(\frac{S^i_u}{B_u}). \tag{51}$$

The dynamics of the discounted price processes $Y^i_t = \frac{S^i_t}{B_t}, i = 1, 2$ are given by

$$dY^i_t = Y^i_{t-}(\sigma^i dW^*_t - \nu^i d(\Pi_t - \lambda^* t)), i = 1, 2.$$

Substituting X^π by $\mathbb{C}$ and dY^1 by means of the last equation in (51) yields

$$\frac{\mathbb{C}(S^1_t, S^2_t, t)}{B_t} = \frac{\mathbb{C}(S^1_0, 0)}{B_0} + \int_0^t \frac{\gamma^1\sigma^1 S^1_{u-} + \gamma^2\sigma^2 S^2_{u-}}{B_u} dW^*_u - \int_0^t \frac{\gamma^1\nu^1 S^1_{u-} + \gamma^2\nu^2 S^2_{u-}}{B_u} d(\Pi_u - \lambda^* u). \tag{52}$$

We are going to compare this representation with the representation we obtain by applying the Ito-formula to $\mathbb{C}(S^1_t, S^2_t, t)$:

$$\mathbb{C}(S^1_t, S^2_t, t) = \mathbb{C}(S^1_0, S^2_0, 0) + \int_0^t \frac{\partial}{\partial x}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)dS^1_u + \int_0^t \frac{\partial}{\partial y}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)dS^2_u$$
$$+ \int_0^t \frac{\partial}{\partial t}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)du + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)d\left\langle S^{1^c}, S^{1^c}\right\rangle_u$$
$$+ \frac{1}{2}\int_0^t \frac{\partial^2}{\partial y^2}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)d\left\langle S^{2^c}, S^{2^c}\right\rangle_u + \int_0^t \frac{\partial^2}{\partial x \partial y}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)d\left\langle S^{1^c}, S^{2^c}\right\rangle_u$$
$$+ \sum_{0<u\leqslant t}\Big[\mathbb{C}(S^1_u, S^2_u, u) - \mathbb{C}(S^1_{u-}, S^2_{u-}, u) - \frac{\partial}{\partial x}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)\Delta S^1_u - \frac{\partial}{\partial y}\mathbb{C}(S^1_{u-}, S^2_{u-}, u)\Delta S^2_u\Big] \tag{53}$$

where S^{i^c} denotes the continuous part of S^i, see also (54). We can substitute

$$\Delta S^i_u = -\nu^i S^i_{u-}\Delta\Pi_u,$$

$$\mathbb{C}(S^1_u, S^2_u, u) - \mathbb{C}(S^1_{u-}, S^2_{u-}, u) = \big[\mathbb{C}(S^1_{u-}(1-\nu^1), S^2_{u-}(1-\nu^2), u) - \mathbb{C}(S^1_{u-}, S^2_{u-}, u)\big]\cdot\Delta\Pi_u.$$

and

$$d\left\langle S^{i^c}, S^{j^c}\right\rangle_u = (\sigma^i\sigma^j)(S^i_{u-}S^j_{u-})du, i, j = 1, 2 \tag{54}$$

to obtain

$$d\mathbb{C}(S^1_t, S^2_t, t) = \frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)dS^1_t + \frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)dS^2_t + \frac{\partial}{\partial t}\mathbb{C}(S^1_{t-}, t)dt$$
$$+ \frac{1}{2}(\sigma^1 S^1_{t-})^2\frac{\partial^2}{\partial x^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)dt + \frac{1}{2}(\sigma^2 S^2_{t-})^2\frac{\partial^2}{\partial y^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)dt$$
$$+ (\sigma^1\sigma^2 S^1_{t-}S^2_{t-})\frac{\partial^2}{\partial x\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)dt + \big[\mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t) - \mathbb{C}(S^1_{t-}, S^2_{t-}, t)\big]\,d\Pi_t$$
$$- \frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)(-\nu^1 S^1_{t-})d\Pi_t - \frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)(-\nu^2 S^2_{t-})d\Pi_t.$$

The last equation yields

$$d\frac{\mathbb{C}(S^1_t, S^2_t, t)}{B_t} = (\frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\frac{S^1_{t-}}{B_t}\sigma^1 + \frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\frac{S^2_{t-}}{B_t}\sigma^2)dW^*_t$$
$$+ \big[\mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t) - \mathbb{C}(S^1_{t-}, S^2_{t-}, t)\big]\,e^{-rt}d(\Pi_t - \lambda^* t)$$
$$+ e^{-rt}\Big(\,\big[\mathbb{C}(S^1_{t-}(1-\nu^1), S^2_{t-}(1-\nu^2), t) - \mathbb{C}(S^1_{t-}, S^2_{t-}, t)\big]\,\lambda^* + rS^1_{t-}\frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)$$
$$+ rS^2_{t-}\frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \frac{\partial}{\partial t}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + \frac{1}{2}(\sigma^1 S^1_{t-})^2\frac{\partial^2}{\partial x^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)$$
$$+ \frac{1}{2}(\sigma^2 S^2_{t-})^2\frac{\partial^2}{\partial y^2}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) + (\sigma^1\sigma^2 S^1_{t-}S^2_{t-})\frac{\partial^2}{\partial x\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t) - r\mathbb{C}(S^1_{t-}, S^2_{t-}, t)$$
$$+\frac{\partial}{\partial x}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\nu^1\lambda^* S^1_{t-} + \frac{\partial}{\partial y}\mathbb{C}(S^1_{t-}, S^2_{t-}, t)\nu^2\lambda^* S^2_{t-}\Big)\,dt.$$

The comparison of this representation with equation (52) completes the proof. $\square$