Markovian short rates in a forward rate model with a general class of Lévy processes*

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Abstract: Short rates of interest are considered within in the term structure model of Eberlein-Raible [6] driven by a Lévy process. It is shown that they are Markovian if and only if the volatility function factorizes. This extends results of Caverhill [5] for the Wiener process and of Eberlein, Raible [6] for Lévy processes with a restricting property to the most general class of Lévy processes being possible within this model. As new examples compound Poisson processes and bilateral gamma processes are included, in particular variance gamma processes in the sense of Madan [14], Madan, Senata [15].

Key words: term structure of interest rates, Markovian spot rates, Lévy processes, Eberlein-Raible-model, bilateral gamma processes, variance gamma processes

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1 Introduction

In Eberlein, Raible [6] a term structure model was studied that can be described as follows. Suppose $T^* > 0$ is fixed and for any T with $0 < T \leq T^*$ there is a zero coupon bond on the market with maturity time T and price $P(t,T)$ at time $t \leq T$. Assume the bond prices $P(t,T)$ satisfy

$$P(t,T) = P(0,T) \cdot \beta(t) \cdot \frac{\exp\Big[\int\limits_0^t \sigma(s,T)dL_s\Big]}{E\Big[\exp\Big(\int\limits_0^t \sigma(s,T)dL_s\Big)\Big]} \quad , \quad 0 \leq t \leq T \leq T^* \tag{1}$$

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and

$$P(T,T) = 1, \qquad 0 \le T \le T^*, \tag{2}$$

where $\beta(\cdot)$ is a numeraire, $\sigma(t,T), \quad 0 \le t \le T \le T^*$, is a deterministic function being positive and bounded.
$L := (L_t, t \in [0, T^*])$ is assumed to be a Lévy process with the Lévy-measure F satisfying

$$\int\limits_{|x|>1} \exp(ux)F(dx) < \infty \tag{3}$$

for all u from an open interval $I = (u_0, u_1)$ including zero. (This is to ensure the finiteness of the expectation in (1) and all moments of L_t.) For calculating prices of derivatives within term structure models it is of interest under which conditions the short rate process $r(\cdot)$ is Markovian. This and similar questions were studied in the framework of the Heath-Jarrow-Morton model by several authors, for example Caverhill [4], Bhar, Chiarella [2], Inui, Kijima [11], Ritchken, Sankarasubramanian [16]. Eberlein, Raible proved in [6] that the short rate of interests $r(t), t \in [0, T^*]$, in their model (for definitions see Chapter 2) is a Markov process if and only if the volatility function $\sigma_2(t,T) :== \frac{\partial}{\partial T}\sigma(t,T)$ factorizes:

$$\sigma_2(t,T) = \tau(t)\zeta(T) \quad , \quad 0 \le t \le T \le T^*,$$

for some functions $\tau(\cdot)$ and $\zeta(\cdot)$ on $[0, T^*]$.
On proving this property they impose the following additional assumption denoted here by (ER):

$$|E\exp(iuL_1)| \le C \cdot \exp(-\gamma |u|^\eta) \quad , \quad u \in R_1 \tag{ER}$$

for some positive constants C, γ, η.

This condition is satisfied for example for Wiener processes, normal inverse Gaussian Lévy processes, stable processes and hyperbolic Lévy processes. On the other hand, it does not hold for compound Poisson processes, gamma processes and finite sums of independent exemplars of them.

In this paper we are going to show that the mentioned characterization of Markovian short rates holds for each Lévy process with property (3), without assuming (ER).

As one of the examples of Lévy processes satisfying (3) but not (ER) the bilateral gamma process defined as the difference of independent gamma processes is considered in some detail. All of its marginal distributions are bilateral gamma and their expectations, variations, skewness and excess can be explicitely calculated. These distributions are semiheavy tailed and leptokurtosic, unimodal and self-decomposable. Based on these Lévy processes the short rate and forward rate stochastic differential equations turn out to have relatively simple coefficients. As special cases appear the variance gamma processes considered for example in Madan [14], Madan, Senate [15].

2 The Eberlein-Raible model

In this paper we often use Lévy processes L and their path behaviour. To fix the notation and for reminding we will summarize the definition and some properties of such processes. For details and proofs the reader is refered to Sato [19], for example.

Let t be a fixed positive number. A real-valued stochastic process $L = (L_s, 0 \leq s \leq t)$ is said to be a Lévy process (on $[0,t]$), if it has stationary independent increments, the trajectories are right continuous and have limits from the left as well as it holds $L_0 = 0$. For every Lévy process L there is a uniquely determined triplet (γ, σ^2, F) with $\gamma \in R_1, \sigma^2 \geq 0$ and F being a σ-finite measure on $R_1 \setminus \{0\}$ with

$$\int_{|x|>\varepsilon} \frac{|x|}{1+x^2} F(dx) < \infty$$

for every $\varepsilon > 0$ such that

$$E\exp(iuL_s) = \exp[s\Big(iu\gamma - \tfrac{\sigma^2}{2}u^2 + \int_{R_1\setminus\{0\}} (\mathrm{e}^{iux} - 1 - \frac{iux}{1+x^2})F(dy)\Big)], \quad u \in R_1, s \in [0,t]$$

(γ, σ^2, F) is called the *generating triplet of* L. It can be shown that for every $\varepsilon > 0$ the process $L^\varepsilon := (L^\epsilon_s, s \in [0,t])$ defined by

$$L^\varepsilon_s := \sum_{s\leq t} \triangle L_s \cdot 1\!\!1_{\{|\triangle L_s|>\varepsilon\}} \quad , \quad s \in [0,t]$$

with $\triangle L_s := L_s - L_{s-o}$ forms a compound Poisson process with jump intensity $\lambda_\varepsilon := F(R_1 \setminus [-\varepsilon, \varepsilon])$ and jump size distribution $\lambda_\varepsilon^{-1} F(\cdot \cap (R_1 \setminus [-\varepsilon, \varepsilon]))$ if $\lambda_\varepsilon > 0$.

The limit

$$L^0_s := \lim_{\varepsilon\downarrow 0} \Big(L^\varepsilon_s - \int_{R_1\setminus[-\varepsilon,\varepsilon]} \frac{x}{1+x^2} dF(x)\Big), \quad s \in [0,t]$$

exists uniformly in s and forms a Lévy process with generating triplet $(\gamma, \sigma^2, 0)$. The process $W := L - L^0$ is a Wiener process with diffusion coefficient σ^2 and drift γ, i.e. a continuous Lévy process with $W_1 \sim N(\gamma, \sigma^2)$. W is called degenerated if $\sigma^2 = 0$.
Using this notation we obtain

Lemma 2.1.: *(Lévy-Itô decomposition, see Sato [18], Chapter 4)*
Every Lévy process $(L_s, s \in [0,t])$ *has the decomposition*

$$L_s = W_s + L_s^0 \quad , \quad s \in [0,t],$$

where W *and* L^0 *just defined are independent, and for every* $\varepsilon > 0$*, the processes* $W, L^0 - L^\varepsilon$*, and* L^ε *are mutually independent.*

Here W is called the Gaussian and L^0 the purely discontinuous part of L.

As the next step we summarize some definitions and results without proofs from Eberlein, Raible [6] that will be used in the sequel.

Assume that up to some finite time horizon T^* a zero coupon bond is available on the market for every time T of maturity . The dynamics of the price process $P(t,T), t \in [0,T]$, is supposed to be described by equation (1) together with the boundary condition (2). The integrals in (1) are defined as follows. For every continuously differentiable function f on $[0,t]$ we put

$$\int_0^t f(s) dL_s := f(t)L_t - \int_0^t f'(s) L_s ds.$$

The following assumptions are supposed to be valid throughout this paper.

Assumptions 2.1:

(i) The initial bond prices $P(0,T), T \in [0,T^*]$, are given deterministic functions being positive and twice differentiable with respect to T.

(ii) $L = (L_t, t \in [0,T^*])$ is a Lévy process with the cumulant generating function $\vartheta(u) := \log E \exp(uL_1)$ defined and being finite on an open interval $I = (u_0, u_1)$ including zero.

(iii) The Lévy-measure F of L satisfies condition (3).

(iv) $\sigma(s,t)$ is defined on $\triangle := \{(s,t) : 0 \le s \le t \le T^*\}$, continuously differentiable in s and twice continuously differentiable in t. Moreover, it holds $\sigma(s,t) > 0$ for $s < t, \sigma(t,t) \equiv 0, t \in [0,T^*]$, as well as $\sup\limits_{0 \le s \le T \le T^*} \sigma(s,t) < u_1$.

Note that if f is continuously differentiable function on $[0,t]$ having values in the interval I only, then it follows

$$E \exp\left[\int_0^t f(s) dL_s\right] = \exp\left[\int_0^t \vartheta(f(s)) ds\right]$$

.
The discounted processes

$$\tilde{P}(t,T) = \beta^{-1}(t)P(t,T) \quad , \quad 0 \le t \le T, \quad T \in [0,T^*],$$

are martingales with respect to $(\mathcal{A}_s, s \in [0,T])$, where $\mathcal{A}_s$ is the σ-Algebra generated by $(L_{s'}, s' \le s)$.

The *forward rates* $f(t,T)$ with maturity T and the *short rates* $r(t)$ are defined as usual by

$$f(t,T) := -\frac{\partial}{\partial T} \log P(t,T), \quad 0 \le t \le T \le T^*$$

and

$$r(t) := f(t,t), \quad 0 \le t \le T^*$$

respectively.

For every T with $T \le T^*$ the forward rate process $(f(t,T), t \in [0,T])$ satisfies the equation

$$f(t,T) = f(0,T) + \int_0^t \vartheta'(\sigma(s,T))\sigma_2(s,T)ds - \int_0^t \sigma_2(s,T)dL_s, 0 \le t \le T, \tag{4}$$

Hereby the functions $\vartheta'(u)$ and $\sigma_2(s,T)$ are defined as

$$\vartheta'(u) := \frac{d}{du}\vartheta(u), \quad u \in I, \quad \sigma_2(s,T) := \frac{\partial}{\partial T}\sigma(s,T), \quad 0 \le s \le T \le T^*$$

respectively. $\sigma_2(s,T)$ is called the *volatility function.*
From (4) one gets

$$r(t) = f(0,t) + \int_0^t \vartheta'(\sigma(s,t))\sigma_2(s,t)ds - \int_0^t \sigma_2(s,t)dL_s, \quad t \in [0,T^*]. \tag{5}$$

If L is a standard Wiener process we have $\vartheta'(u) = u$ and the model satisfies the classical Heath-Jarrow-Morton condition on the drift coefficient of the forward rate processes, see Heath, Jarrow, Morton [10].

For the numeraire $\beta(t)$ one gets necessarily

$$\beta(t) = \exp\Big(\int_0^t r(s)ds\Big) \quad , \quad t \in [0,T^*]. \tag{6}$$

(For the proofs see Eberlein, Raible [6].)

Finally, using the martingale property of $\tilde{P}(\cdot, T)$, the equality (6) and $P(T,T) = 1$ one can conclude

$$\tilde{P}(t,T) = E(\exp(-\int_0^T r_s ds)|\mathcal{A}_t) \quad , \quad t \in [0,T],$$

that means

$$P(t,T) = E\Big(\exp(-\int_t^T r_s ds)|\mathcal{A}_t\Big), \quad t \in [0,T]. \tag{7}$$

This expression and analogue formulas for the prices of contingent claims sometimes become much more simple and can be evaluated explicitely, if $(r_t, \mathcal{A}_t, t \in [0, T^*])$ forms a Markov process.

3 Markovian short rates

Now let us turn to the question, under which conditions on $\sigma(t,T)$ the short rate process $(r(t), t \in [0, T^*])$ given by (5) is a Markov process.

We will prove the following theorem, that generalizes a result of Eberlein, Raible [6] who derived if under the additional assumption (ER).

Theorem 3.1: *Suppose $(L_t, t \in [0, T^*])$ is a non identical zero Lévy process such that the Assumptions 2.1 hold.*
Then the short rate process $(r(t), t \in [0, T^*])$ is Markovian if and only if the volatility function $\sigma_2(t,T)$ factorizes as follows:

$$\sigma_2(t,T) = \tau(t)\zeta(T) \quad , \quad 0 \leq t \leq T \leq T^* \tag{8}$$

for some continuously differentiable functions τ and ζ from $[0, T^*]$ into $(0, \infty)$.

Proof: The arguments of the proof follow the line of Eberlein, Raible [6], but we will use other properties of the underlying Lévy process than in [6]. The essential change is made in Lemma 3.1. below.

Firstly let us note that because the function $\sigma(s,t)$ is deterministic $r(\cdot)$ is Markovian if and only if

$\Big(\int\limits_0^t \sigma_2(s,t)dL_s, t \in [0,T^*]\Big)$ it is.

Lemma 3.1: *Assume $L = (L_s, s \in [0,t])$ is a nonidentical zero Lévy process and f_1, f_2 are continuously differentiable, nonconstant functions from $[0,t]$ into R_1. If f_1 and f_2 are affine independent, then the distribution of*

$$X^L := (X_1^L, X_2^L) = \Big(\int\limits_0^t f_1(s)dL_s, \int\limits_0^t f_2(s)dL_s\Big)$$

has a nonzero absolutely continuous part with respect to the Lebesgue measure λ_2 on R_2.

Note: Assume f_1, f_2 are affine independent. Under the condition (ER), Eberlein, Raible [6] show that the distribution of the random vector X^L defined in Lemma 3.1. has a density and use this fact to derive the desired result (8). We do not go this way. We will prove the result by making use of the inner structure of Lévy processes with jumps. In the general case considered here the distribution of X^L has not necessarily a density, but a nontrivial absolutely continuous part, that facilitates the further steps of the proof of Theorem 3.1.

Proof: For every nondegenerated Wiener process $L = (W_s, s \in [0,t])$ the lemma is an easy consequence of properties of the Gaussian distributions.

Indeed, in this case X^L turns out to be Gaussian with

$$\mathrm{Var}(X_2^L - cX_1^L) = \sigma^2 \int\limits_0^t \Big(f_2(s) - cf_1(s)\Big)^2 ds$$

which is positive for any real c by assumption. Thus $(\mathrm{Cov}\ (X_1^L, X_2^L))^2 < \ \mathrm{Var}\ (X_1^L)\ \mathrm{Var}\ (X_2^L)$, this means that, X_1^L and X_2^L have a common density.

In particular for every Lévy process having a nonzero Gaussian part the Lemma 3.1 holds. This can be easily derived from Lemma 2.1 on noting that both parts of the decomposition $L = W + L^0$ are independent. Now suppose $L = (L_t, 0 \le t \le T^*)$ is a nonidentical zero Lévy process whose Gaussian part is zero. We divide the remaining proof into three steps and show firstly, that the assertion is valid for Poisson processes, secondly that it holds for compound Poisson processes and thirdly, that it is true for general Lévy processes having no nonzero Gaussian part.

1^{st} step: Let L be a Poisson process with intensity $\lambda > 0$ and jump times $\tau_1, \tau_2 \cdots$. Define a map $\phi = \phi(u, v)$ from $[0, t]^2$ into R^2 by

$$\phi(u, v) := (f_1(u) + f_1(v), f_2(u) + f_2(v)), \quad (u, v) \in [0, t]^2.$$

By assumption on f_1 and f_2 the mapping ϕ is continuously differentiable with a nonsingular Jacobian

$$D_{u,v} := \det \begin{pmatrix} f'(u) & f'(v) \\ g'(u) & g'(v) \end{pmatrix} \neq 0$$

at least in an open neighbourhood U of some point (u_0, v_0) from $(0, t)^2$.
We can assume that ϕ maps U one-onto-one to an open neighbourhood V of $\phi(u_0, v_0)$ and that the inverse mapping ϕ^{-1} on V is continuously differentiable, see e.g. Förster [17]. Because of $\phi(u, v) = \phi(v, u)$ for all $u, v \in [0, t]$ it is no restriction to suppose that U is symmetric: $(u, v) \in U$ if and only if $(v, u) \in U$. In particular $U \cap \triangle_t$ has a positive Lebesgue measure, where $\triangle_t := \{(u, v) \in R_2 | 0 \leq u \leq v \leq t\}$.
Using L is Poisson, it follows for $C := \{\tau_2 \leq t < \tau_3\}$ that $P(C) > 0$. Moreover, it is well-known that under the condition C the vector (τ_1, τ_2) has a strictly positive density $h(u, v)$ on $\triangle_t$, actually it is uniformly distributed under the condition C. Consequently, for $B := \{(\tau_1, \tau_2) \in U\}$ we get $P(B|C) > 0$. This implies $P(B \cap C) > 0$ and for every Borel set A of R_2 we may infer

$$\begin{aligned} P(\phi(\tau_1, \tau_2) \in A | B \cap C) \cdot P(B|C) &= \\ P(\{\phi(\tau_1, \tau_2) \in A\} \cap B | C) &= \\ \int_{\phi^{-1}(A) \cap U} h(u, v) du dv = \int_{\phi^{-1}(A)} 1\!\!1_U(u, v) h(u, v) du dv &= \\ \int_A h(\phi^{-1}(x, y)) 1\!\!1_V(x, y) D^{-1}_{\phi^{-1}(x,y)} dx dy. & \end{aligned}$$

Thus conditioned on $B \cap C$ the random vector $\phi(\tau_1, \tau_2)$ has a density.

Now observe that $\phi(\tau_1, \tau_2) = X^L$ on $B \cap C$. Therefore the measure $Q(\cdot)$ on R_2 defined by

$$Q(\cdot) := P(X^L \in \cdot | B \cap C) \cdot P(B \cap C),$$

forms a nonzero absolutely continuous part of the distribution of X^L.

2^{nd} step: Assume L is a compound Poisson process with $L_t = \sum_{k=1}^{N_t} Z_k$, where $N = (N_t, t \geq 0)$ is a Poisson process with intensity $\lambda > 0$ and jump times $\tau_1, \tau_2, \cdots$, and $(Z_k, k \geq 1)$ is a sequence of mutually independent and independent of N identically distributed random variables with distribution $\nu(dz)$. It is no restriction to suppose $\nu(\{0\}) = 0$.

Denote by $F_{z_1,z_2}(u,v)$ a version of the distribution function of X^L under $Z_1 = z_1, Z_2 = z_2$ and $C := \{\tau_2 \le t < \tau_3\}$. Because of the independence of (Z_1, Z_2) and N we get

$$F_{z_1,z_2}(u,v) = P(X_1^L \le u, X_2^L \le v | Z_1 = z_1, Z_2 = z_2, \{\tau_2 \le t < \tau_3\})$$

$$= P(z_1 f_1(\tau_1) + z_2 f_1(\tau_2) \le u, z_1 f_2(\tau_1) + z_2 f_2(\tau_2) \le v | \tau_2 \le t < \tau_3) \qquad \nu \otimes \nu - a.s.$$

It follows by an analogue procedure as in the first step, that $F_{z_1,z_2}(u,v)$ has a nontrivial absolutely continuous part $\nu \otimes \nu - a.s.$. (We have used that z_1 and z_2 are unequal zero ν -a.s.) Because $P(X_1^L \le u, X_2^L \le v|C)$ is the mixing of $F_{z_1,z_2}(u,v)$ with respect to the mixing measure $\nu \otimes \nu(dz_1, dz_2)$, the same holds for the distribution of X^L under the condition $C = \{\tau_2 \le t < \tau_3\}$. Consequently, the lemma is also valid for compound Poisson Processes.

3^{rd} part: Now suppose that L is an arbitrary Lévy process having jumps. Then for some $\varepsilon > 0$ the process L is the sum of the nonzero compound Poisson process $L^{(0)} := L^\varepsilon L^\varepsilon$ and an independent of $L^{(0)}$ Lévy process $L^{(1)} := L - L^\varepsilon$ (we use the notation of Chapter 2).

Then by the second step of this proof the assertion of the lemma holds for $L^{(0)}$ and consequently also for L, because $\left(\int_0^t f_1(s) dL_s^{(i)}, \int_0^t f_2(s) dL_s^{(i)}\right), i = 0, 1$ are mutually independent. Thus the lemma is proved.

Now let us continue the proof of Theorem 3.1.

Lemma 3.2 *Assume $L = (L_s, 0 \le s \le t)$ is a non identically zero Lévy process, f_1, f_2 are two continuously differentiable functions on $[0,t]$ and there exists a Borel function G with*

$$\int_0^t f_2(s) dL_s = G(\int_0^t f_1(s) dL_s) \qquad P - a.s. \qquad (9)$$

Then it holds

$$f_2(\cdot) = c f_1(\cdot) \text{ on } [0,t]$$

for some $c \in R_1$.

Proof: With the notation of Lemma 3.1. we start with any nongenerated Wiener process $L = (W(s), 0 \leq s \leq t)$. Formula (9) implies in this case

$$E(X_2^L | X_1^L) = G(X_1^L)$$

and because (X_1^L, X_2^L) is Gaussian, the function G is affine, that means

$$G(x) = cx + d \qquad P^{X_1^L} - a.e.$$

for some real c and with $d := E(X_2^L - X_1^L)$.

Thus it follows

$$\int_0^t (cf_1(s) - f_2(s))^2 ds = \mathrm{Var}(cX_1^L - X_2^L) = 0,$$

and, consequently, $f_2 = cf_1$.

Now let $L = (L_s, 0 \leq s \leq t)$ be a Lévy process having jumps. From assumption (9) it can be easily obtained that the distribution of $X^L = (X_1^L, X_2^L)$ has no nontrivial absolutely continuous part with respect to λ_2. Using Lemma 3.1 it follows that f_1 and f_2 must be affine dependent, that means

$$f_2(s) = c_1 f_1(s) + c_0 \quad , \quad s \in [0, t] \tag{10}$$

for some real numbers c_0, and c_1.

To finish the proof of Lemma 3.2 it suffices to show the following

Lemma 3.3 Under the assumptions of Lemma 3.2 the constant c_0 in equation (10) can be chosen equal zero.

Proof: We have already shown that (10) holds. If f_1 were a constant then f_2 would be a constant too. Thus we suppose f_1 is not constant.

Assume $c_0 \neq 0$. Then for the Borel function H with $H(x) = c_0^{-1} \cdot (G(x) - C_1 x)$ it holds

$$H(\int_0^t f_1(s) dL_s) = L_t. \qquad P - a.s. \tag{11}$$

From (11) it follows that f_1 is a constant, which contradicts the assumption. Thus c_0 has to be zero.
The proof of the just made conclusion about f_1 can be found in Küchler [13].

Now the proof of Lemma 3.2 is complete. We return to the proof of Theorem 3.1.

Assume that $r(.)$ is Markovian with respect to $(\mathcal{A}_t)$. Then it follows from (5) and because $\sigma(s,t)$ is deterministic that

$$X_t := \int_0^t \sigma_2(s,t)dL_s, \quad t \in [0,T^*]$$

is also Markovian with respect to $(\mathcal{A}_t)$. This implies

$$E(X_u|\mathcal{A}_t) = E(X_u|X_t), P - a.s., \quad 0 \le t < u \le T^*. \tag{12}$$

From, the definition of X_t and the independence of the increments of L it follows for all fixed t and u with $0 \le t < u \le T^*$ on the one hand

$$E(X_u|\mathcal{A}_t) = E\Big(\int_0^t \sigma_2(s,u)dL_s|\mathcal{A}_t\Big) + E\Big(\int_t^u \sigma_2(s,u)dL_s|\mathcal{A}_t\Big)$$

$$= \int_0^t \sigma_2(s,u)dL_s + E\Big(\int_t^u \sigma_2(s,u)dL_s\Big)$$

and on the other hand

$$E(X_u|X_t) = E\Big(\int_0^t \sigma_2(s,u)dL_s|X_t\Big) + E\Big(\int_t^u \sigma_2(s,u)dL_s\Big).$$

Inserting these equations into (12) we get for any choice of t and u with $0 \le t < u \le T^*$

$$\int_0^t \sigma_2(s,u)dL_s = G_{u,t}\Big(\int_0^t \sigma_2(s,t)dL_s\Big) \quad P - a.s. \tag{13}$$

where $G_{u,t}(\cdot)$ is a certain Borel function, depending on u and t.

From (13) we conclude, that the distribution of the random vector $\Big(\int_0^t \sigma_2(s,u)dL_s, \int_0^t \sigma_2(s,t)dL_s\Big)$ cannot have a nontrivial absolutely continuous part with respect to λ_2. Thus, by Lemma 3.1 and Lemma 3.2 $\sigma_2(\cdot,u)$ is a scalar multiple of $\sigma_2(\cdot,t)$ on $[0,t]$. Hence for some nonnegative $\xi = \xi(t,u)$ depending on (t,u) we have

$$\sigma_2(\cdot, u) = \xi \sigma_2(\cdot, t) \text{ on } [0, t]. \tag{14}$$

This equation holds for all t, u with $0 \le t < u \le T^*$.
By assumptions 2.1 we have $\sigma_2(\cdot, v) \not\equiv 0$ for every v from $[0, T^*]$. Together with (14) this leads to

$$\xi(t, u) > 0 \quad , \quad 0 \le t < u \le T^*.$$

Therefore we have for all s, t, T with $0 \le s \le t < T \le T^*$

$$\sigma_2(s, t) = \frac{\sigma_2(s, T)}{\xi(t, T)} \quad , \quad s \in [0, t].$$

Now by defining

$$\tau(s) := \sigma_2(s, T^*) \text{ and } \zeta(t) := (\xi(t, T^*))^{-1}$$

one gets

$$\sigma_2(s, t) = \tau(s)\zeta(t) \quad , \quad 0 \le s \le t \le T^*. \tag{15}$$

Using assumption 2.1 (iv) this finishes the proof of Theorem 3.1.

The following Corollary is a consequence of the preceding proofs. Because it is not needed in the sequel the proof is omitted.

Corollary 3.1: *Assume L is a nonidentical zero Lévy process and f, g are continuously differentiable function on $[0, t]$ with the property that for no a, b with $0 \le a < b \le t$ the functions $f 1\!\!1_{[a,b]}$ and $g 1\!\!1_{[a,b]}$ are affine dependent.*
Then $X^L := \Big(\int_0^t f(s) dL_s, \int_0^t g(s) dL_s \Big)$ has a density if and only if L is not a compound Poisson process. If L is compound Poisson it holds

$$\begin{aligned} P(X^L \in A) &= P(\tau_1 > t) 1\!\!1_A(0,0) \\ &+ P(\tau_1 \le t < \tau_2) \cdot (\lambda_1 \otimes \nu)\{(s, z) : (zf(s), zg(s)) \in A\} \\ &+ P(\tau_2 \le t) \int_A h(x, y) dx dy \end{aligned}$$

for some probability density h on R^2. Here λ_1 and ν denote the Lebesgue measure on R_1 and the distribution of the jump size of L, respectively.

4 The short and the forward rate equations

In this chapter we derive a differential equation for the short rate process $r(t), 0 \leq t \leq T^*$, in the Markovian case and express the forward rates $f(t,T)$, in terms of the short rate $r(t)$. The proofs and formula (21) are taken from Eberlein, Raible [6] and partially added here for the sake of completeness. The results will be refered to in the next chapter. Starting from equation (5) we obtain

$$r(t) = f(0,t) + \int_0^t \frac{\partial}{\partial t}\vartheta(\sigma(s,t))ds - \int_0^t \sigma_2(s,t)dL_s \tag{16}$$

and

$$\begin{aligned} dr(t) &= \Big(\frac{\vartheta}{\vartheta t}f(0,t) + \vartheta'(0)\cdot\sigma_2(t,t) + \int_0^t \frac{\vartheta^2}{\vartheta t^2}(\vartheta(\sigma(s,t))ds\Big)dt \\ &- d\int_0^t \sigma_2(s,t)dL_s. \end{aligned} \tag{17}$$

Using the representation (13) we get for the last term

$$\begin{aligned} -d\Big(\int_0^t \tau(s)dL_s\cdot\zeta(t)\Big) &= -\zeta(t)\tau(t)dL_t - \int_0^t \tau(s)dL_s d\zeta(t) \\ &= -\sigma_2(t,t)dL_t - \int_0^t \sigma_2(s,t)dL_s\cdot\frac{\zeta'(t)}{\zeta(t)}dt \end{aligned} \tag{18}$$

Formulas (16) - (18) yield

$$dr(t) = \Big(\frac{\partial}{\partial t}f(0,t) + \vartheta'(0)\sigma_2(t,t) + \int_0^t \frac{\partial^2}{\partial t^2}\vartheta(\sigma(s,t))ds\Big)dt$$

$$-\frac{\zeta'(t)}{\zeta(t)}\Big(f(0,t) + \int_0^t \frac{\partial}{\partial t}(\vartheta(\sigma(s,t))ds - r(t)\Big)dt - \sigma_2(t,t)dL_s. \tag{19}$$

From (4), (19), as well as the factorization of $\sigma_2(s,T)$ it follows that

$$
\begin{aligned}
f(t,T) &= f0,T) + \int_0^t (\vartheta'(\sigma(s,T))\sigma_2(s,T)ds - \frac{\zeta(T)}{\zeta(t)}\int_0^t \sigma_2(s,t)dL_: \\
&= f(0,T) + \int_0^t (\vartheta'(\sigma(s,T)) - \vartheta'(\sigma(s,t))\sigma_2(s,T)ds + \\
&+ \frac{\zeta(T)}{\zeta(t)}(r(t) - f(0,t)). \qquad (20)
\end{aligned}
$$

If the volatility structure $\sigma(s,t)$ is stationary in the sense

$$
\sigma(s,t) = \tilde{\sigma}(t-s) \quad , \quad 0 \leq s \leq t \leq T^*
$$

for some function $\tilde{\sigma}$ in $[0,T^*]$, then one of the two cases following necessarily hold:

$$
\begin{aligned}
\sigma(s,t) &= \frac{\hat{\sigma}}{a}\left(1 - \mathrm{e}^{-a(t-s)}\right) \text{ or} \qquad (21) \\
\sigma(s,t) &= \hat{\sigma} \cdot (t-s)
\end{aligned}
$$

for some real constants $\hat{\sigma} > 0$ and $a \neq 0$, as well as for all s,t with $0 \leq s \leq t \leq T^*$.

5 Example: The bilateral gamma process

As mentioned above, the condition (ER) is restrictive and excludes Lévy processes like Poisson processes and gamma processes. In order to illustrate the extension provided by Theorem 3.1 here we consider as an example a class of Lévy processes being a slight generalization of the so called variance gamma processes (see e.g. Madan [14]).
Note that a gamma process $L = (L_t, t \geq 0)$ with parameters $\alpha > 0$ and $\lambda > 0$ is defined to be a Lévy process having for each positive t the marginal density

$$
\begin{aligned}
f_t(x;\alpha,\lambda) &= f_1(x;\alpha t,\lambda) \text{ with} \\
f_1(x;\alpha,\lambda) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\mathrm{e}^{-\lambda x}1\!\!1_{(0,\infty)}(x), \quad x \in R_1. \qquad (22)
\end{aligned}
$$

As usual $\Gamma(\alpha)$ denotes Eulers gamma function for positive values of α. The generating triplet of L is given by $(0,0,F)$ with $F(dx) = \frac{\alpha}{x}\mathrm{e}^{-\lambda x}1\!\!1_{(0,\infty)}(x)dx,\ x \in R_1$, and the cumulant generating function $\vartheta(\dot{})$ of L is defined on $(-\infty,\lambda)$ and can be expressed there as follows:

$$
\vartheta(u) = \alpha \log \frac{\lambda}{\lambda - u}. \qquad (23)
$$

The trajectories of L are strictly increasing, and move by jumps only. Indeed with $\triangle L(s) := L(s) - L(s-)$ one has

$$L(t) = \sum_{s \le t} \triangle L(s), \quad t > 0,$$

In every interval (s,t) with $0 < s < t$ there occur infinitely many jumps.
Let $(\Gamma^+ := (\Gamma^+_t, t \ge 0)$ and $\Gamma^- := (\Gamma^-_t, t \ge 0)$ two independent gamma processes with parameters α^+, λ^+ and α^-, λ^- respectively (all positive).

Put

$$\Gamma_t := \Gamma^+_t - \Gamma^-_t, \quad t \ge 0. \tag{24}$$

Then $\Gamma := (\Gamma_t, t \ge 0)$ is a Lévy process which we call *bilateral gamma process with parameters* $(\alpha^+, \alpha^-, \lambda^+, \lambda^-)$. The bilateral gamma processes have a series of properties making them interesting for theory and practice of mathematical finance, in particular for the term structure model described in the preceding chapters. Some of these properties are presented here.

Obviously, bilateral gamma processes move by jumps only, where the positive jumps are executed by Γ^+, the negative jumps by Γ^-. We shall denote the marginal distributions P^{Γ_t} of Γ *bilateral gamma distributions.* The process is selfdecomposable because the density f of its Lévy measure F has the property that $k(x) := |x| f(x)$ is monotone increasing (decreasing) on $(-\infty, 0)$ (on $(0, \infty)$ respectively), see e.g. Sato [19], p. 403. Consequently, every marginal distribution P^{Γ_t} is unimodal (dto. p. 404).
Denote by f_t the density of P^{Γ_t}. The mode of P^{Γ_t} equals zero if $\alpha^+ + \alpha^- \le 1$. In this case it holds $f_t(0-) = f_t(0+) = \infty$, see Sato [19] Remark 53.10.
From (24) and the indpendence of Γ^+ and Γ^- follows that the Lévy-measure of Γ is given by

$$F(dx) = \left[\frac{\alpha_+}{x}\, \mathrm{e}^{-\lambda_+ x} 1\!\mathrm{I}_{(0,\infty)}(x) + \frac{\alpha_-}{-x}\, \mathrm{e}^{\lambda_- x} 1\!\mathrm{I}_{(-\infty,0)}(x) \right] dx, \quad x \in R_1. \tag{25}$$

Using (23) and (24) the cumulant generating function $\vartheta(\cdot)$ can be explicitely expressed as

$$\vartheta(u) \;=\; \alpha^+ ln\Big(\frac{\lambda^+}{\lambda^+ - u} \Big) + \alpha^- ln\Big(\frac{\lambda^-}{\lambda^- + u} \Big) \qquad u \in I := (-\lambda^-, \lambda^+). \tag{26}$$

Thus we have

$$\vartheta'(u) = \Big(\frac{\alpha^+}{\lambda^+ - u} - \frac{\alpha^-}{\lambda^- + u} \Big) \quad , \quad u \in (-\lambda^-, \lambda^+) \tag{27}$$

which yields simple explicite coefficients in the equations (16)-(20) for $r(t)$ and $f(t,T)$ respectively.

Denoting by $\kappa_n(t)$ the n-th order cumulant of Γ_t we obtain

$$\begin{aligned}\kappa_n(t) &= t\cdot\frac{d^n}{du^n}\vartheta(u)|_{u=0} = \\ &= t\cdot n!\Big(\frac{\alpha^+}{(\lambda^+)^n}+(-1)^n\frac{\alpha^-}{(\lambda^-)^n}\Big) \quad , \qquad n\geq 1, t>0.\end{aligned}$$

In particular one can specify

the expectation

$$E\Gamma_t = \kappa_1(t) = \Big(\frac{\alpha^+}{\lambda^+}-\frac{\alpha^-}{\lambda^-}\Big)\cdot t,$$

the variance

$$\text{Var}\,\Gamma_t = \kappa_2(t) = \Big(\frac{\alpha^+}{(\lambda^+)^2}+\frac{\alpha^-}{(\lambda^-)^2}\Big)\cdot t,$$

the Charliers skewness

$$\gamma_1(\Gamma_t) = \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{2\Big(\frac{\alpha^+}{(\lambda^+)^3}-\frac{\alpha^-}{(\lambda^-)^3}\Big)}{\Big(\frac{\alpha^+}{(\lambda^+)^2}+\frac{\alpha^-}{(\lambda^-)^2}\Big)^{\frac{3}{2}}}\cdot t^{-\frac{1}{2}},$$

as well as the excess

$$\gamma_2(\Gamma_t) = \frac{\kappa_4}{\kappa_2^2} = \frac{6\Big(\frac{\alpha^+}{(\lambda^+)^4}+\frac{\alpha^-}{(\lambda^-)^4}\Big)}{\Big(\frac{\alpha^+}{(\lambda^+)^2}+\frac{\alpha^-}{(\lambda^-)^2}\Big)^{2}}\cdot t^{-1}.$$

It follows that the bilateral Gamma distribution is *leptokurtosic.*

Using (22),(24) and the independence of Γ^+ and Γ^- the density $f_t(x;\alpha^+,\alpha^-,\lambda^+,\lambda^-)$ of Γ_t for $x>0$ is evaluated as

$$\begin{aligned}&f_t(x;\alpha^+,\alpha^-,\lambda^+,\lambda^-) = f_1(x;\alpha^+t,\alpha^-t,\lambda^+,\lambda^-) \text{ with}\\ &f_1(x;\alpha^+,\alpha^-,\lambda^+,\lambda^-) = \\ &= \frac{(\lambda^-)^{\alpha^-}(\lambda^+)^{\alpha^+}}{(\lambda^++\lambda^-)^{\alpha^-}}\cdot\frac{x^{\alpha^+-1}\mathrm{e}^{-\lambda^+x}}{\Gamma(\alpha^+)\Gamma(\alpha^-)}\int_0^\infty v^{\alpha^--1}\Big(1+\frac{v}{x(\lambda^++\lambda^-)}\Big)^{\alpha^+-1}dv. \qquad (28)\end{aligned}$$

For $x<0$ we have $P(\Gamma_t<x) = P(-\Gamma_t>-x) = 1-P(\Gamma_t<-x)$ and, consequently, due to $-\Gamma = \Gamma^-\Gamma^+$

$$f_t(x;\alpha^+,\alpha^-,\lambda^+,\lambda^-) = f_t(-x;\alpha^-,\alpha^+,\lambda^-,\lambda^+). \tag{29}$$

The right hand of (28) equals to

$$= \frac{(\lambda^-)^{\alpha^-}(\lambda^+)^{\alpha^+}}{(\lambda^+ + \lambda^-)^{\frac{\alpha^+ + \alpha^-}{2}}} \frac{x^{\frac{\alpha^+ + \alpha^-}{2}} - 1}{\Gamma(\alpha^+)} \mathrm{e}^{-\frac{x}{2}(\lambda^+ - \lambda^-)} W_{\frac{\alpha^+ - \alpha^-}{2}, \frac{\alpha^+ + \alpha^- - 1}{2}}(x(\lambda^+ + \lambda^-)),$$

where $W_{\nu\mu}(z)$ denotes the so-called *Whittaker function* (see for example Ryshik, Gradstein [18] pages 398-402). For all $\nu, \mu \in R_1$ with $\mu - \nu > -\frac{1}{2}$ the function $W_{\nu\mu}(z)$ is defined on $(0, \infty)$ by

$$W_{\nu\mu}(z) := \frac{z^{\mu+\frac{1}{2}}\mathrm{e}^{-\frac{z}{2}}}{\Gamma(\mu - \nu + \frac{1}{2})} \int_0^\infty \mathrm{e}^{-zt} t^{\mu-\nu-\frac{1}{2}} (1+t)^{\mu+\lambda-\frac{1}{2}} dt, \quad z > 0.$$

(If there is no cause for confusion on the parameters we abbreviate $f_t(x) := f_t(x;\alpha^+,\alpha^-,\lambda^+,\lambda^-)$.) An easy substitution shows

$$W_{\nu\mu}(z) = \frac{z^{\nu}\mathrm{e}^{-\frac{z}{2}}}{\Gamma(\mu - \nu + \frac{1}{2})} \int_0^\infty \mathrm{e}^{-v} v^{\mu-\nu-\frac{1}{2}} (1+\frac{v}{z})^{\mu+\nu-1} dv, \quad z > 0.$$

Now the connection between the density f_t from (28) and the Whittaker function is obvious. The asymptotic behaviour for $z \to \infty$ is given by

$$W_{\nu,\mu}(z) \sim \mathrm{e}^{-\frac{z}{2}} z^{\nu} H(z) \tag{30}$$

with

$$H(z) = 1 + \sum_{k=1}^{\infty} \frac{\left[\mu^2 - (\nu - \frac{1}{2})^2\right]\left[\mu^2 - (\nu - \frac{3}{2})^2\right] \cdots \left[\mu^2(\nu - k + \frac{1}{2})^2\right]}{k! z^k}.$$

(See Ryshik, Gradstein [18], p. 400.)

Obviously $H(z) \sim 1$ for $z \uparrow \infty$. Thus from (28) - (30), we obtain for $x \uparrow \infty$

$$f_1(x) \sim \frac{(\lambda^-)^{\alpha^-}(\lambda^+)^{\alpha^+}}{(\lambda^+ + \lambda^-)^{\alpha^-}} \frac{x^{\alpha^+ - 1}}{\Gamma(\alpha^+)} \mathrm{e}^{-\lambda^+ x} \tag{31}$$

as well as for $x \downarrow -\infty$

$$f_1(x) \sim \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\alpha^+}} \; \frac{(-x)^{\alpha^- - 1}}{\Gamma(\alpha^-)} \mathrm{e}^{\lambda^- x}. \tag{32}$$

In particular it turns out that the density of Γ_t is *semiheavy tailed*.

The short rate equation (16) now turns into the explicite form

$$r(t) = f(0,t) - \int_0^t \Big(\frac{\alpha^+}{\lambda^+ - \sigma(s,t)} - \frac{\alpha^-}{\lambda^- + \sigma(s,t)} \Big) \sigma_2(s,t) ds - \int_0^t \sigma_2(s,t) dL_s$$

and for the forward rates $f(t,T)$ we infer from (20)

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^\infty \Big(\frac{\alpha^+}{\lambda^+ - \sigma(s,T)} - \frac{\alpha^+}{\lambda^+ - \sigma(s,t)} - \frac{\alpha^-}{\lambda^- + \sigma(s,T)} + \frac{\alpha^-}{\lambda^- + \sigma(s,t)} \Big) ds \\ &+ \frac{\zeta(T)}{\zeta(t)} (r(t) - f(0,t)). \end{aligned}$$

The case $\alpha^+ = \alpha^- =: \alpha$ has some special properties. Indeed we obtain for the marginal density f_1 of Γ_1:

$$\begin{aligned} f_1(x) = \frac{1}{\Gamma(\alpha)} \Big(\frac{\lambda^+ \lambda^-}{\lambda^+ + \lambda^-} \Big)^\alpha |x|^{\alpha - 1} \, \mathrm{e}^{-\frac{|x|}{2}(\lambda^+ - \lambda^-)} \sqrt{\frac{|x|(\lambda^+ + \lambda^-)}{\pi}} \\ \cdot K_{\alpha - \frac{1}{2}} \Big(\frac{|x|}{2} (\lambda^+ + \lambda^-) \Big) \end{aligned} \tag{33}$$

because of $W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}} K_\mu(\frac{z}{2})$ (Ryshik, Gradstein [17], p. 401). Here K_μ denotes the Bessel function of the third kind and order μ given by the integral representation

$$K_\mu(z) = \frac{1}{2} \int_o^\infty u^{\mu - 1} \mathrm{e}^{-\frac{1}{2} z (u + u^{-1})} du, \quad z > 0, \mu \in R_1.$$

Thus f_1 is a *generalized hyperbolic density* with parameters

$$(\lambda, \alpha, \beta, \delta, \mu) = (\alpha, \frac{\lambda^+ + \lambda^-}{2}, \frac{\lambda^+ - \lambda^-}{2}, 0, 0)$$

in the terminology of Barndorff-Nielsen [1].

From this fact it is clear, that Γ_1 has a variance-mean mixed normal distribution whose mixing measure is given by the gamma density

$$\frac{(\sqrt{\lambda^+\lambda^-})^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, \mathrm{e}^{-\sqrt{\lambda^+\lambda^-}x}, \qquad x > 0.$$

Precisely, we have

$$\mathfrak{L}(\Gamma_1) = N(\beta\sigma^2, \sigma^2) \underset{\sigma^2}{\wedge} \Gamma(\alpha, (\lambda^+\lambda^-)^{\frac{1}{2}}) \tag{34}$$

with $\beta := \frac{\lambda^- - \lambda^+}{2}$, see Barndorff-Nielsen [1]. Here $\mathfrak{L}(\Gamma_1)$ denotes the distribution law of Γ_1.

Putting $\lambda := \frac{\lambda^+ + \lambda^-}{2}$ it is easily seen that for fixed positive α and λ the densities $f_t(x; \alpha, \lambda, \beta)$ form an exponential family of distributions with respect to β with $|\beta| < \lambda$

$$f_t(x; \alpha, \lambda, \beta) = f_1(x; \alpha t, \lambda, \beta) =$$

$$= C(\alpha, \beta, \lambda) \cdot |x|^{\alpha-1} \mathrm{e}^{-\beta|x|} \sqrt{|x|} K_{\alpha-\frac{1}{2}}(\lambda|x|) \quad , \quad x \in R_1$$

and with the normalizing constant

$$C(\alpha, \beta, \lambda) := 2 \cdot (\pi^{\frac{1}{2}}\Gamma(\alpha))^{-1} \left(\frac{\lambda}{2}\right)^{\alpha+\frac{1}{2}} \left(1 - \left(\frac{\beta}{\lambda}^2\right)\right)^{\alpha}$$

For $\beta = 0$ i.e. $\lambda^+ = \lambda^- = \lambda$, we get the (symmetric) *variance gamma densities* $f_t(x; \alpha, \lambda) = f_1(x; \alpha t, \lambda)$ with

$$f_1(x; \alpha, \lambda) = C(\alpha, \lambda)|x|^{\alpha-1} \sqrt{|x|} K_{\alpha-\frac{1}{2}}(\lambda|x|) \quad , \quad x \in R_1$$

and with

$$C(\alpha, \lambda) = 2\pi^{\frac{1}{2}}\Gamma(\alpha)(\frac{\lambda}{2})^{\alpha+\frac{1}{2}}.$$

The cumulant generating function here simplifies to

$$\vartheta(u) \;=\; \alpha ln\left(\frac{\lambda^2}{\lambda^2 - u^2}\right) \quad , \quad |u| < \lambda$$

with

$$\vartheta'(u) \;=\; \frac{2\alpha u}{\lambda^2 - u^2} \quad , |u| < \lambda$$

and the equations (16) and (20) turn into

$$r(t) = f(0, t) - 2\alpha \int_0^t \frac{\sigma_2(s, t)}{\lambda^2 - \sigma_2(s, t)} ds - \int_0^t \sigma_2(s, t) dL_s$$

and

$$f(t,T) = f(0,T) + 2\alpha \int_0^T \Big(\frac{1}{\lambda^2 - \sigma^2(s,T)} - \frac{1}{\lambda^2 - \sigma^2(s,t)} \Big) ds$$
$$+ \frac{\zeta(T)}{\zeta(t)} (r(t) - f(0,t)) \quad , \qquad 0 \le t \le T,$$

respectively.

The variance gamma process was has been used in mathematical finance and was studied in detail for example by Madan [14], Madan, Seneta [15].

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