On integrals with respect to Lévy processes*

Uwe Küchler
Humboldt University Berlin, Germany
Institute of Mathematics

Abstract
Assume $L$ is a non-deterministic real valued Lévy process and $f$ is a smooth function on $[0,t]$. If for some Borel function $H$ $P$-almost sure the equality

$$H\left( \int_{[0,t]} f(s) dL_s \right) = L_t$$

holds, then $f$ is constant on $[0,t]$.

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1 Introduction

In modeling term structures of interest rates driven by Lévy processes $L = (L_s, s \geq 0)$ arises the question, if for a fixed positive $t$ and for a given deterministic continuously differentiable function $f$ on $[0,t]$ there may exist a (deterministic) Borel function $H$ with

$$H\left( \int_{[0,t]} f(s) dL_s \right) = L_t \quad P - a.s. \quad (1)$$

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Authors address: Humboldt- Universität zu Berlin, Institut für Mathematik, Unter den Linden 6, Sitz: Adlershof, D-10099 Berlin, Germany; e-mail: kuechler@mathematik.hu-berlin.de
See Küchler, Naumann [3]. Obviously (1) holds with $H(x) = c^{-1}x$ if $f$ equals a nonzero constant $c$. In this note we will show that for any non-deterministic Lévy process $L$ the existence of a measurable function $H$ with (1) implies $f$ to be constant.

Note that under assumption (1) the support of the distribution of the random vector $(I_t, L_t)$ with $I_t = \int f(s) dL_s$ is a zero set with respect to the (two-dimensional) Lebesgue measure $\lambda_2$. Therefore the assertion immediately follows if one can ensure that $(I_t, L_t)$ has a common density with respect to $\lambda_2$. This is the case if $L$ is a Wiener process with positive diffusion coefficient and arbitrary drift. Indeed, if $f$ is not a constant then $(F_t, L_t)$ is two-dimensional Gaussian distributed with a regular covariance matrix. For other Lévy processes $L$ Eberlein and Raible [1] proved that the following condition on the characteristic function of $L_t$ is sufficient for $(I_t, L_t)$ to have a common density: There exist real constants $C, \gamma, \eta > 0$ such that

$$|E[\exp(iu L_t)]| \leq C \cdot \exp(-\gamma |u|^\eta) \quad \forall u \in \mathbb{R}$$

(2)

But not for every Lévy process $L$ the vector $(I_t, L_t)$ has a common density, consider for example Poisson processes. Analyzing the arguments above it turns out to be enough for our purpose to show that the common distribution of $(I_t, L_t)$ has a nonzero absolutely continuous part with respect to $\lambda_2$ if $f$ is not equal to a constant. The key point of the proof of the theorem below is to show that this property holds for every non-deterministic Lévy process $L$.

## 2 Lévy processes

Assume $L = (L_s, s \geq 0)$ to be a real valued Lévy process defined on some probability space $(\Omega, \mathfrak{F}, P)$. This means

(i) $P(L_0 = 0) = 1$.

(ii) $L$ has independent and stationary increments.

(iii) All trajectories $(L_s(\omega), s \geq 0), \omega \in \Omega$, are cadlag, i.e. continuous from the right and having limits from the left.

Examples of Lévy process are

- the Wiener process $W$ with drift $\mu$ and diffusion $\sigma^2 > 0$, in this case $W_t - W_s$ is $N(\mu(t - s), \sigma^2(t - s))$-distributed, and all trajectories are continuous,
- the Poisson process \( N \) with jump intensity \( \lambda > 0 \), in this case \( N_t - N_s \) is Poisson-distributed with parameter \( \lambda(t-s) \) and the trajectories are piecewise constant, non-decreasing, jumping at times \( \tau_k, k \geq 1 \), with jump size one where \( (\tau_k - \tau_{k-1}), k \geq 1 \), with \( \tau_0 = 0 \) are mutual independent exponential distributed with parameter \( \lambda \) random variables.

- the compound Poisson process \( Y_t = \sum_{k=1}^{N_t} Z_k \), where \( N = (N_s, s \geq 0) \) is a Poisson process with jump intensity \( \lambda > 0 \) and the \( (Z_k, k \geq 1) \) are independent identically distributed random variables the distribution function \( F \) of which satisfies \( F(0+) = F(0-) \). Moreover, the \( (Z_k, k \geq 1) \) and \( N \) are independent. The \( Z_k \) form the jump sizes of \( (Y_t, t \geq 0) \).

We summarize some well known facts on Lévy processes that will be used below. For proofs and further details see for example Sato [4]. In general, to every Lévy process \( L \) is assigned a uniquely determined characteristic triple \( \mathcal{L} := (\mu, \sigma^2, \nu) \) where \( \mu \in \mathbb{R}, \sigma^2 \geq 0 \) and \( \nu \) is a measure on the real axis \( \mathbb{R} \) satisfying

\[
\nu(\{0\}) = 0, \quad \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} \nu(dx) < \infty \quad (3)
\]

such that it holds

\[
E \exp(iyL_t) = \exp[t \cdot \psi(y)], \quad t \geq 0, y \in \mathbb{R}
\]

with

\[
\psi(y) = i\mu y - \frac{\sigma^2}{2} y^2 + \int_{-\infty}^{\infty} (e^{iyx} - 1 - \frac{iyx}{1 + x^2}) \nu(dx), y \in \mathbb{R}.
\]

Conversely, to any triple \( \mathcal{L} = (\mu, \sigma^2, \nu) \) with (3) there is a uniquely in law determined Lévy process \( L \) having \( \mathcal{L} \) as its characteristic triple. If \( \sigma^2 + \nu(R) = 0 \) it holds \( L_s = \mu s, s \geq 0 \). We call this case the deterministic one. If \( \nu \equiv 0 \) then we obtain a Wiener process with \( \mu \) as drift and with \( \sigma^2 \) as diffusion coefficient.

Let \( L \) be a Lévy process. For any \( \varepsilon > 0 \) one can decompose \( L \) into two mutual independent Lévy processes \( L^{<\varepsilon} \) and \( L^{>\varepsilon} \) defined by

\[
L^{>\varepsilon}_s := \sum_{u \leq s} \mathbb{I}_{\{|\Delta L_u| > \varepsilon\}} \Delta L_u, s \geq 0, \text{ where}
\]

\[
\Delta L_u := L_u - L_{u-0},
\]

and

\[
L^{\leq\varepsilon}_s := L_s - L^{>\varepsilon}_s, \quad s \geq 0.
\]
The process \( L^> \) is a compound Poisson process with jump intensity \( \lambda_e := \nu(R\backslash (-\varepsilon, \varepsilon]) \) and jump size distribution \( F_\varepsilon(dz) := \lambda_e^{-1} \nu(dz) \cdot \mathbb{1}_{R\backslash (-\varepsilon, \varepsilon]}(z) \).
If \( f \) is a continuously differentiable function on \([0, t]\) we define the integral
\[
\int_{[0, t]} f(s) dL_s \text{ by}
\]
\[
\int_{[0, t]} f(s) dL_s := f(t) L_t - \int_{[0, t]} L_s \frac{d}{ds} f(s) ds
\]
and denote it shortly by \( I_f \).
Note that \( L \) is a (possibly degenerated to the deterministic case if \( \sigma^2 = 0 \)) Wiener process if and only if \( L^> \equiv 0 \) for all \( \varepsilon > 0 \).

3 Results

The result of this note is the following theorem,

**Theorem 3.1.** Fix a positive number \( t \), let \( f \) be a real valued continuously differentiable function on \([0, t]\) and assume \( L = (L_s, s \geq 0) \) to be a non-deterministic Lévy process. If for some Borel function \( H \) it holds

\[
H\left( \int_{[0, t]} f(s) dL_s \right) = L_t \quad P - a.s.
\]

then \( f \) is necessarily a constant.

**Proof.** Let us firstly consider the case that \( L \) is a Wiener process with parameters \( \mu \in R \) and \( \sigma^2 > 0 \). Then \( (I_t, L_t) = (\int_0^t f(s) dL_s, L_t) \) is a Gaussian vector.
Its distribution is degenerated by assumption (4). Thus its covariance matrix is singular, because of \( \sigma^2 > 0 \) this implies \( \left( \int_{[0, t]} f(s) ds \right)^2 = t \cdot \int_{[0, t]} f^2(s) ds \). By the Cauchy-Schwarz inequality this is only possible if \( f \) is a constant.

As a second step we assume that \( L \) is a compound Poisson process with jump times \( \tau_1, \tau_2, \cdots \) and jumps sizes \( Z_k \) at time \( \tau_k, k \geq 1 \). Denote the distribution function of \( Z_k \) by \( F \). Let \( z \) be a point of increase of \( F \). This means \( P(z- \varepsilon < Z_1 < z+ \varepsilon) > 0 \) for all \( \varepsilon > 0 \).

We may suppose \( z > 0 \), otherwise consider \(-L \) instead of \( L \). If \( f \) is not a constant function, based on the assumptions on \( f \) we find a subinterval \( U := (u_0, u_1) \) of \((0, t)\) with \( u_0 < u_1 \) and \( f'(x) \neq 0 \) at all points \( x \) from \( U \). In particular \( f' \) is either strictly positive or strictly negative on \( U \). Thus \( f \) is strictly monotone on \( U \) and consequently maps \( U \) one-to-one on an interval \( V = (v_0, v_1) \) with \( v_0 < v_1 \).
Without restriction we can assume \( v_0 > 0 \), otherwise restrict \( U \) to a smaller open non-void interval and/or consider, if it is needed, \(-f\) instead of \( f\).

We continue the proof by choosing a positive \( \varepsilon \) such that

\[
\varepsilon < z \cdot \frac{c-1}{c+1}
\]

with \( c := (v_1/v_0)^{1/2} \). For every \( n \geq 1 \) put

\[
C_{\varepsilon,n} := \{ \omega \in \Omega \mid Z_k(\omega) \in (z-\varepsilon, z+\varepsilon), k = 1, \ldots, n \}
\]

and

\[
D_n := \{ \omega \in \Omega : u_0 < \tau_1(\omega) < \ldots < \tau_n(\omega) < u_1, \tau_{n+1}(\omega) < t \}.
\]

Because of the special choice of the point \( z \) and the independence of the \( Z_k, k \geq 1 \) we have \( P(C_{\varepsilon,n}) > 0 \). The jump times \( \tau_n, n \geq 1 \) have the property that the differences \( \tau_{k+1} - \tau_k, k \geq 1 \), are independent and exponentially distributed. This implies in particular \( P(D_n) > 0 \). Now using the independence of the \( (Z_k, k \geq 1) \) from the \( (\tau_k, k \geq 1) \) we conclude that

\[
P(D_n \cap C_{\varepsilon,n}) = P(D_n)P(C_{\varepsilon,n}) > 0.
\]

We have \( P \)-almost surely for all \( \omega \in D_n \cap C_{\varepsilon,n} \)

\[
I_t = \int_{[0,t]} f(s) dL_s = \sum_{k=1}^{n} Z_k f(\tau_k) \in (n(z-\varepsilon)v_0, n(z+\varepsilon)v_1)
\]

Consequently, the distribution of \( I_t \) given \( D_n \cap C_{\varepsilon,n} \) can be expressed as

\[
P(I_t \in B \mid D_n \cap C_{\varepsilon,n}) = \frac{1}{P(C_{\varepsilon,n})} \int_{(z-\varepsilon,z+\varepsilon)^n} P \left( \sum_{k=1}^{n} z_k f(\tau_k) \in B \mid D_n \right) F(dz_1) \cdots F(dz_n)
\]

where \( B \) is any Borel set. Here we have used once again the independence of the \( (Z_k, k \geq 1) \) from the \( (\tau_k, k \geq 1) \).

At this point we formulate and prove an auxiliary result on the distribution of \( \sum_{k=1}^{n} z_k f(\tau_k) \) as a lemma.

**Lemma 3.1.** Assume \( n \geq 1 \), as well as \( z_k > 0, k = 1, \ldots, n \), to be fixed. Then the distribution of \( S_n := \sum_{k=1}^{n} z_k f(\tau_k) \) given \( D_n \) has a density being strictly positive on

\[
\mathcal{J} := \left( v_0 \sum_{k=1}^{n} z_k, v_1 \sum_{k=1}^{n} z_k \right).
\]
Proof. Given \( D_n \) := \( \{ \tau_n \leq t < \tau_{n+1} \} \) the random vector \((\tau_1, \tau_2, \ldots, \tau_n)\) has a strictly positive density on \( \Delta_n := \{ (s_1, \ldots, s_n) : 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq t \} \), indeed it is uniformly distributed thereon (see for example Sato [4], Chapter 1.3). This implies immediately that given \( D_n \) the vector \((\tau_1, \ldots, \tau_n)\) is uniformly distributed on \( \Delta_n \cap (u_0, u_1)^n \).

Put \( S_n(s_1, \ldots, s_n) := \sum_{k=1}^n z_k f(s_k) \). The function \( G \) defined by \( G(s_1, \ldots, s_n) = (s_1, s_2, \ldots, s_n, S_n(s_1, \ldots, s_n)) \) maps \( \Delta_n \cap (u_0, u_1)^n \) continuously differentiable one-to-one on \( (\Delta_{n-1} \cap (u_0, u_1)^{n-1}) \times J \). This is a consequence of the supposed smoothness and the strict monotonicity of the mapping \( f \) from \((u_0, u_1)\) onto \((v_0, v_1)\). Thus, given \( D_n \) the vector \( G(\tau_1, \ldots, \tau_n) = (\tau_1, \ldots, \tau_{n-1}, \sum_{k=1}^n z_k f(\tau_k)) \) has a density being strictly positive on \((\Delta_{n-1} \cap (u_0, u_1)^{n-1}) \times J \) which can be expressed as

\[
\gamma \cdot \Pi_{\Delta_n \cap (u_0, u_1)^n} (G^{-1}(v_1, v_2, \ldots, v_n))(z_n | f'(v_n))^{-1}
\]

where \( \gamma \) is the normalizing constant. Consequently given \( D_n \) the random variable \( S_n = S_n(\tau_1, \ldots, \tau_n) \) has a density \( \psi_{S_n}(v) \) equal to

\[
\gamma z_n^{-1} \cdot \int_{\Delta_{n-1} \cap (u_0, u_1)^{n-1}} \Pi_{\Delta_{n-1} \cap (u_0, u_1)^{n-1}} (G^{-1}(v_1, \ldots, v_{n-1}, v)) \, dv_1, \ldots, dv_{n-1} \cdot |f'(v)|^{-1} \cdot \Pi_J(v), \quad v \in R,
\]

which is strictly positive on \( J \). Thus the lemma is proved.

Corollary 3.1: Given \( D_n \cap C_{\varepsilon,n} \) the integral \( I_t \) has a density being strictly positive at least on the non-void interval \((nv_0(z + \varepsilon), nv_1(z - \varepsilon))\).

Proof. We use the Lemma 3.1 to express the integrand in formula (8)

\[
P\left( \sum_{k=1}^n z_k f(\tau_k) \in B \mid D_n \right) = \int_B \psi_{S_n}(v; z_1, \ldots, z_n) \, dv
\]

(9)

Inserting (9) into (8) and changing the order of integration we get

\[
P(I_t \in B \mid D_n \cap C_{\varepsilon,n}) \cdot P(C_{\varepsilon,n}) = \int_B \left( \int_{(z - \varepsilon, z + \varepsilon)^n} \psi_{S_n}(v; z_1, \ldots, z_n) F(dz_1) \ldots F(dz_n) \right) \, dv
\]

Thus given \( D_n \cap C_{\varepsilon,n} \) the integral \( I_t \) has a density which is strictly positive on \((nv_0(z + \varepsilon), nv_1(z - \varepsilon))\).
That this interval is non void follows from assumption (5). Indeed, (5) implies $z + \varepsilon < c(z - \varepsilon)$ as well as $z > \varepsilon$ and thus we have

$$\frac{v_1(z - \varepsilon)}{v_0(z + \varepsilon)} = c^2 (\frac{z - \varepsilon}{z + \varepsilon})^2 \cdot \frac{z + \varepsilon}{z - \varepsilon} > 1.$$  

(10)

This completes the proof of the corollary. \qed

We continue the proof of the theorem. By assumption we have

$$H(I_l) = L_\ell \quad P - a.s.$$  

For any $n \geq 1$ this implies

$$H(I_l) = \sum_{k=1}^{n} Z_k \in (n(z - \varepsilon), n(z + \varepsilon)) \quad P(\cdot|D_n \cap C_{\varepsilon,n}) - a.s.$$  

(11)

Now choose two positive integers $l$ and $m$ with

$$\frac{z + \varepsilon}{z - \varepsilon} < \frac{l}{m} < c^2 \frac{z - \varepsilon}{z + \varepsilon}$$  

(12)

which is possible because of (10).

Introduce for any $n \geq 1$ the two intervals

$$V_n := (n(z - \varepsilon), n(z + \varepsilon)) \quad \text{and} \quad W_n := (mw_0(z + \varepsilon), mw_1(z - \varepsilon)).$$

By construction of $l$ and $m$ we get from (12) $l > m$ and

$$\underline{w} := lw_0(z + \varepsilon) < mw_1(z - \varepsilon) := \overline{w}.$$  

(13)

Thus it holds

$$W_l \cap W_m = (\underline{w}, \overline{w}) \neq \emptyset.$$  

(14)

From Corollary 3.1. and (11) it follows for $n = l$

$$H(v) \in V_l \quad \text{Lebesgue - a.e. on } W_l$$

and for $n = m$

$$H(v) \in V_m \quad \text{Lebesgue - a.e. on } W_m.$$
Thus (14) implies

$$H(v) \in V_l \cap V_m \quad \text{for all } v \text{ from } (\underline{w}, \bar{w}).$$  \hfill (15)

But from the first inequality of (12) we have \(l > m\) and \(l(z_0 + \varepsilon) > m(z + \varepsilon)\), this means \(V_l \cap V_m = \emptyset\). This is a contradiction to (15). Consequently, the assumption, that \(f\) is not constant cannot be valid.

The proof of the theorem is finished for \(L\) being a compound Poisson process or a Wiener process, both non degenerated.

In the third part of the proof we assume that \(L\) is a general non-deterministic Lévy process. We choose an \(\varepsilon > 0\) such that \(L_{t}^{> \varepsilon}\) is non-trivial. (If no such \(\varepsilon\) exists then \(L\) necessarily is a Wiener process, and for this case the proof was given at the beginning.)

By assumption we have

$$H(I_t^{> \varepsilon} + I_t^{\leq \varepsilon}) = L_t^{> \varepsilon} + L_t^{\leq \varepsilon} \quad \text{P-a.s.}$$  \hfill (16)

with

$$I_t^{> \varepsilon} := \int_{[0,t]} f(s)dL_s^{> \varepsilon}, \quad I_t^{\leq \varepsilon} := \int_{[0,t]} f(s)dL_s^{\leq \varepsilon}.$$

We know that \((I_t^{> \varepsilon}, L_t^{> \varepsilon})\) is independent of \((I_t^{\leq \varepsilon}, L_t^{\leq \varepsilon})\).

Therefore from (16) we get

$$H(I_t^{> \varepsilon} + y) = L_t^{> \varepsilon} + z \quad \text{P-a.s.}$$  \hfill (17)

for \(P(I_t^{> \varepsilon}, I_t^{\leq \varepsilon})\) - almost all \((y, z)\). Thus there exists at least one pair \((y, z)\) such that (17) holds. As a consequence from the second part of the proof we obtain

$$\tilde{H}(I_t^{> \varepsilon}) = L_t^{> \varepsilon} \quad \text{P-a.s.}$$  \hfill (18)

for the Borel function \(\tilde{H}(x) := H(x+y) - z\). Because \(L_t^{> \varepsilon}\) is a nontrivial compound Poisson process, the function \(f\) has to be constant. Now the proof of the theorem is complete. \(\square\)
References


