A Benchmark Model for Financial Markets

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Abstract. This paper introduces a benchmark model for financial markets, which is based on the unique characterization of a benchmark portfolio that is chosen to be the growth optimal portfolio. The general structure of risk premia for asset prices and portfolios is derived. Furthermore, the short rate is obtained as an average of appreciation rates. The benchmark model is shown to be locally arbitrage free, however, it still permits some form of arbitrage. Finally, a subclass of arbitrage free contingent claim prices is derived.

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1 Introduction to Benchmark Pricing

Various alternative methodologies for the modelling of asset prices and financial markets have been proposed in the literature. The Capital Asset Pricing Model (CAPM), which is a mean-variance one-period equilibrium model of exchange, see Sharpe (1964), Lintner (1965) and Mossin (1966), has been designed to model asset price dynamics. This model has been crucial for the understanding of the relationship between mean and variance of returns in equilibrium. Merton (1973) developed an intertemporal CAPM from portfolio selection behaviour of investors maximizing equilibrium expected utility.

Of particular importance in the theory and practice of derivative pricing has been the Arbitrage Pricing Theory (APT), originated by Ross (1976) and further developed in an extensive literature, including Harrison & Kreps (1979), Harrison & Pliska (1981), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991), Delbaen & Schachermayer (1997) and Yan (1998). The APT in its standard version relies on the existence of an equivalent martingale measure. A closely related approach uses the state price density or state price deflator, see, for instance, Constatinides (1992), Duffie (1996) or Rogers (1997), which also leads to an arbitrage free pricing methodology with reference to an equivalent martingale measure.

In this paper we aim to model a certain form of arbitrage and still to obtain the key features of the CAPM and APT without the standard assumption needed to ensure the existence of an equivalent martingale measure. We will not use expected utility maximization, equilibrium arguments or an equivalent martingale measure. Instead, we start from the concept of a *growth optimal portfolio* (GOP), originally developed by Kelly (1956) and further developed in a stream of literature leading to Long (1990), Artzner (1997), Bajeux-Besnainou & Portait (1997), Karatzas & Shreve (1998), Platen (2000) and Heath & Platen (2001). The GOP is also known as the numeraire portfolio and appears in the APT as the inverse state price deflator. The state price deflator has been independently suggested for the modelling of financial and insurance markets by Bühlmann (1992, 1995) and Bühlmann, Delbaen, Embrechts & Shiryaev (1998).

In Long (1990) and subsequent papers on the numeraire portfolio it has been assumed that any asset price that is benchmarked by the GOP is a martingale. We extend this to a benchmark methodology that requires only benchmarked asset prices to be local martingales. This weaker assumption will be sufficient to construct a financial market model with risk premia that are independent of the chosen denomination. The resulting benchmark model provides a more general modelling framework than the CAPM and APT. In the case when all benchmarked price processes are martingales, then the benchmark methodology is equivalent to the APT.

The benchmark methodology allows us to model some form of arbitrage, which is likely to appear in emerging and maturing markets that are frequently subject to shocks or turbulence. Under the APT, no-arbitrage prices are formed directly from conditional expectations, which require participants to have perfect knowledge of the probabilistic dynamics of asset prices. To achieve this they would need to have a correct model that is always exactly calibrated. One must admit that this is a strong assumption. For all of the above reasons we will relax the restrictive assumptions of the APT.

A further advantage of the benchmark methodology is that derivative pricing, Value at Risk analysis, portfolio optimization, calibration, estimation, filtering and other risk management tasks can be performed under one and the same probability measure, the real world probability measure. The benchmark approach can be extended to include asset price dynamics that are modeled as semimartingales incorporating both predictable and inaccessible jumps. Under such a model we are in a better position to analyze and manage the combination of market, credit, operational, liquidity, insurance and other risks in an integrated framework.

The paper demonstrates under appropriate assumption that the GOP is the only reference portfolio that when used as benchmark, generates risk premia with a structure that is independent of the considered denomination. The market portfolio, which is used in the CAPM, is qualitatively similar to the GOP but, in general, quantitatively different. Unlike the GOP, the market portfolio is not directly linked to contingent claim pricing. Furthermore, we will see that the GOP, when used as a benchmark, is the only portfolio that transforms all nonnegative benchmarked price processes into supermartingales. This relates us directly to the standard notion of no-arbitrage.

For a given asset its traded price is in practice a result of a process that matches supply and demand. The difference between a benchmarked traded price and any corresponding expected future benchmarked value of this traded asset is nonnegative and represents an arbitrage amount, in case it is not zero. Such arbitrage amounts naturally exist under the benchmark model. They do not arise under the APT. The benchmark pricing methodology does not require the existence of an equivalent martingale measure.

The subclass of benchmarked arbitrage free portfolios is uniquely determined by corresponding conditional expectations under the real world probability measure. If this subclass covers all traded prices, then the benchmark model is arbitrage free and the results obtained are consistent with those obtained from the well-known APT.

2 Continuous Multi-Asset Market

We consider a multi-asset market with $d+1$ primary assets. The uncertainty in this market is generated by d independent standard Wiener processes $W^1, \dots, W^d$ defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ under the usual conditions, see Karatzas & Shreve (1988). We choose $d+1$ primary assets and d Wiener processes because our aim in this paper is to describe the complete market case. The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0,T]}$ is the augmentation under P of the natural filtration $\mathcal{A}^W$ generated by the vector $W = \{W(t) = (W^1(t), \dots, W^d(t))^\top,\ t \in [0,T]\}$ of independent, standard Wiener processes. For such a multi-asset market, we derive in the following section relationships that naturally exist between volatilities and risk premia.

2.1 Savings Accounts

Let us denote by $B^j(t)$ the savings account price at time t of the jth primary asset, when denominated in units of this asset. For the jth primary asset the jth savings account is assumed to satisfy the differential equation

$$dB^j(t) = B^j(t)\, f^j(t)\, dt \tag{2.1}$$

for $t \in [0,T]$ with $B^j(0) = 1$, where the jth short rate is predictable and such that

$$\int_0^T |f^j(s)|\, ds < \infty \tag{2.2}$$

a.s. for $j \in \{0, 1, \dots, d\}$. The jth short rate process $f^j = \{f^j(t),\ t \in [0,T]\}$ characterizes the *evolution of the time value* of the jth primary asset, $j \in \{0, 1, \dots, d\}$. Here $B^0(t)$ denotes the domestic savings account price and $f^0(t)$ the corresponding domestic short rate at time t. Other primary asset prices can be interpreted, for instance, as prices of shares, currencies and commodities, when delivered today or after a prescribed time period. This means a rollover savings account for one year zero coupon bonds can represent a primary asset, where the one year forward rate is the corresponding short rate.

The i,jth *exchange price* $X^{i,j}(t)$ describes the number of units of the ith primary asset that are exchanged at time $t \in [0,T]$ for one unit of the jth primary asset, $i, j \in \{0, 1, \dots, d\}$. For currencies the exchange price is the exchange rate. The quantity

$$S^{i,j}(t) = X^{i,j}(t)\, B^j(t) \tag{2.3}$$

denotes the value of the jth savings account at time $t \in [0,T]$ when measured in units of the ith primary asset $i, j \in \{0, 1, \dots, d\}$. Thus, $S^{0,j}(t)$ is the value of the savings account of the jth primary asset at time t when expressed in units of the domestic currency.

We assume in the following that a strong solution of the SDE

$$dS^{i,j}(t) = S^{i,j}(t) \left(a^{i,j}(t)\, dt + \sum_{k=1}^{d} b^{i,j,k}(t)\, dW^k(t) \right) \tag{2.4}$$

exists and is pathwise uniquely determined for $t \in [0, T]$ with $S^{i,j}(0) > 0$ for $i, j \in \{0, 1, \dots, d\}$, see Karatzas & Shreve (1988). We express without loss of generality the i, jth *appreciation rate* $a^{i,j}(t)$ as the sum

$$a^{i,j}(t) = f^i(t) + p^{i,j}(t) \tag{2.5}$$

for all $t \in [0, T]$ and $i, j \in \{0, 1, \dots, d\}$. Here the i, jth *risk premium* $p^{i,j}(t)$ is the expected return that an investor receives in excess of the ith short rate $f^i(t)$ for the risk incurred through holding the jth savings account in the ith denomination. The i, j, kth *volatility* $b^{i,j,k}(t)$ measures at time t the proportional fluctuations of the price of the savings account of the jth primary asset with respect to the kth Wiener process when this asset is denominated in units of the ith asset. To ensure the existence of the stochastic integrals in (2.4), the i, jth risk premium process $p^{i,j}$ and i, j, kth volatility process $b^{i,j,k}$ are assumed to be predictable and such that

$$\int_0^T \left\{ |p^{i,j}(s)| + (b^{i,j,k}(s))^2 \right\} ds < \infty \tag{2.6}$$

a.s. for all $i, j \in \{0, 1, \dots, d\}$ and $k \in \{1, 2, \dots, d\}$.

2.2 Portfolios and Strategies

Let us denote by $S^i = \{S^i(t) = (S^{i,0}(t), \dots, S^{i,d}(t))^\top,\ t \in [0, T]\}$ the vector process of savings accounts of primary assets expressed in units of the ith denomination. We call a stochastic process $\delta = \{\delta(t) = (\delta^0(t), \dots, \delta^d(t))^\top,\ t \in [0, T]\}$ a *strategy*, if δ is predictable and S^i-integrable for all $i \in \{0, 1, \dots, d\}$, see Karatzas & Shreve (1988). Here $\delta^j(t)$ is the number of units of the savings account of the jth primary asset that are held at time $t \in [0, T]$ in a corresponding portfolio, $j \in \{0, 1, \dots, d\}$. For a strategy δ, we denote by $V^i_\delta(t)$ the value of the corresponding *portfolio* at time t when measured in units of the ith primary asset such that

$$V^i_\delta(t) = \delta(t)\, S^i(t)^\top \tag{2.7}$$

for $t \in [0, T]$ and $i \in \{0, 1, \dots, d\}$. Note that for any strategy δ with a.s. strictly positive portfolio value $V^j_\delta(s) > 0$ for all $s \in [0, T]$ one can express the i, jth exchange price $X^{i,j}(t)$ at time t in the form

$$X^{i,j}(t) = \frac{V^i_\delta(t)}{V^j_\delta(t)} \tag{2.8}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\dots,d\}$. From (2.3) the savings account of the jth primary asset in the ith denomination can therefore be expressed as

$$S^{i,j}(t) = \frac{V^i_\delta(t)}{V^j_\delta(t)}\, B^j(t) \tag{2.9}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\dots,d\}$. A strategy δ is called *self-financing* if

$$dV^i_\delta(t) = \sum_{j=0}^{d} \delta^j(t)\, dS^{i,j}(t) \tag{2.10}$$

for all $t \in [0,T]$ and $i \in \{0,1,\dots,d\}$. Changes in the value of the portfolio are exactly matched by the corresponding gains from trade. By considering self-financing strategies one acknowledges the *conservation of value*. For a given self-financing strategy δ, let $\pi^j_\delta(t)$ denote the jth *proportion* of the value of the corresponding portfolio, which is invested at time t in the jth savings account. This proportion is given by the relation

$$\pi^j_\delta(t) = \frac{\delta^j(t)\, B^j(t)}{V^j_\delta(t)} = \delta^j(t)\, \frac{S^{i,j}(t)}{V^i_\delta(t)} \tag{2.11}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\dots,d\}$, where the second equation follows from (2.9). Note that by (2.7), for any given strategy δ the proportions always add to one, that is

$$\sum_{j=0}^{d} \pi^j_\delta(t) = 1 \tag{2.12}$$

for all $t \in [0,T]$.

For a given self-financing strategy δ, which has a.s. strictly positive corresponding portfolio value $V^i_\delta(t) > 0$ for all $t \in [0,T]$, $i \in \{0,1,\dots,d\}$, we introduce the corresponding i,kth *portfolio volatility*

$$\sigma^{i,k}_\delta(t) = \sum_{r=0}^{d} \pi^r_\delta(t)\, b^{i,r,k}(t) \tag{2.13}$$

for $k \in \{1,2,\dots,d\}$ and the ith *portfolio risk premium*

$$p^i_\delta(t) = \sum_{r=0}^{d} \pi^r_\delta(t)\, p^{i,r}(t). \tag{2.14}$$

The expression (2.13) characterizes the portfolio volatility with respect to the kth Wiener process under the ith denomination. For the above portfolio we obtain from (2.10), (2.4) and (2.11) with (2.13) the SDE

$$dV^i_\delta(t) = V^i_\delta(t) \left(\left\{ f^i(t) + p^i_\delta(t) \right\} dt + \sum_{k=1}^{d} \sigma^{i,k}_\delta(t)\, dW^k(t) \right) \tag{2.15}$$

for $t \in [0,T]$ and $i \in \{0,1,\dots,d\}$.

2.3 Growth Optimal Portfolio

From (2.15), (2.14) and (2.13) we obtain by application of the Itô formula for the logarithm of the domestic value $V_\delta^0(t)$ of an a.s. strictly positive portfolio the SDE

$$d\log(V_\delta^0(t)) = g_\delta^0(t)\,dt + \sum_{k=1}^{d} \sigma_\delta^{0,k}(t)\,dW^k(t) \tag{2.16}$$

with *domestic growth rate*

$$g_\delta^0(t) = f^0(t) + \sum_{j=0}^{d} \pi_\delta^j(t)\,p^{0,j}(t) - \frac{1}{2}\sum_{k=1}^{d}\left(\sum_{j=0}^{d} \pi_\delta^j(t)\,b^{0,j,k}(t)\right)^2 \tag{2.17}$$

for $t \in [0,T]$. We now choose in our multi-asset market a self-financing benchmark portfolio. In particular, we construct it to be such that its long term growth cannot be outperformed by any other portfolio. This portfolio is known as *growth optimal portfolio* (GOP), discovered by Kelly (1956). It achieves the maximum domestic growth rate at each time $t \in [0,T]$.

In the following, we denote a self-financing strategy that generates such a GOP by $\underline{\delta} = \{\underline{\delta}(t) = (\underline{\delta}^0(t), \dots, \underline{\delta}^d(t))^\top,\, t \in [0,T]\}$. A necessary condition for achieving the maximum domestic growth rate by a self-financing portfolio is obtained by setting the partial derivatives of the quadratic form $g_\delta^0(t)$ with respect to $\pi_\delta^j(t)$, $j \in \{1,2,\dots,d\}$, in (2.17) equal to zero. In particular, it follows that the proportions $\pi_{\underline{\delta}}^0(t), \dots, \pi_{\underline{\delta}}^d(t)$ must satisfy the linear system of equations

$$p^{0,j}(t) = \sum_{r=0}^{d} \pi_{\underline{\delta}}^r(t) \left(\sum_{k=1}^{d} b^{0,j,k}(t)\,b^{0,r,k}(t)\right), \tag{2.18}$$

for all $j \in \{1,2,\dots,d\}$ and $t \in [0,T]$ together with relation (2.12). This leads us to the formulation of the following assumption.

Assumption 2.1 *There exists a unique vector of proportions* $\pi_{\underline{\delta}}(t) = (\pi_{\underline{\delta}}^0(t), \dots, \pi_{\underline{\delta}}^d(t))^\top$*, which satisfies equation* (2.18) *for all* $j \in \{1,2,\dots,d\}$ *together with* (2.12) *for Lebesgue-almost every* $t \in [0,T]$.

This assumption is satisfied if and only if the matrix $\beta(t) = [\beta^{r,j}(t)]_{r,j=0}^d$ with

$$\beta^{r,j}(t) = \begin{cases} 1 & \text{for} \quad r = 0 \\ \sum_{k=1}^{d} b^{0,r,k}(t)\,b^{0,j,k}(t) & \text{for} \quad r \in \{1,2,\dots,d\} \end{cases} \tag{2.19}$$

for $j \in \{0,1,\dots,d\}$ is *invertible* for Lebesgue-almost every $t \in [0,T]$. Then we have

$$\pi_{\underline{\delta}}(t) = (\pi_{\underline{\delta}}^0(t), \dots, \pi_{\underline{\delta}}^d(t))^\top = \beta^{-1}(t)\,p^0(t) \tag{2.20}$$

with

$$p^0(t) = (1, p^{0,1}(t), \dots, p^{0,d}(t))^\top \tag{2.21}$$

for $t \in [0,T]$.

Theorem 2.2 *Under Assumption 2.1, there exists a unique GOP, satisfying in its ith denomination the SDE*

$$dV^i_{\underline{\delta}}(t) = V^i_{\underline{\delta}}(t) \left(\left(f^i(t) + \sum_{k=1}^{d} \left(\sigma^{i,k}_{\underline{\delta}}(t) \right)^2 \right) dt + \sum_{k=1}^{d} \sigma^{i,k}_{\underline{\delta}}(t)\, dW^k(t) \right). \tag{2.22}$$

The i, j, kth volatility has the form

$$b^{i,j,k}(t) = \sigma^{i,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t). \tag{2.23}$$

The i, jth risk premium is given by

$$p^{i,j}(t) = \sum_{k=1}^{d} \sigma^{i,k}_{\underline{\delta}}(t) \left(\sigma^{i,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t) \right) \tag{2.24}$$

and the GOP-volatilities satisfy the system of equations

$$\sum_{j=0}^{d} \sigma^{j,k}_{\underline{\delta}}(t)\, \pi^j_{\underline{\delta}}(t) = 0 \tag{2.25}$$

together with the normalization condition (2.12) *for* $t \in [0,T]$, $i, j \in \{0, 1, \dots, d\}$ *and* $k \in \{1, 2, \dots, d\}$.

The proof of the above theorem is given in Appendix A.

Note from (2.22) that the dynamics of the GOP has the same form for each denomination. Also with the resulting specifications (2.23) and (2.24) the dynamics of the savings accounts for the primary assets given by (2.4) has the same structure for each denomination.

3 Benchmark Model

Let us now establish the *benchmark model* by reparameterizing the model in terms of GOP-volatilities.

3.1 Asset Price Dynamics in the Benchmark Model

We first formulate the following assumption, which as we will see, can be used to replace Assumption 2.1.

Assumption 3.1 *There exist predictable processes* $\sigma^{i,k} = \{\sigma^{i,k}(t),\, t \in [0,T]\}$ *with*

$$\int_0^T (\sigma^{i,k}(s))^2 < \infty \tag{3.1}$$

a.s. for all $i \in \{0,1,\dots,d\}$ *and* $k \in \{1,2,\dots,d\}$, *that form the* volatility matrix $v(t) = [v^{k,i}(t)]_{k,i=0}^d$ *with*

$$v^{k,i}(t) = \begin{cases} 1 & \text{for} \quad k = 0 \\ \sigma^{i,k}(t) & \text{for} \quad k \in \{1,2,\dots,d\} \end{cases} \tag{3.2}$$

for all $t \in [0,T]$, *which is assumed to be* invertible *for Lebesgue-almost-every* $t \in [0,T]$.

The invertibility of the matrix $v(t)$ does not impose too much of a restriction on the asset price dynamics. To satisfy this condition, one needs to model a sufficiently diverse set of nonredundant primary assets. According to the structures established by Theorem 2.2, it is easy to see that the SDE (2.4) for the savings account of the jth primary asset, when denominated in units of the ith primary asset, can be written as

$$\begin{aligned} dS^{i,j}(t) &= S^{i,j}(t) \left(\left[f^i(t) + \sum_{k=1}^d \sigma^{i,k}(t) \left(\sigma^{i,k}(t) - \sigma^{j,k}(t)\right) \right] dt \right. \\ &\qquad \left. + \sum_{k=1}^d (\sigma^{i,k}(t) - \sigma^{j,k}(t))\, dW^k(t) \right) \end{aligned} \tag{3.3}$$

for all $t \in [0,T]$ and $i,j \in \{0,1,\dots,d\}$. A multi-asset market with all savings accounts $S^{i,j}$, $i,j \in \{0,1,\dots,d\}$ satisfying the above SDE and Assumption 3.1, is called a *benchmark model*.

Thus the benchmark model is obtained by specifying the i,j,kth savings account volatility $b^{i,j,k}(t)$ by the difference $\sigma_{\underline{\delta}}^{i,k}(t) - \sigma_{\underline{\delta}}^{j,k}(t)$ of GOP-volatilities and the i,jth risk premium in the form (2.24).

It can be shown under Assumption 3.1 and with all assets $S^{i,j}$, $i,j \in \{0,1,\dots,d\}$, satisfying the SDE (3.3), that there exists the vector of proportions

$$\begin{aligned} \pi_{\underline{\delta}}(t) &= (\pi_{\underline{\delta}}^0(t), \dots, \pi_{\underline{\delta}}^d(t))^\top \\ &= v^{-1}(t)\,(1,0,\dots,0)^\top, \end{aligned} \tag{3.4}$$

which satisfies the equation (2.18) together with (2.12). This allows us to replace Assumption 2.1 by Assumption 3.1. Note that if Assumption (2.1) is *not* satisfied we do *not* have a GOP.

Obviously, due to Theorem 2.2 the i,kth GOP-volatility is given by

$$\sigma_{\underline{\delta}}^{i,k}(t) = \sigma^{i,k}(t) \tag{3.5}$$

for $t \in [0,T]$, $i \in \{0,1,\ldots,d\}$ and $k \in \{1,2,\ldots,d\}$.

The different denominations of the GOP specify a benchmark model. These can be obtained from the GOP-volatilities, the short rates and the initial values of the denominations of the GOP. The above analysis shows us that the GOP is the central building block for the benchmark model.

3.2 Short Rates

According to (2.22) and (3.5), the short rate plays an important role in the ith denomination $V^i_{\underline{\delta}}$ of the GOP and consequently by (2.5) also in the ith appreciation rates. From equations (2.5) and (2.24), it follows that

$$f^i(t) = a^{i,j}(t) - \sum_{k=1}^{d} \sigma^{i,k}(t) \left(\sigma^{i,k}(t) - \sigma^{j,k}(t)\right) \tag{3.6}$$

for all $t \in [0,T]$ and $j \in \{1,2,\ldots,d\}$ and $i \in \{0,1,\ldots,d\}$. Thus, in the case $i=0$, the domestic short rate $f^0(t)$ can be obtained from the appreciation rate of any savings account process when denominated in domestic currency if the corresponding GOP-volatilities are known.

In addition, if we set in (3.6) $i=0$ and weight the appreciation rates $a^{0,j}(t)$ for $j \in \{1,2,\ldots,d\}$ by the corresponding GOP-proportions $\pi^j_{\underline{\delta}}(t)$, then the following statement follows by (3.6) and (2.25).

Remark 3.2 *The domestic short rate takes the form*

$$f^0(t) = \sum_{j=1}^{d} \left(\frac{\pi^j_{\underline{\delta}}(t)}{1-\pi^0_{\underline{\delta}}(t)} \right) a^{0,j}(t) - \sum_{k=1}^{d} \frac{(\sigma^{0,k}(t))^2}{1-\pi^0_{\underline{\delta}}(t)} \tag{3.7}$$

for $t \in [0,T]$.

Recall from (2.25) and (3.5), that the proportions of the GOP are determined by the GOP-volatilities. Therefore, the appreciation rates of the risky domestic assets together with the GOP-volatilities fully determine the domestic short rate. There is no freedom in a benchmark model to set the domestic short rate or to choose it exogenously. In the case when $\pi^0_{\underline{\delta}}(t)$ is close to zero, then by (3.7) the short rate is approximately the average of the appreciation rates of the risky domestic primary assets minus the sum of the squared volatilities of the domestic GOP. This provides an intuitive explanation for the nature of the domestic short rate. In a different framework a similar observation has been pointed out by Reiß, Schoenemakers & Schweizer (2001).

3.3 Risk Premium

Inspection of the SDE (2.22) for $i = 0$ shows that the appreciation rate of the domestic GOP must equal the short rate plus its squared volatility. In terms of returns, this feature can be interpreted as the optimal mean-variance property, emphasized in the seminal work by Markowitz (1959).

There has been a long standing debate between theorists and practitioners on how risk premia for asset prices should be modeled. This is referred to as the risk premium puzzle. Based on the use of the market portfolio, the CAPM provides some response to this problem, obtained under relatively strong equilibrium and utility based assumptions. However, the CAPM does not yield the same structure for the risk premia when one considers asset prices under different denominations. Under the benchmark approach, by using the GOP as numeraire, the same form for the risk premia, see (2.24), is obtained for *each* denomination. In the benchmark model, the risk premium is proportional to the covariance between the return of the given asset or portfolio and that of the corresponding denomination of the GOP, see (2.24) together with (3.3) and (2.22). To some extent this allows us to recover a key property of the CAPM under the benchmark model, where the covariance between the domestic return of the given asset and that of the market portfolio are defined to be the domestic risk premium.

3.4 Example for the Benchmark Model

In Platen (2000) an example for the benchmark model, the Minimal Market Model, with stochastic volatility is given. To provide a further example for the benchmark model one can simply choose deterministic GOP-volatilities and short rates, which results in a general Black-Scholes benchmark model.

Let us now consider another example with a volatility matrix $v(t)$, see (3.2), that can be directly inverted. This case arises when all volatilities of the domestic denomination of the GOP are a.s. strictly positive and the other GOP-volatilities are such that $\sigma^{0,k}(t) > 0$, $\sigma^{k,i}(t) = 0$ for $i \neq k$ and $\sigma^{k,k}(t) < 0$ for $t \in [0, T]$ and $k, i \in \{1, 2, \ldots, d\}$. For this special volatility structure it follows from (2.25) and (3.5) that

$$\pi^0_{\underline{\delta}}(t)\, \sigma^{0,k}(t) + \pi^k_{\underline{\delta}}(t)\, \sigma^{k,k}(t) = 0 \tag{3.8}$$

for $t \in [0, T]$ and $k \in \{1, 2, \ldots, d\}$. Thus one obtains the relation

$$\pi^k_{\underline{\delta}}(t) = -\frac{\sigma^{0,k}(t)}{\sigma^{k,k}(t)}\, \pi^0_{\underline{\delta}}(t) \tag{3.9}$$

for $k \in \{1, 2, \ldots, d\}$, where according to (2.12)

$$\pi^0_{\underline{\delta}}(t) = \frac{1}{1 - \sum_{k=1}^{d} \frac{\sigma^{0,k}(t)}{\sigma^{k,k}(t)}} \tag{3.10}$$

for $t \in [0,T]$. Consequently, in this case we obtain explicit expressions for the proportions of the GOP.

Using the above special volatility structure with constant volatilities $\sigma^{0,1}$ and $\sigma^{1,1}$, and short rates f^0 and f^1, we obtain the following *two asset Black-Scholes market*. For this example the domestic GOP $V^0_{\underline{\delta}}$, domestic savings account B^0 and savings account $S^{0,1}$ of the other asset satisfy the SDEs

$$\begin{aligned} dV^0_{\underline{\delta}}(t) &= V^0_{\underline{\delta}}(t)\left(\left[f^0 + (\sigma^{0,1})^2\right] dt + \sigma^{0,1}\, dW^1(t)\right), \\ dB^0(t) &= B^0(t)\, f^0\, dt, \\ dS^{0,1}(t) &= S^{0,1}(t)\left(\left[f^0 + \sigma^{0,1}\,(\sigma^{0,1} - \sigma^{1,1})\right] dt + (\sigma^{0,1} - \sigma^{1,1})\, dW^1(t)\right) \end{aligned} \tag{3.11}$$

for $t \in [0,T]$, respectively. Note that the domestic risk premium for $S^{0,1}$ is proportional to the covariance between the domestic returns of $V^0_{\underline{\delta}}$ and $S^{0,1}$. On the other hand, the prices of these securities, when expressed in units of the other asset, satisfy the SDEs

$$\begin{aligned} dV^1_{\underline{\delta}}(t) &= V^1_{\underline{\delta}}(t)\left(\left[f^1 + (\sigma^{1,1})^2\right] dt + \sigma^{1,1}\, dW^1(t)\right), \\ dS^{1,0}(t) &= S^{1,0}(t)\left(\left[f^1 + \sigma^{1,1}\,(\sigma^{1,1} - \sigma^{0,1})\right] dt + (\sigma^{1,1} - \sigma^{0,1})\, dW^1(t)\right), \\ dB^1(t) &= B^1(t)\, f^1\, dt \end{aligned} \tag{3.12}$$

for $t \in [0,T]$, respectively. Here again, the risk premium for $S^{0,1}$ is proportional to the covariance between the returns of $V^1_{\underline{\delta}}$ and $S^{1,0}$. In general, the CAPM would not generate such risk premia unless the GOP coincides with the market portfolio. Note that by (3.9) and (3.10), the GOP-proportions are given by the expressions

$$\pi^0_{\underline{\delta}}(t) = \frac{1}{1 - \frac{\sigma^{0,1}}{\sigma^{1,1}}} \quad \text{and} \quad \pi^1_{\underline{\delta}}(t) = \frac{1}{1 - \frac{\sigma^{1,1}}{\sigma^{0,1}}}. \tag{3.13}$$

For illustration, let us simulate this two asset Black-Scholes market over a ten year period, that is $T = 10$, with domestic short rate $f^0 = 0.05$, short rate of the other primary asset $f^1 = 0.05$ and volatilities $\sigma^{0,1} = -\sigma^{1,1} = 0.1$. Here we set, for simplicity, $B^0(0) = S^{0,1}(0) = S^{1,0}(0) = B^1(0) = 1$. Trajectories for the domestic GOP $V^0_{\underline{\delta}}(t)$ and the savings accounts $B^0(t)$ and $S^{0,1}(t)$ are shown in Figure 1. The corresponding securities in the denomination of the other primary asset are plotted in Figure 2. Note that only the Wiener process W^1 drives this two asset market dynamics. The paths of $S^{0,1}$ and V^0 as well as $S^{1,0}$ and V^1 appear to be positively correlated, whereas $S^{0,1}$ and $S^{1,0}$, as well as V^0 and V^1, are negatively correlated.

4 Arbitrage under the Benchmark Model

We now demonstrate that the benchmark model permits, in general, some form of arbitrage, which can be directly expressed.

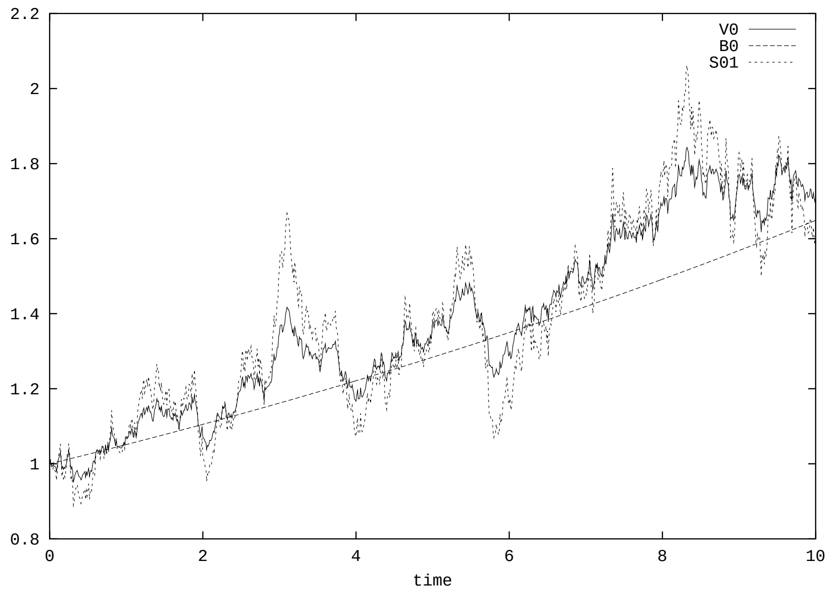

Figure 1: $V^0_{\underline{\delta}}$ and savings accounts B^0 and $S^{0,1}$ in the domestic market.

4.1 Benchmarked Savings Accounts

We introduce the jth *benchmarked savings account process* $\hat{S}^j = \{\hat{S}^j(t),\, t \in [0,T]\}$, which is obtained by using the GOP as benchmark for the jth savings account, that is

$$\hat{S}^j(t) = \frac{B^j(t)}{V^j_{\underline{\delta}}(t)} = \frac{S^{i,j}(t)}{V^i_{\underline{\delta}}(t)} \tag{4.1}$$

for $t \in [0,T]$, $i, j \in \{0,1,\dots,d\}$, see (2.9). We assume that the quadratic variation of the jth benchmarked savings account remains finite, that is, $\langle \hat{S}^j \rangle_T < \infty$ a.s., for all $j \in \{0,1,\dots,d\}$. By application of the Itô formula, one obtains from (4.1), (2.1) (2.22) and (3.5), for $\hat{S}^j(t)$, the SDE

$$d\hat{S}^j(t) = -\hat{S}^j(t) \sum_{k=1}^{d} \sigma^{j,k}(t)\, dW^k(t) \tag{4.2}$$

for all $t \in [0,T]$ and $j \in \{0,1,\dots,d\}$. Note that this SDE is driftless, which means that $\hat{S}^j$ is an $(\underline{\mathcal{A}}, P)$-local martingale, $j \in \{0,1,\dots,d\}$ and since the benchmarked savings account $\hat{S}^j$ is nonnegative, it is an $(\underline{\mathcal{A}}, P)$-*supermartingale*, see Karatzas & Shreve (1988).

4.2 Locally Arbitrage Free Portfolios

Let us denote by $\hat{S} = \{\hat{S}(t) = (\hat{S}^0(t), \dots, \hat{S}^d(t))^\top,\, t \in [0,T]\}$ the vector process of benchmarked savings accounts. For a given self-financing strategy $\delta = \{\delta(t) =$

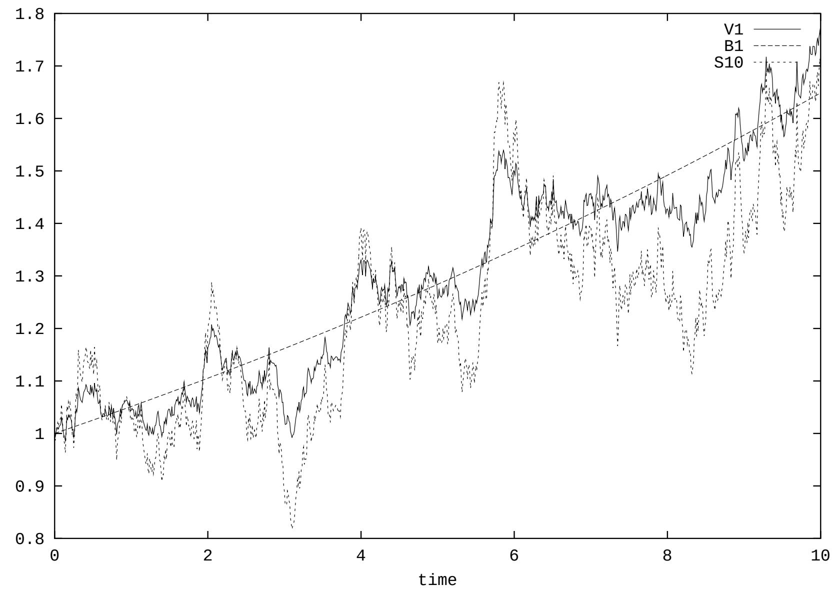

Figure 2: $V^1_{\underline{\delta}}$, B^1 and $S^{1,0}$ in units of the other primary asset.

$(\delta^0(t), \ldots, \delta^d(t))^\top$, $t \in [0,T]\}$, which is assumed to be $\hat{S}$-integrable, see Karatzas & Shreve (1988), we introduce the corresponding *benchmarked portfolio process* $\hat{V}_\delta = \{\hat{V}_\delta(t),\, t \in [0,T]\}$ given by

$$\hat{V}_\delta(t) = \sum_{j=0}^{d} \delta^j(t)\, \hat{S}^j(t) \tag{4.3}$$

for $t \in [0,T]$. Thus it follows from (2.7) and (4.1) that

$$\hat{V}_\delta(t) = \frac{V^i_\delta(t)}{V^i_{\underline{\delta}}(t)} \tag{4.4}$$

for $t \in [0,T]$ and $i \in \{0,1,\ldots,d\}$.

Similar to Jamshidian (1997), we say that a benchmarked portfolio process $\hat{V}_\delta$, associated with an $\hat{S}$-integrable, self-financing strategy δ, is *locally arbitrage free* if $\hat{V}_\delta$ is an $(\underline{\mathcal{A}}, P)$-local martingale. This allows us to prove the following result.

Lemma 4.1 *In a benchmark model, any benchmarked portfolio process $\hat{V}_\delta$ that corresponds to an $\hat{S}$-integrable, self-financing strategy δ is locally arbitrage free.*

Proof: For given $i \in \{0,1,\ldots,d\}$ and an $\hat{S}$-integrable, self-financing strategy δ

we obtain from (4.4) and (2.10) by the Itô formula with (4.1) the SDE

$$
\begin{aligned}
d\hat{V}_\delta(t) &= \frac{1}{V^i_{\underline{\delta}}(t)}\, dV^i_\delta(t) + V^i_\delta(t)\, d\left(\frac{1}{V^i_{\underline{\delta}}(t)}\right) + d\left\langle V^i_\delta, \frac{1}{V^i_{\underline{\delta}}}\right\rangle_t \\
&= \sum_{j=0}^{d} \delta^j(t) \left\{ \frac{dS^{i,j}(t)}{V^i_{\underline{\delta}}(t)} + S^{i,j}(t)\, d\left(\frac{1}{V^i_{\underline{\delta}}(t)}\right) + d\left\langle S^{i,j}, \frac{1}{V^i_{\underline{\delta}}}\right\rangle_t \right\} \\
&= \sum_{j=0}^{d} \delta^j(t)\, d\left(\frac{S^{i,j}(t)}{V^i_{\underline{\delta}}(t)}\right) \\
&= \sum_{j=0}^{d} \delta^j(t)\, d\hat{S}^j(t) \qquad (4.5)
\end{aligned}
$$

for $t \in [0,T]$. Thus by (4.2), $\hat{V}_\delta$ is an $(\underline{\mathcal{A}}, P)$-local martingale. □

Since any nonnegative local martingale is a supermartingale, see Karatzas & Shreve (1988), we can state the following result as a direct consequence.

Lemma 4.2 *In a benchmark model, any nonnegative benchmarked portfolio process $\hat{V}_\delta$, which corresponds to an $\hat{S}$-integrable, self-financing strategy δ, is an $(\underline{\mathcal{A}}, P)$-supermartingale.*

This fundamental property of the benchmark model is based on its structure that is determined via the denominations of the GOP.

4.3 Arbitrage Amounts

For an $\hat{S}$-integrable, self-financing strategy δ with corresponding a.s. nonnegative, square integrable, benchmarked portfolio process $\hat{V}_\delta$ and a stopping time $\bar{T} \in [0,T]$ we define at time $t \in [0,\bar{T}]$ the corresponding *benchmarked arbitrage amount* as the difference

$$\hat{A}^{\bar{T}}_\delta(t) = \hat{V}_\delta(t) - E\left(\hat{V}_\delta(\bar{T}) \,\middle|\, \mathcal{A}_t\right). \qquad (4.6)$$

Note that the benchmarked arbitrage amount $\hat{A}^{\bar{T}}_\delta(t)$ is a.s. nonnegative for all $t \in [0,\bar{T}]$ due to the supermartingale property of $\hat{V}_\delta$. We assume here that the above $\hat{V}_\delta$ corresponds to actually traded benchmarked prices.

We call a square integrable, benchmarked portfolio process $\hat{V}_\delta$, corresponding to an $\hat{S}$-integrable, self-financing strategy δ, *arbitrage free* if it has arbitrage amount $\hat{A}^{\bar{T}}_\delta(t) = 0$ for all stopping times $\bar{T} \in [0,T]$ and $t \in [0,\bar{T}]$. Obviously, due to (4.6), a benchmarked arbitrage free portfolio process is a square integrable $(\underline{\mathcal{A}}, P)$-martingale. Due to a well-known property of the expected quadratic variation of square integrable, local martingales, see Karatzas & Shreve (1988), one can use the following result to distinguish between benchmarked portfolios that are arbitrage free and those that are not.

Lemma 4.3 *A square integrable, benchmarked portfolio process $\hat{V}_\delta$, which corresponds to an $\hat{S}$-integrable self-financing strategy δ is arbitrage free if and only if it has finite expected quadratic variation, that is $E(\langle \hat{V}_\delta \rangle_T) < \infty$.*

For a given benchmark model, we denote by Φ the *set of benchmarked arbitrage free portfolio processes.* The Martingale Representation Theorem, see Karatzas & Shreve (1988), allows us to establish the following result.

Theorem 4.4 *For any benchmarked arbitrage free portfolio process $u \in \Phi$ there exists a progressively measurable process $x_u = \{x_u(t) = (x_u^1(t), \dots, x_u^d(t))^\top$, $t \in [0,T]\}$ such that*

$$E\left(\int_0^T \left(x_u^k(s)\right)^2 \, ds\right) < \infty \tag{4.7}$$

for $k \in \{1, 2, \dots, d\}$ and

$$u(t) = u(0) + \sum_{k=1}^{d} \int_0^t x_u^k(s) \, dW^k(s) \tag{4.8}$$

for $t \in [0,T]$, where u is a.s. continuous. Furthermore, if the integrands $\tilde{x}_u^k$, $k \in \{1, 2, \dots, d\}$, are any other progressively measurable processes, satisfying (4.7) and (4.8), then

$$\int_0^T \sum_{k=1}^{d} \left|x_u^k(s) - \tilde{x}_u^k(s)\right|^2 \, ds = 0 \tag{4.9}$$

a.s.

The above theorem allows us to establish a *unique* representation for any benchmarked arbitrage free portfolio process. Note that in a benchmark model where not all benchmarked portfolio processes are arbitrage free, Theorem 4.4 still applies to the subclass Φ of arbitrage free portfolios. Only for this subclass we obtain what is typically provided by the APT.

In the benchmark framework one has the freedom to model the formation of primary asset prices as a consequence of demand, supply and other market forces. These prices may not be formed via conditional expectations as is strictly required by the APT. Arbitrage amounts can be explicitly expressed. Note however that these would be typically minimized as a market matures and models become more accurate.

4.4 Standard Arbitrage

There exists an extensive literature on various important notions relating to no-arbitrage, see, for instance, Harrison & Kreps (1979), Harrison & Pliska (1981), Föllmer & Sondermann (1986), Föllmer & Schweizer (1991) or Delbaen & Schachermayer (1994), The following definition is similar to the standard no-arbitrage condition formulated, for instance, in Karatzas & Shreve (1998). Note that we consider benchmarked portfolios and not domestic savings account discounted portfolios. Also the following no-arbitrage definition applies to portfolios and not to the overall model. Finally, it refers to the real world measure and not some equivalent martingale measure.

We say that a nonnegative benchmarked portfolio process $\hat{V}_\delta$ that corresponds to a self-financing strategy δ, which is $\hat{S}$-integrable, is *arbitrage free in the standard sense* if $\hat{V}_\delta$ is an $(\underline{\mathcal{A}}, P)$-supermartingale.

This means, that a portfolio that is not arbitrage free in the standard sense, allows us to generate strictly positive wealth out of nothing with strictly positive probability as can be shown in the classical manner, see, for instance, Karatzas & Shreve (1998).

From the supermartingale property established by Lemma 4.2, we obtain the following result.

Remark 4.5 *In a benchmark model any nonnegative portfolio process $\hat{V}_\delta$ with corresponding $\hat{S}$-integrable, self-financing strategy δ is arbitrage free in the standard sense.*

4.5 Prices for Contingent Claims

Let us now define a *maturity date* $\bar{T} \in [0, T]$ as a stopping time. For a given maturity date $\bar{T} \in [0, T]$ we call an $\mathcal{A}_{\bar{T}}$-measurable, square integrable random variable $\hat{H}_{\bar{T}}$ a *benchmarked contingent claim.* Using Proposition 4.18 in Karatzas & Shreve (1988), the following theorem can be directly obtained.

Theorem 4.6 *For any contingent claim $\hat{H}_{\bar{T}}$, there exists a unique progressively measurable vector process x_u satisfying (4.7), such that*

$$\hat{H}_{\bar{T}} = E\left(\hat{H}_{\bar{T}} \,\middle|\, \mathcal{A}_t\right) + \sum_{k=1}^{d} \int_t^{\bar{T}} x_u^k(s)\, dW^k(s) \tag{4.10}$$

a.s. for $t \in [0, \bar{T}]$.

The benchmarked arbitrage free portfolio value for the benchmarked contingent claim $\hat{H}_{\bar{T}}$ is the conditional expectation

$$\hat{V}_{\delta_{\hat{H}_{\bar{T}}}}(t) = E\left(\hat{H}_{\bar{T}} \,|\, \mathcal{A}_t\right) \tag{4.11}$$

for $t \in [0, \bar{T}]$, where $\delta_{\hat{H}_{\bar{T}}}$ is the strategy that is determined by the representation (4.10). This strategy can be explicitly determined in a Markovian multi-factor version of the benchmark model, see Heath & Platen (2001). For the subclass of arbitrage free portfolio processes the above theorem gives us access to the corresponding *arbitrage free prices* of contingent claims via conditional expectations. It thus allows us to obtain arbitrage free derivative prices when the risk neutral methodology fails. This case arises, for instance, when some benchmarked savings accounts are strict local martingales. In particular, when the Radon-Nikodym derivative process $\Lambda = \{\Lambda(t) = \frac{\hat{S}^0(t)}{\hat{S}^0(0)}, \, t \in [0,T]\}$ is a strict local martingale, then we do not have an equivalent martingale measure, which is essential for the APT. The benchmark approach provides us with arbitrage free prices, even when the equivalent martingale measure does not exist.

As shown in Heath & Platen (2001), in a complete market, as considered in this paper, the above benchmark model permits perfect hedging also in cases where arbitrage arises. However, the hedge is no longer unique. In a market that is made incomplete by not allowing trade in certain assets, the Föllmer & Schweizer decomposition (1991), see also Föllmer & Sondermann (1986), provides in the arbitrage free case a natural hedge that is related to local risk minimization.

If in our benchmark model all benchmarked portfolio processes are arbitrage free, then the APT can be applied. Consequently, in this setting the benchmark approach yields the same results as the APT.

Conclusion

This paper presents a benchmark model for financial markets, which is constructed on the basis of the different denominations of the growth optimal portfolio. The risk premia for primary assets and portfolios have been identified and do not depend on the denomination. Furthermore, the domestic short rate is determined as a functional of an average of appreciation rates. It has been shown that the benchmark model is locally arbitrage free and that all nonnegative benchmarked portfolio prices are supermartingales. Contrary to the classical equivalent martingale measure approach, the benchmark framework permits a certain form of arbitrage. Arbitrage amounts can be expressed as they occur after shocks or market turbulence in emerging and maturing markets.

A Appendix

At first we establish a lemma about some symmetry in volatilities.

Lemma A.1 *For any self-financing strategy δ, which has an a.s. strictly positive corresponding portfolio process, the corresponding portfolio volatilities satisfy the relation*

$$\sigma_\delta^{i,k}(t) - \sigma_\delta^{j,k}(t) = b^{i,j,k}(t) = -b^{j,i,k}(t) \tag{A.1}$$

for $t \in [0,T]$, $i,j \in \{0,1,\ldots,d\}$ and $k \in \{1,2,\ldots,d\}$.

Proof: Using (2.15), one obtains by the Itô formula together with (2.3), (2.8) and (2.1) the SDE

$$\begin{aligned} dS^{i,j}(t) &= d\left(\frac{V_\delta^i(t)\,B^j(t)}{V_\delta^j(t)}\right) \\ &= S^{i,j}(t)\left(\left\{f^i(t) + \sum_{r=0}^{d} \pi_\delta^r(t)\,(p^{i,r}(t) - p^{j,r}(t))\right.\right. \\ &\qquad \left. + \sum_{k=1}^{d} \sigma_\delta^{j,k}(t)\,\left(\sigma_\delta^{j,k}(t) - \sigma_\delta^{i,k}(t)\right)\right\} dt \\ &\qquad \left. + \sum_{k=1}^{d} \left(\sigma_\delta^{i,k}(t) - \sigma_\delta^{j,k}(t)\right) dW^k(t)\right) \end{aligned} \tag{A.2}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\ldots,d\}$. A comparison of the SDEs (A.2) and (2.4) reveals the symmetry (A.1) in the portfolio volatilities. Note that the i,j,kth volatility $b^{i,j,k}(t)$ is independent of the strategy δ. □

Proof of Theorem 2.2

Under Assumption 2.1, there exists only one unique vector of proportions that maximizes the quadratic form (2.17). Therefore the GOP is uniquely determined by the resulting unique vector of proportions $\pi_{\underline{\delta}}(t)$ with corresponding self-financing strategy $\underline{\delta}$. Note that (2.23) follows directly from (A.1). From (2.18) and (2.13) we see that

$$\begin{aligned} \sum_{j=0}^{d} p^{0,j}(t)\,\pi_{\underline{\delta}}^j(t) &= \sum_{k=1}^{d}\left(\sum_{r=0}^{d} \pi_{\underline{\delta}}^r(t)\,b^{0,r,k}(t)\right)^2 \\ &= \sum_{k=1}^{d}\left(\sigma_{\underline{\delta}}^{0,k}(t)\right)^2 \end{aligned} \tag{A.3}$$

for $t \in [0,T]$. It follows from (2.15) and (A.3) that the obtained GOP must satisfy the SDE

$$dV^0_{\underline{\delta}}(t) = V^0_{\underline{\delta}}(t)\left(\left[f^0(t) + \sum_{k=1}^{d}(\sigma^{0,k}_{\underline{\delta}}(t))^2\right]dt + \sum_{k=1}^{d}\sigma^{0,k}_{\underline{\delta}}(t)\,dW^k(t)\right) \tag{A.4}$$

for $t \in [0,T]$. In addition, from (2.18) with (2.23) and (2.13) it can be concluded that the $0,j$th domestic risk premium is

$$p^{0,j}(t) = \sum_{k=1}^{d}\left(\sigma^{0,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t)\right)\sigma^{0,k}_{\underline{\delta}}(t) \tag{A.5}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$. Thus one obtains for $S^{0,j}$ by (2.4), (2.23) and (A.5) the SDE

$$\begin{aligned} dS^{0,j}(t) \;&=\; S^{0,j}(t)\left(\left[f^0(t) + \sum_{k=1}^{d}\sigma^{0,k}_{\underline{\delta}}(t)\left(\sigma^{0,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t)\right)\right]dt\right. \\ &\qquad \left. + \sum_{k=1}^{d}\left(\sigma^{0,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t)\right)dW^k(t)\right) \end{aligned} \tag{A.6}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$. By application of the Itô formula together with (2.3), (2.8), (A.4) and (A.6) the SDE for $V^j_{\underline{\delta}}(t)$ becomes

$$\begin{aligned} dV^j_{\underline{\delta}}(t) \;&=\; d\left(\frac{V^0_{\underline{\delta}}(t)\,B^j(t)}{S^{0,j}(t)}\right) \\ &=\; V^j_{\underline{\delta}}(t)\left(\left[f^j(t) + \sum_{k=1}^{d}\left(\sigma^{j,k}_{\underline{\delta}}(t)\right)^2\right]dt + \sum_{k=1}^{d}\sigma^{j,k}_{\underline{\delta}}(t)\,dW^k(t)\right) \end{aligned} \tag{A.7}$$

for $t \in [0,T]$ and $j \in \{0,1,\ldots,d\}$, which yields (2.22). Furthermore, by application of the Itô formula using (2.3), (2.8), (A.4), (A.7) and (2.1) it follows for $S^{i,j}(t)$ the SDE

$$\begin{aligned} dS^{i,j}(t) \;&=\; S^{i,j}(t)\left(\left[f^i(t) + \sum_{k=1}^{d}\sigma^{i,k}_{\underline{\delta}}(t)\left(\sigma^{i,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t)\right)\right]dt\right. \\ &\qquad \left. + \sum_{k=1}^{d}\left(\sigma^{i,k}_{\underline{\delta}}(t) - \sigma^{j,k}_{\underline{\delta}}(t)\right)dW^k(t)\right) \end{aligned} \tag{A.8}$$

for $t \in [0,T]$ and $i,j \in \{0,1,\ldots,d\}$. A comparison of the drift coefficients of the SDEs (A.8) and (2.4) shows that the i,jth risk premium must be of the form

(2.24). We compute from the self-financing property (2.10) with (A.8) the SDE for $V^i_{\underline{\delta}}(t)$ to obtain

$$dV^i_{\underline{\delta}}(t) = V^i_{\underline{\delta}}(t) \left(\left[f^i(t) + \sum_{k=1}^{d} \sigma^{i,k}_{\underline{\delta}}(t) \left(\sigma^{i,k}_{\underline{\delta}}(t) - \sum_{j=0}^{d} \pi^j_{\underline{\delta}}(t)\, \sigma^{j,k}_{\underline{\delta}}(t) \right) \right] dt + \sum_{k=1}^{d} \left(\sigma^{i,k}_{\underline{\delta}}(t) - \sum_{j=0}^{d} \pi^j_{\underline{\delta}}(t)\, \sigma^{j,k}_{\underline{\delta}}(t) \right) dW^k(t) \right) \tag{A.9}$$

for $t \in [0, T]$ and $i \in \{0, 1, \ldots, d\}$. A comparison of the diffusion and drift coefficients of the SDEs (A.9) and (A.7) reveals that the proportions of the obtained GOP must satisfy (2.25). □

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