On $L^2$-stability of solutions of
Linear Stochastic Delay Differential Equations

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Version: Berlin, 03.12.2003

Abstract

Stochastic Delay Differential Equations (SDDE) are Stochastic Functional Differential Equations with important applications. It is of interest to characterize the $\mathcal{L}^2$-stability (stability of second moments) of solutions of SDDE. For the class of linear, scalar SDDE we can show that second moment function of the solution satisfies a partial differential equation (PDE) with time delay and derive a characteristic equation from it determining the asymptotic behaviour of the second moments. Additionally we derive a necessary criterion for weak stationarity of solutions of linear SDDE.

Keywords: SDDE, SFDE, stochastic delay equations, stability, characteristic equation

AMS 2000 Subject Classification: 34K50, 34F05, 60Hxx

1 Introduction

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space provided with an increasing right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,\infty)}$. Stochastic Delay Differential Equations (SDDE)

$$
\begin{align*}
    dX(t) &= F(t,X_t)\,dt + G(t,X_t)\,dW(t), \quad t \in [0,\infty), \\
    X(t) &= \xi(t) \quad t \in [-\tau,0].
\end{align*}
$$

where

$(W(t),\mathcal{F}_t)_{t \in [0,\infty)}$ is a standard Brownian motion,

$\tau \in [0,\infty)$ is the non-negative delay-time,

$X_t = \{X(s) \mid s \in [t-\tau,t]\} \subset \mathcal{C}([-\tau,0],\mathbb{R}), t \in [0,\infty)$ is a segment at $t$,

$\xi \in \mathcal{C}([-\tau,0],\mathbb{R})$ is a $\mathcal{F}_0$-measurable random variable,

$F,G \in \mathcal{C}(\mathbb{R} \times \mathcal{C}([-\tau,0],\mathbb{R}),\mathcal{C}([-\tau,0],\mathbb{R}))$ are parameter functions.

are a standard tool in the modelling of realistic phenomena in biology, physics, economics in particular in presence of time-delayed feedback processes.

It is of interest to investigate the long term behaviour of solutions of SDDE in form of the time evolution of the second moment of solutions, that is $m_2^X(t) = \mathbb{E}[|X(t)|^2]$ for $t \to \infty$. For general SDDE like (1.1) it is hard to achieve exact results due to possible non-linearity of the parameter functions $F, G$ and due to the memory. However, in the case of linear, time-homogeneous SDDE one can consider the so-called comoment function $K_2^X(t,s) = \mathbb{E}[X(t)X(s)]$, into which $m_2^X$ is embedded. Using Fubini theorems and the linearity of integrals we can show that $K_2^X$ satisfies a certain PDE with time delay which is called (time-continuous) amplified system. In addition to this we will show that solutions of amplified systems have some special properties. One of the important properties is that its solutions are completely determined by its values on a diagonal „strip“ $D_\tau^2 = \{ (t,s) \mid s,t \in [-\tau,\infty), |t-s| \leq \tau \}$. This allows to use results from the semigroup theory to derive a characteristic equation in (complex)

*This work was financially supported by Deutsche Forschungsgemeinschaft (SFB 373)

Thanks to Prof.U.Klöcher, Dr.M.Reiß and Prof.C.T.H.Baker from Humboldt University of Berlin and Prof.C.T.H.Baker from the University of Manchester for fruitful discussions and helpful comments.
eigenvalues and corresponding eigenfunctions. The real part of the eigenvalues determines the behaviour of $m(t)$ for $t \to \infty$. A second result from the investigation of the amplified system is a necessary criterion for weak stationarity of solutions of linear, scalar SDDE.

As an example we want to consider a particular linear, scalar SDDE:

$$dX(t) = \left( a_0 X(t) + a_1 X(t - \tau) \right) dt + \left( b_0 X(t) + b_1 X(t - \tau) \right) dW(t), \quad t \in [0, \infty),$$

with real parameters $a_0$, $a_1$, $b_0$, $b_1$ and its discretizations. This will give us an idea how the amplified system for (1.3) looks like. Furthermore this provides us with an important link to the time-evolution of those stochastic recurrence relations which are the result from the application of numerical methods to linear SDDE. Let us choose here the numerical method "Euler explicit" and set the discretization step width to $h = \tau$. Then we obtain a numerical solution $Y = \{Y_n\}_{n \in \mathbb{N}}$ satisfying

$$Y_{n+1} = Y_n + h \left( a_0 Y_n + a_1 Y_{n-1} \right) + \sqrt{h} \left( b_0 Y_n + b_1 Y_{n-1} \right) \epsilon_{n+1}, \quad n \in \mathbb{N},$$

where $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a sequence of i.i.d. $N(0,1)$-distributed random variables. As in [4] was pointed out the second moments of $\{Y_n\}_{n \in \mathbb{N}}$ satisfy a deterministic recurrence relation - the so-called (time-discrete) amplified system. Following the notations in [4] it is:

$$Z_{n+1} = AZ_n$$

where

$$Z_{n+1} = \left( E[Y_{n+1}^2], E[Y_{n+1}Y_n], E[Y_{n+1}] \right)^T,$$

$$A = \begin{pmatrix}
(1 + a_0 h)^2 + b_0^2 h & 2(1 + a_0 h)a_1 h + 2b_0 b_1 h & (a_1 h)^2 + b_1^2 h \\
1 + a_0 h & a_1 h & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (1.6)$$

The first row of this recurrence reads as

$$E[Y_{n+1}^2] = E[Y_n^2] + h \left( 2a_0 + b_0^2 \right) E[Y_n^2] + 2(a_1 + b_0 b_1) E[Y_n Y_{n-1}] + b_1^2 E[Y_{n-1}^2] + h^2 \left( a_0^2 + a_1^2 \right) E[Y_n^2] + 2a_0 a_1 E[Y_n Y_{n-1}] + a_1^2 E[Y_{n-1}^2]$$

Summing up both sides over the first $n$ recurrence elements results in

$$E[Y_{n+1}^2] = E[Y_n^2] + \sum_{k=0}^{n} \left( 2a_0 E[Y_k^2] + 2a_1 E[Y_k Y_{k-1}] + b_0^2 E[Y_k^2] + 2b_0 b_1 E[Y_k Y_{k-1}] + b_1^2 E[Y_{k-1}^2] \right) h$$

$$+ \sum_{k=0}^{n} \left( (a_0^2 + b_0^2) E[Y_k^2] + 2a_0 a_1 E[Y_k Y_{k-1}] + a_1^2 E[Y_{k-1}^2] \right) h.$$ \quad (1.8)

Assuming the equidistant lattice $(t_0 - \tau) + \tau n$ and considering the step functions $\sum_{n=0}^{\infty} Y_n 1_{[t_n, t_{n+1})}$, equation (1.8) is equivalent to

$$E[X_h(t_{n+1})^2] = E[X_h(t_0)^2]$$

$$+ 2\sum_{k=0}^{n} \left( a_0 E[X_h(t_k)^2] + a_1 E[X_h(t_k) X_h(t_{k-1})] \right) (t_{k+1} - t_k)$$

$$+ \sum_{k=0}^{n} \left( b_0^2 E[X_h(t_k)^2] + 2b_0 b_1 E[X_h(t_k) X_h(t_k - \tau)] + b_1^2 E[X_h(t_k - \tau)^2] \right) (t_{k+1} - t_k)$$

$$+ h \sum_{k=0}^{n} \left( (a_0^2 + b_0^2) E[X_h(t_k)^2] + 2a_0 a_1 E[X_h(t_k) X_h(t_{k-1})] + a_1^2 E[X_h(t_k - \tau)^2] \right) (t_{k+1} - t_k)$$

$$= E[X_h(t_0)^2]$$

$$+ 2 \int_{t_0}^{t_{n+1}} \left( a_0 E[X_h(s)^2] + a_1 E[X_h(s) X_h(s - \tau)] \right) ds$$

$$+ \int_{t_0}^{t_{n+1}} \left( b_0^2 E[X_h(s)^2] + 2b_0 b_1 E[X_h(s) X_h(s - \tau)] + b_1^2 E[X_h(s - \tau)^2] \right) ds$$

$$+ h \int_{t_0}^{t_{n+1}} \left( (a_0^2 + b_0^2) E[X_h(s - \tau)^2] + 2a_0 a_1 E[X_h(s) X_h(s - \tau)] + a_1^2 E[X_h(s - \tau)^2] \right) ds. \quad (1.9)$$
If we let \( h \) converge down to zero in (1.9) and take into account that then step-function \( X_h \) converges to the exact solution \( X \) of (1.3) in the Delfour-Mitter space \( \mathcal{M}([-\tau,T],\mathbb{R}) \) (see [6]) we expect \( X \) to satisfy:

\[
E[X(t)^2] = E[X(0)^2] + \int_0^t 2a_0E[X(s)^2] + 2a_1E[X(s)X(s-\tau)] \, ds \\
+ \int_0^t b_0^2E[X(s)^2] + 2b_1E[X(s-\tau)^2] + b_1^2E[X(s-\tau)^2] \, ds.
\]

(1.10)

The second row of the recurrence relation (1.5) reads as

\[
E[Y_{n+1}Y_n] = E[Y_n^2] + h(a_0 E[Y_n^2] + a_1 E[Y_nY_{n-1})].
\]

(1.11)

Here summing up does not lead to the cancellation of summands on both sides. Furthermore, increasing the dimension of the discrete problem by letting \( h \) tending to 0 leads to possibly infinite sums. However we observe that (1.12) is equivalent to

\[
\frac{1}{h}(E[Y_{n+1}Y_n] - E[Y_n^2]) = (a_0 E[Y_n^2] + a_1 E[Y_nY_{n-1})]
\]

or

\[
\frac{1}{h}(E[X(t+h)X(t)] - E[X(t)^2]) = 2(a_0E[X(t)^2] + a_1E[X(t)X(t-\tau)]) \]

\[
+ ((b_0^2E[X(t)^2] + 2b_1E[X(t)X(t-\tau)] + b_1^2E[X(t-\tau)^2])
\]

(1.13)

\[
+ h(2a_0a_1)E[X(t)X(t-\tau)] + a_0^2E[X(t-\tau)^2].
\]

(1.14)

The equations (1.13) and (1.14) show that for the numerical method „Euler explicit“ the recurrence relation (1.5) of the second order moment of the generated numerical solution \( Y \) can be understood as a system of equations involving particular difference quotients. Like (1.10) this leads to a conjecture for a (time-continuous) amplified system of the linear SDDE: a PDE with time-delay satisfied by the comoment function of the exact solution \( X \) of (1.7).

In this paper we do not want to consider the relation between time-discretized and time-continuous amplified systems. Instead our focus lies on the time-continuous amplified systems. In section 2 we derive the PDE with time-delay which is satisfied by the comoment function \( K_2X \) of the exact solution. We also consider existence and uniqueness of solutions of this PDE and its smoothness properties. In section 3 we introduce a strong continuous semigroup associated to the (time-continuous) amplified system as well as a characteristic equation in eigenvalues and eigenfunctions of its generator. In section 4 we derive necessary criteria for the weak stationarity of solutions of scalar, linear SDDE. Section 5 provides some additional knowledge which is used. Furthermore it contains the proofs of statements which are too long or not instructive in their respective sections.

2 The amplified system embedding the second moments of a linear SDDE

As usual we assume a probability space \((\Omega,\mathcal{F},\mathbb{P})\) which is provided with an increasing, right continuous filtration \( \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,\infty)} \). Let us consider the linear SDDE

\[
\begin{align*}
\frac{dX(t)}{dt} &= \int_0^t X(s+u) \, da(u) \, dt + \int_{-\tau}^0 X(s+u) \, db(u) \, dW(s), \quad t \in (0,\infty), \\
X(t) &= \xi(t), \quad t \in [-\tau,0].
\end{align*}
\]

(2.1)

where

\[
\begin{align*}
(W(t),\mathcal{F}_t)_{t \in [0,\infty)} & \text{ is a standard Brownian motion,} \\
\xi & \in \mathcal{C}([-\tau,0],\mathbb{R}) \text{ is a } \mathcal{F}_0\text{-measurable random variable,} \\
a,b & \in ([-\tau,0],\mathbb{R}) \text{ are functions of bounded variation,} \\
\tau & \in [0,\infty) \text{ is the non-negative delay-time.}
\end{align*}
\]

(2.2)
Here we are interested in the asymptotic behaviour of the second moments of solutions of (2.1). That is we are interested in behaviour of the second moment’s function of $X$

$$m_X^2: \begin{cases} [-\tau, \infty) & \rightarrow \mathbb{R} \quad \text{for } t \in [-\tau, \infty) \rightarrow m_X^2(t) := E[X(t)^2]. \end{cases}$$

(2.3)

for $t \rightarrow \infty$. In the second subsection it turns out that the time derivative of $m_X^2$ on $[0, \infty)$ does not depend on the values of $m_X^2$ and time only. Due to the memory in (2.1) it depends on the values of the commoment function (of order 2) of $X$

$$K_X^2: \begin{cases} [-\tau, \infty)^2 & \rightarrow \mathbb{R} \quad \text{for } t, r \in [-\tau, \infty) \rightarrow K_X^2(t, r) := E[X(t)X(r)]. \end{cases}$$

(2.4)

So instead of considering $m_X^2$ we investigate the commoment function $K_X^2$ and will see in a later subsection that $K_X^2$ satisfies a PDE with time delay. In the following this PDE is referred to as the amplified system as it describes the dynamics of $K_X^2$ and as $m_X^2$ is “embedded” into $K_X^2$ by $m_X^2(t) = K_X^2(t, t)$. In a further subsection we describe the existence and uniqueness of solutions of the respective PDE. In the last subsection we collect further properties of the commoment function $K_X^2$.

But before starting to consider $m_X^2$ and $K_X^2$, in the next subsection we provide basic results on existence and uniqueness of solutions of (2.1) and existence of its moments in a first subsection.

### 2.1 Existence and uniqueness of solutions of linear SDDE

There are some standard references about existence and uniqueness theorems. We present here basic results from [6]. Recall, that (1.1) in fact is an integral equation

$$X(t) = X(0) + \int_0^t F(s, X_s) \, ds + \int_0^t G(s, X_s) \, dW(s), \quad t \in [0, \infty),$$

(2.5)

with the stochastic Itô-integral and satisfying (1.2).

**Definition 2.1**

A solution of (1.1) and (2.5), respectively, is then a stochastic process $(X_t, [\tau, \infty))$ which is adapted to $\mathcal{F}_{[0, \infty)}$ and satisfies (1.1).

Let us introduce the following spaces.

**Definition 2.2**

Let $[0, T] \in [-\tau, \infty)$, $\mathbb{F}_{[0, T]} := \{ \mathcal{F}_t \}_{t \in [\tau, \infty]}$. Define:

(i) Let $f \in \mathbb{F}_{[0, T]} \times \Omega, \mathbb{R}$ be adapted to $\mathbb{F}_{[0, T]}$.

$$||f||_{\mathbb{M}^2([0, T], \mathbb{R})} := E[\int_0^T |f(s)|^2 \, ds]$$

$$||f||_{\mathbb{M}^2_{sup}([0, T], \mathbb{R})} := E[\sup_{s \in [0, T]} |f(s)|^2]$$

(ii) $\mathbb{M}^2([0, T], \mathbb{R}) := \{ f \in [0, T] \times \Omega | f \text{ adapted to } \mathbb{F}_{[0, T]} \text{ and } ||f||_{\mathbb{M}^2([0, T], \mathbb{R})} < \infty \}$

(iii) $\mathbb{M}^2_{sup}([0, T], \mathbb{R}) := \{ f \in [0, T] \times \Omega | f \text{ adapted to } \mathbb{F}_{[0, T]} \text{ and } ||f||_{\mathbb{M}^2_{sup}([0, T], \mathbb{R})} < \infty \}$

Then the following theorem provides conditions for the existence and uniqueness of solutions of (1.1).

**Theorem 2.3**

Let the initial function $\xi \in C([-\tau, 0], \mathbb{R})$, the parameter functions $F, G$ satisfy:

(i) (Lipschitz condition)

$$\forall T \in (0, \infty) \exists K_{T, n} \in (0, \infty) \text{ such that } \forall \varphi, \psi \in C([-\tau, 0], \mathbb{R}) \text{ with } ||\varphi||_{C([-\tau, 0], \mathbb{R})} \vee ||\psi||_{C([-\tau, 0], \mathbb{R})}$$

$$\forall t \in [0, T]: |F(t, \varphi) - F(t, \psi)|^2 \vee |G(t, \varphi) - G(t, \psi)|^2 \leq K_{T, n} ||\varphi - \psi||^2_{C([-\tau, 0], \mathbb{R})}$$

(ii) (Growth condition)

$$\forall T \in (0, \infty) \exists K_T \in (0, \infty) \text{ such that } \forall \varphi, \psi \in C([-\tau, 0], \mathbb{R}), \forall t \in [0, T]:$$

$$|F(t, \varphi)|^2 \vee |G(t, \varphi)|^2 \leq K_T (1 + ||\varphi||^2_{C([-\tau, 0], \mathbb{R})})$$

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Then it holds:

1. The SDDE \((1.1)\) has a unique solution \(X\).
2. \(\forall T \in (0, \infty): X \in \mathcal{M}^2([-\tau, T], \mathbb{R})\) and \(X \in \mathcal{M}^2_{\text{sup}}([-\tau, T], \mathbb{R})\).

In the equation (2.1) the parameter functions \(F\) and \(G\) are defined by:

\[
\forall \varphi \in \mathcal{C}([-\tau,0], \mathbb{R}), t \in [0, \infty): F(t, \varphi) = \int_{-\tau}^{t} \varphi(u) da(u) \quad \text{and} \quad G(t, \varphi) = \int_{-\tau}^{0} \varphi(u) db(u).
\]

As \(a, b\) are functions of bounded variation the conditions 2.3.(i) and 2.3.(ii) are satisfied. Let \(K_{a,b}^{2} = 2 (|a|^{2} \vee |b|^{2}) \). Then \(\forall \varphi, \psi \in \mathcal{C}([-\tau,0], \mathbb{R}), t \in [0, \infty)\) it holds:

\[
\|F(t, \varphi) - F(t, \psi)\|^{2} \leq 2 (|a|^{2} \vee |b|^{2}) \|\varphi - \psi\|_{C([-\tau,0], \mathbb{R})}^{2},
\]

\[
\|G(t, \varphi) - G(t, \psi)\|^{2} \leq 2 (|a|^{2} \vee |b|^{2}) \|\varphi - \psi\|_{C([-\tau,0], \mathbb{R})}^{2}.
\]

Hence by 2.3.(1), 2.1 has a unique solution \(X\). Due to 2.3.(2), any solution \(X\) as second moments and comoments.

### 2.2 \(m_{2}^{X}, K_{2}^{X}\) and the amplified system

When investigating the time evolution of second moments of solutions \(X\) of (2.1) then it is natural to ask whether \(m_{2}^{X}(t) = E[X(t)^{2}]\) satisfies a kind of differential equation. Following this approach we have a closer look at the difference quotient

\[
\frac{1}{h}(m_{2}^{X}(t+h) - m_{2}^{X}(t)) = \frac{1}{h}(E[X(t+h)^{2} - X(t)^{2}]) \tag{2.6}
\]

where \(h \in [0, \infty), t \in (0, \infty)\) and try to determine its limit for \(h \downarrow 0\). The main reason why the existence of a limit of (2.6) for \(h\) can be expected is that \(X(t+h)^{2} - X(t)^{2}\) can be represented as a quadratic polynomial of \(X(t)\) and of the state increments \(\Delta X(t,h) = X(t+h) - X(t)\). However, as \(X\) is a solution of the SDDE (2.1) \(\Delta X(t,h)\) can be represented as a sum of integrals resulting from the SDDE (2.6). More precisely it holds:

\[
\frac{1}{h}(E[X(t+h)^{2} - X(t)^{2}]) = \frac{1}{h}E[X(t+h)X(t)] - \frac{1}{h}E[X(t)X(t+h)]
\]

\[
= \frac{1}{h}E[(2X(t) + \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds + \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s))] - \frac{1}{h}E[(X(t) + \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds + \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s))]
\]

\[
= \frac{1}{h}(2E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds + 2E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s)) - E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds + E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s)]
\]

\[
+ 2E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s) - 2E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s)
\]

\[
+ E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds + E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s)]
\]

\[
+ 2E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s) + 2E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s)]
\]

\[
+ E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s) + E[X(t)] \int_{t}^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s)].
\]
In the following we consider each of the five summands of the last equation in (2.7) separately and compute the respective limits for \( h \downarrow 0 \).

(i) \[ \frac{1}{h} \mathbb{E} \left[ \frac{X(t)}{t} \int_{t}^{t+h} X(s+u) da(u) \right] ds = \frac{1}{h} \mathbb{E} \left[ \int_{t}^{t+h} X(t+s) da(s) \right] \]

\[ = \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(t) X(s+u) \right] da(u) ds \]

\[ \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[ X(t) \int_{t}^{t+h} X(s+u) da(u) \right] ds = \mathbb{E} \left[ X(t) X(t+u) \right] da(u), \quad (2.8) \]

(ii) \[ \frac{1}{h} \mathbb{E} \left[ t^{t+h} \int_{t}^{t+h} X(s+u) db(u) dW(s) \right] = \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(t) \int_{t}^{t+h} X(s+u) dW(s) db(u) \right] ds \]

\[ = \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(t) \int_{t}^{t+h} X(s+u) dW(s) db(u) \right] ds \]

\[ = \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(t) \int_{t}^{t+h} X(s+u) dW(s) \right] db(u) \]

\[ = \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(t) \left( \int_{t}^{t+h} X(s+u) dW(s) \right) \right] db(u) \]

\[ \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[ X(t) \int_{t}^{t+h} X(s+u) db(u) dW(s) \right] = 0, \quad (2.9) \]

(iii) \[ \frac{1}{h} \mathbb{E} \left[ t^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds \right] \]

\[ \leq \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(s+u) \int_{t}^{t+h} X(r+u) da(u) \right] ds dr \]

\[ \leq \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(s+u)^2 \right]^{1/2} \mathbb{E} \left[ \int_{t}^{t+h} X(r+u) da(u) \right]^{1/2} ds dr, \]

\[ \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[ t^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds \right] \]

\[ \leq \lim_{h \downarrow 0} \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(s+u)^2 \right]^{1/2} \mathbb{E} \left[ \int_{t}^{t+h} X(r+u) da(u) \right]^{1/2} ds dr, \]

\[ \leq 0, \quad (2.10) \]

(iv) \[ \frac{1}{h} \mathbb{E} \left[ t^{t+h} \int_{t}^{t+h} X(s+u) da(u) ds \right] \]

\[ = \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ t^{t+h} \int_{t}^{t+h} X(s+u) X(r+u) db(v) da(u) \right] ds \]

\[ \leq \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ X(s) X(s+u) \right] \mathbb{E} \left[ X(r+u) dW(r) db(v) \right] da(u) ds \]

\[ \leq \lim_{h \downarrow 0} \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ \int_{t}^{t+h} X(s+u) \right] \mathbb{E} \left[ X(r+u) dW(r) \right] db(v) da(u) ds \]

\[ \leq \lim_{h \downarrow 0} \frac{1}{h} \int_{t}^{t+h} \mathbb{E} \left[ \int_{t}^{t+h} X(s+u)^2 \right]^{1/2} \mathbb{E} \left[ \int_{t}^{t+h} X(r+u)^2 \right]^{1/2} dW(r) db(v) da(u) ds, \]

\[ \leq 0, \quad (2.11) \]
(v) \[ \mathbb{E} \left[ \int_{t-h}^{t} X(s+u) \, dB(u) \, dW(s) \right] = \frac{1}{h} \int_{t-h}^{t} \mathbb{E} \left[ X(s+u) \, dB(u)^2 \right] \, ds \]

\[ \lim_{h \to 0} \mathbb{E} \left[ \int_{t-h}^{t} X(s+u) \, dB(u) \, dW(s) \right] = \int_{-t}^{t} \mathbb{E} \left[ X(s+u)X(s+v) \right] \, dB(u) \, dB(v) \, ds , \]

(2.12)

The same considerations can be taken for the difference quotient

\[ \frac{1}{h} \left( m_X^2(t-h) - m_X^2(t) \right) = \frac{1}{h} \left( \mathbb{E} [ X(t-h)^2 ] - \mathbb{E} [ X(t)^2 ] \right) . \]

(2.13)

where \( t \in (0, \infty) \), \( h \in [0, t) \) and we obtain:

**Lemma 2.4**

Let \( X \) be a solution of (2.1), \( t \in [0, \infty) \).

Then it holds:

(i) \( \lim_{h \to 0} \frac{1}{h} \left( \mathbb{E} [ X(t+h)^2 ] - \mathbb{E} [ X(t)^2 ] \right) \]

(ii) \( \lim_{h \to 0} \frac{1}{h} \left( \mathbb{E} [ X(t-h)^2 ] - \mathbb{E} [ X(t)^2 ] \right) \]

(iii) \( \lim_{h \to 0} \frac{1}{h} \left( \mathbb{E} [ X(t+h)^2 ] - \mathbb{E} [ X(t)^2 ] \right) \]

Proof:

(i) Follows from (2.8) - (2.12).

(ii) Deferred to the appendix.

(iii) Combine (i) and (ii).

The previous lemma shows, that the derivative of the function \( m_X^2 \) at \( t \in (0, \infty) \) does not depend only on \( \{ \mathbb{E} [ X(r)^2 ] \}_{r \in [-t, t]} \) (\( m_X^2 \) on \( [t-\tau, t] \)) but as well on all moments \( \{ \mathbb{E} [ X(t)X(r) ] \}_{r \in [-\tau, t]} \). So in order to characterize \( m_X^2 \) one necessarily has to consider \( K_X^2 \).

**Remark 2.5**

(i) This result is natural. The time evolution of second moments of sequences \( \{ X_n \}_{n \in -k(h)+\mathbb{N}} \), where \( k(h) = \tau/h \in \mathbb{N} \) represents the length of memory and \( h \in (0, \infty) \) a step width, generated by a certain class of numerical schemes (see [4]) can be described exactly by the recurrence relation (1.5).

This recurrence relation is a recurrence relation which does not include only \( \{ \mathbb{E} [ X_n ] \}_{n \in -k(h)+\mathbb{N}} \). Instead it is a recurrence relation on the vector of all moments \( \{ \mathbb{E} [ X_{n-r}X_{n-j} ] \}_{i,j=0, \ldots, k(h)} \). So the previous lemma generalizes this result to the time-continuous case.

(ii) In the introductory section it was observed that for the numerical scheme „Euler explicit“ the linear recurrence relation can be restated as a representation of difference quotients \( \{ \mathbb{E} [ X_{n+1}^2 - \mathbb{E} [ X_n^2 ] ]/h \) and \( \{ \mathbb{E} [ X_{n+1}X_{n-r} ] - \mathbb{E} [ X_nX_{n-r} ] ]/h \}, \( r \in \{ 0, \ldots, k(h) \} \). So the previous lemma generalizes this result to the time-continuous case.

Hence following the observations of remark 2.5(ii) we consider now in addition to the previous difference quotients \( m_X^2 \) or equivalently of \( K_X^2 \) in direction (1,1) the difference quotients

\[ \frac{1}{h} ( K_X^2(t+h,r) - K_X^2(t,r) ) = \frac{1}{h} ( \mathbb{E} [ X(t+h)X(r) ] - \mathbb{E} [ X(t)X(r) ] ) \]

(2.14)

where \( t \in [0, \infty) \), \( r \in [-\tau, t] \), \( h \in (-t, \infty) \). In order to show the existence of the limit of (2.14) for \( h \downarrow 0 \) we proceed as before and represent \( \Delta X(t,h) \) with integrals coming from the SDDE (2.1). It is easy to see that for \( h \) with \( |h| < |t-r| \) it holds:

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\[
\begin{align*}
\frac{1}{h} & (E[X(t+h)X(r)] - E[X(t)X(r)]) \\
& = \frac{1}{h} E\left[ \int_{t}^{t+h} \int_{-\tau}^{0} X(s+u) \, du \, ds + \int_{t}^{t+h} \int_{-\tau}^{0} X(s) \, dW(s) \right] \, X(r) \\
& = \frac{1}{h} \left( \int_{t}^{t+h} \int_{-\tau}^{0} E[X(s+u)X(r)] \, du \, ds + \int_{t}^{t+h} \int_{-\tau}^{0} E[X(r)E[\int_{t}^{t+h} X(s) \, dW(s) | F_{t}] ] \, du \right) \\
& = \frac{1}{h} \left( \int_{t}^{t+h} \int_{-\tau}^{0} E[X(s+u)X(r)] \, du \, ds + \int_{t}^{t+h} \int_{-\tau}^{0} E[X(r)0 \, dW(s) | F_{t}] ] \, du \right) \\
& = \frac{1}{h} \left( \int_{t}^{t+h} \int_{-\tau}^{0} E[X(s+u)X(r)] \, du \, ds \right)
\end{align*}
\]

\[
\lim_{h \to 0} \frac{1}{h} (E[X(t+h)X(r)] - E[X(t)X(r)]) = \int_{-\tau}^{0} E[X(t+h)X(r)] \, da(u).
\] (2.15)

The same procedure can be applied to show the existence and to determine the limit of the difference quotient

\[
\frac{1}{h} (K_{2}^{X}(t+h) - K_{2}^{X}(t)) = \frac{1}{h} E[X(t+h)X(r)] - E[X(t)X(r)]
\] (2.16)

where \( t \in (0, \infty) \), \( r \in [-\tau, t] \), \( h \in [0, t - r] \). This allows us to conclude:

**Lemma 2.6**

Let \( X \) be a solution of (2.1).

Then it holds:

(i) \( \lim_{h \to 0} \frac{1}{h} (E[X(t+h)X(r)] - E[X(t)X(r)]) = \int_{-\tau}^{0} E[X(t+h)X(r)] \, da(u), \quad t \in [0, \infty), \ r \in [-\tau, t], \)

(ii) \( \lim_{h \to 0} \frac{1}{h} (E[X(t-h)X(r)] - E[X(t)X(r)]) = \int_{-\tau}^{0} E[X(t-h)X(r)] \, da(u), \quad t \in (0, \infty), \ r \in [-\tau, t], \)

(iii) \( \lim_{h \to 0} \frac{1}{h} (E[X(t+h)X(r)] - E[X(t)X(r)]) = \int_{-\tau}^{0} E[X(t+h)X(r)] \, da(u), \quad t \in (0, \infty), \ r \in [-\tau, t]. \)

**Proof:**

(i) Follows from (2.15).

(ii) Followed to the appendix.

(iii) Follows from (i) and (ii).

The properties of moments of \( E[X(t)X(s)], t, s \in [-\tau, \infty) \), which have been presented in lemma 2.4 and lemma 2.6 can be extended by the observation that

\[
\forall t_{1}, t_{2} \in [-\tau, \infty]: \quad E[X(t_{1})X(t_{2})] = E[X(t_{2})X(t_{1})],
\]

\[
\forall t_{1}, t_{2} \in [-\tau, 0]: \quad E[X(t_{1})X(t_{2})] = E[\xi(t_{1})\xi(t_{2})].
\] (2.17)

So collecting the results of lemma 2.4, lemma 2.17 and (2.17) we see that \( K_{2}^{X} \) satisfies the system

\[
\frac{d}{dx} K(x,x)_{x=t} = 2 \int_{-\tau}^{0} K(t+u,t) \, du = \frac{d}{dx} K(x,x)_{x=0+} = 2 \int_{-\tau}^{0} K(u,t) \, du, \quad t \in (0, \infty),
\]

\[
\frac{d}{dx} K(x,x)_{x=t} = \frac{d}{dx} K(x,x)_{x=0+} = \frac{d}{dx} K(x,r)_{x=t} = \frac{d}{dx} K(x,r)_{x=0+} = \frac{d}{dx} K(r,t) \, da(u), \quad t \in (0, \infty), r \in [-\tau, t],
\]

\[
K(t_{1}, t_{2}) = K(t_{2}, t_{1}), \quad t_{1}, t_{2} \in [-\tau, \infty),
\]

\[
K(t_{1}, t_{2}) = k(t_{1}, t_{2}), \quad t_{1}, t_{2} \in [-\tau, 0]
\] (2.18)
with the initial function \( k(t_1, t_2) = \mathbb{E}[X(t_1), X(t_2)], t_1, t_2 \in [-\tau, 0]. \)

**Definition 2.7**

The system (2.18) of equations is called **amplified system**.

A function \( K \in [-\tau, \infty)^2, \mathbb{R} \) satisfying (2.18) is called **solution of (2.18)**.

A function \( k = K_{[\cdot \tau, 0]} \in ([\cdot \tau, 0]^2, \mathbb{R}) \) is called initial function (of the amplified system).

### 2.3 Solutions of the amplified system

In the previous subsection we considered the comoment function \( K_X^2 \) of a solution \( X \) of (2.1) and observed that it satisfies the amplified system (2.18). In this subsection we want to consider amplified systems (2.18) and collect some properties their solutions. First of all we see that the solutions of (2.18) are symmetric functions. This makes it sometimes convenient to consider solutions of (2.18) on only selected subsets of \( \mathbb{R}^2 \). That is why we introduce the set \( K_{\tau, \geq}(c) \) along with further helpful sets.

**Definition 2.8**

Let \( c \in \mathbb{R} \).

Define \( \Delta(\mathbb{R}^2) := \{ (t, t) \in \mathbb{R}^2 | t \in \mathbb{R} \} \),

\[ K^2_{\tau, \Delta}(c) := \{ (t_1, t_2) \in [c, \infty)^2 | \} \),

\[ K^2_{\tau, >}(c) := \{ (t_1, t_2) \in [c, \infty)^2 | t_1 > t_2 \} \),

\[ K^2_{\tau, \geq}(c) := \{ (t_1, t_2) \in [c, \infty)^2 | t_1 \geq t_2 \} \).

In addition to the symmetry, we see that solutions of the amplified system (2.18) have a special structure. This special structure consists in that the dynamics of solutions of (2.18) on \( K^2_{\tau, \tau} \) can be decomposed into a dynamics along the ray into direction \((1, 1)\) and starting in \((-\tau, -\tau)\) and a dynamics along rays into directions \((1, 0)\) and starting points on the diagonal \( \Delta(\mathbb{R}^2) \cap K^2_{\tau, \tau} \). In order to make this rigorous we introduce the functions:

**Definition 2.9**

\[ y_s(t) = K(t, s), \quad s \in [-\tau, \infty), \quad t \in [s, \infty), \]

\[ z(t) = K(t, t), \quad t \in [-\tau, \infty), \]

and reformulate the amplified system (2.18) as an integral equation, that is for all \((t, s) \in K^2_{\tau, \tau} \):

\[ y_s(t) = z(s) + \int_{s-r}^{t} \int_{s-r}^{t} y_{r+u}(s) da(u) dr + \int_{s-r}^{t} \int_{s-r}^{t} y_{s+r+u}(s) da(u) dr, \quad s \in (0, \infty) \]

\[ y_s(t) = \int_{0}^{t} \int_{0}^{t} y_{s}(r+u) da(u) dr + \int_{0}^{t} \int_{0}^{t} y_{s+r}(r+u) da(u) dr, \quad s \in [-\tau, 0], \quad t \in [0, \infty) \]

\[ y_s(t) = k(t, s), \quad t \in [-\tau, 0]. \]

\[ z(t) = z(s) + 2 \int_{s-r}^{t} \int_{s-r}^{t} y_{r+u}(r) da(u) dr + \int_{s-r}^{t} \int_{s-r}^{t} y_{r+u}(r+u) da(u) da(u), \quad s \in (0, \infty), \]

\[ z(t) = k(t, t), \quad t \in [-\tau, 0]. \]

This shows that for all \( s \in [-\tau, \infty) \) the function \( y_s \) satisfies the deterministic DDE

\[ z'(t) = \int_{s}^{t} z(s) + 2 \int_{s}^{t} y_{s+r}(s) da(u) dr + \int_{s}^{t} y_{s+r}(s+u) da(u) da(u), \quad t \in [s, 0), \]

\[ z(t) = \begin{cases} z(s), & s \in (0, \infty), \quad t = s, \\ y_{s+r}(s), & s \in (0, \infty), \quad t \in [s, -\tau, s), \\ k(t, s), & s \in [-\tau, 0], \quad t \in [-\tau, 0]. \end{cases} \]

In particular we observe that for all \( s \in [-\tau, \infty) \) the functions \( y_s \) solve a linear DDE with the same linear parameter function \( a \) but with specific start functions. For \( s \in (0, \infty) \) the start function of \( y_s \) is \( y_s \) on
the interval $[s - \tau, s]$ and is a function which takes the values $y_r(s)$, $r \in [s - \tau, s]$ and $z(s)$ at $r = s$. For $s \in [-\tau, 0]$ however, the start function of $y_s$ is $y_s$ on the interval $[-\tau, \tau]$, where it is completely determined by the values of the initial function $k$.

Furthermore, the function satisfies the equation

$$
z'(t) = \int_{-\tau}^{0} y_{s+u}(u) \, da(u) + \int_{-\tau}^{0} y_{s+u\vee v}(u) \, db(u) \, db(v), \quad t \in [0, \infty),
$$

$$
z(0) = k(0,0).
$$

Next we consider the question of existence and uniqueness of solutions of the amplified system. In the previous subsection we have seen that if the functions $a,b \in \mathbb{R}$ are suitable parameter functions and $k = \text{E}[\xi(\zeta(\xi))] \in \mathcal{H}$ is a suitable initial function $\xi$ then the corresponding function solution $X(t)$ of (2.1) provides a solution of (2.20). Here we want to find conditions on $a, b, k$ which do not rely on an SDDE and which allow the system (2.20) to have a unique solution.

We start with a simple but helpful lemma.

**Lemma 2.10**

Let $K_1, K_2$ be two solutions of the amplified system (2.20).

Then it holds:

(i) $\forall \lambda_1, \lambda_2 \in \mathbb{R}$: $\lambda_1 K_1 + \lambda_2 K_2$ is a solution of (2.20).

(ii) Let $\Delta := K_2 - K_1$. Then $\Delta$ satisfies the system:

$$
\forall t \in [0, \infty):
\Delta K(t, t) = \Delta K(0, 0) + 2 \int_{-\tau}^{t} \Delta K(u, r) \, da(u) \, dr + \int_{-\tau}^{t} \int_{-\tau}^{r} \Delta K(u + r, v + r) \, db(u) \, db(v) \, dr,
$$

$$
\forall s \in [-\tau, 0], t \in (s \vee 0, \infty):
\Delta K(t, s) = \Delta K(s \vee 0, s) + \int_{s \vee 0}^{t} \Delta K(u + r, s) \, da(u) \, dr.
$$

**Proof:**

(i) Obvious.

(ii) (2.23) is the integral form of (2.1).

2.10. (ii) indicates that a growth bound for the difference of two solutions of (2.20) in terms of the difference of their initial functions can be gained by the application of a Gronwall-type lemma. However, as $\Delta$ is two dimensional, this requires a suitable choice of a one-dimensional "error" function which quantifies the difference $\Delta$ and to which the Gronwall lemma can be applied. The following lemma provides the result.

**Definition 2.11**

Let $K_1(t)$ solution of (2.20). Define

(i) $C(a, b, \tau) := 2 \int_{-\tau}^{0} da(u) + \int_{-\tau}^{0} db(v) < \infty$,

(ii) $\mathcal{K}_{sup}([-\tau, 0]^2, \mathbb{R}) := \{ k \in \mathcal{K} \mid ||k||_{C([-\tau, 0]^2, \mathbb{R})} < \infty \}$, the class of bounded initial functions,

(iii) $\delta_{sup}(t) := \sup \{ |\Delta K(u, v) | \mid u, v \in [-\tau, t] \}$, an error function.

**Lemma 2.12**

Let $K_1, K_2$ be solutions of the amplified system (2.18) with respective initial functions $k_1, k_2$.

$$
k_2 - k_1 \in \mathcal{K}_{sup}([-\tau, 0]^2, \mathbb{R}).
$$

Then it holds:

$$
\forall t, s \in [-\tau, \infty): \quad |K_2(t, s) - K_1(t, s)| \leq \|k_2 - k_1\|_{C([-\tau, 0]^2, \mathbb{R})} C(a, b, \tau)(t \vee s) \vee 0
$$

**Proof:**

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(1) By lemma 2.10(ii) it follows for all \( t \in (0, \infty) \):
\[
|\Delta K(t, t)| \leq \delta_{sup}(0) + 2 \int_0^t \int_{0-t}^0 d[a](u) \delta_{sup}(r) \, dr + \int_0^t \int_{0-t}^0 d[b](u) \, d[b](v) \delta_{sup}(r) \, dr \\
\leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta(r) \, dr.
\]
Hence with lemma 2.10(ii) it also follows for all \( t \geq s \geq 0 \geq -\tau \):
\[
|\Delta K(t, s)| \leq |\Delta K(s \vee 0, s)| + \int_0^t \int_{s \vee 0-t}^0 d[a](u) \delta_{sup}((r + u) \vee s \vee 0) \, dr \\
\leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta_{sup}(r) \, dr + \int_0^t \int_{s \vee 0-t}^0 d[a](u) \delta_{sup}(r + u) \vee s \vee 0 \, dr \\
\leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta_{sup}(r) \, dr + \int_0^t \int_{s \vee 0-t}^0 d[a](u) \delta_{sup}(r) \, dr \\
\leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta_{sup}(r) \, dr.
\]
(3) From (2) we can conclude that for all \( t \in [0, \infty) \):
\[
\sup \{ |\Delta K(t, r)| : r \in [-\tau, \ell] \} \leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta_{sup}(r) \, dr
\]
(4) Let \( t \in [0, \infty) \), \( s \in [-\tau, \ell] \). From (3) we can conclude
\[
\sup \{ |\Delta K(s, r)| : r \in [-\tau, s] \} \leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta_{sup}(r) \, dr
\]
(5) Let \( t \in [0, \infty) \). From (3), (4) we obtain:
\[
\delta_{sup}(t) \leq \delta_{sup}(0) + \int_0^t C(a, b, \tau) \delta_{sup}(r) \, dr
\]
The lemma of Gronwall shows then that \( \forall t \in [0, \infty) : \delta_{sup}(t) \leq \delta_{sup}(0) e^{C(a, b, \tau) t} \).
But then it holds for all \( t \geq (s \vee 0) \geq -\tau \) : \( |\Delta K(t, s)| \leq \delta_{sup}(0) e^{C(a, b, \tau) t} \).
As for all \( t, s \in [-\tau, 0] \) it holds: \( |\Delta K(t, s)| \leq \delta_{sup}(0) \), as \( \Delta K \) is symmetric and as \( K_1([[-\tau, 0]) = k_1 \),
\( K_2([-\tau, 0]) = k_2 \), the proof is complete.
\( \square \)

A reconsideration of (1) in the above proof reveals that the supremum in the factor \( ||k_2 - k_1||_{C([-\tau, 0])} \) is the result of the estimations:
\[
|\Delta K(0, 0) + \int_{t \wedge \tau}^{t \wedge \tau - \tau} \int_{0 - \tau}^{0} \Delta K(u + r, v + r) \, db(u) \, db(v) | \\
\leq |\Delta K(0, 0)| + \int_{0 - \tau}^{0} \int_{0 - \tau}^{0} |\Delta K(u + r, v + r) | \, db(u) \, db(v) \\
|\Delta K(s, 0) + \int_{s \wedge \tau}^{s \wedge \tau - \tau} \int_{0 - \tau}^{0} \Delta K(u + r, s) \, da(u) \, dr | \\
\leq |\Delta K(s, 0)| + \int_{0 - \tau}^{0} \int_{0 - \tau}^{0} |\Delta K(u + r, s) | \, da(u) \, dr,
\]
As for all \( t, s \in [-\tau, 0] \) it holds: \( |\Delta K(t, s)| \leq \delta_{sup}(0) \).

Hence the result of lemma 2.12 can be improved by a more careful choice of the error function and a refined analysis.

**Definition 2.13**

Let \( \Delta K \) solution of (2.20). Define:
(i) \( \forall k \in ([-\tau,0]^2, \mathbb{R}) : \)
\[
\|k\|_{\mathcal{K}_{\alpha,b}([-\tau,0]^2, \mathbb{R})} = \sup_{t,s \in [-\tau,0], \forall \sigma = 0} \left\{ \int k(t,s) \int \right\} \\
+ 2 \sup_{s \in [-\tau,0]} \left\{ \int \int k(r - r, s) |d\sigma| dr \right\} \\
+ \int \int \int k(r - r, r + v) |d\sigma d\sigma d\sigma| dr .
\]

(ii) \( \mathcal{K}_{\alpha,b}([-\tau,0]^2, \mathbb{R}) \)
\[
:= \{ k \in ([-\tau,0]^2, \mathbb{R}) | \forall 0 \leq t \leq s \geq -\tau : k(t,s) = k(s,t), \forall 0 \leq s \geq -\tau : k(s,s) \in \mathcal{L}_1([-\tau,0], \mathbb{R}, d|\sigma|), k \in \mathcal{L}_1([-\tau,0]^2, \mathbb{R}, d|\sigma| d|\sigma|), \|k\|_{\mathcal{K}_{\alpha,b}([-\tau,0]^2, \mathbb{R})} < \infty \},
\]

(iii) \( \delta(t) = \begin{cases} 
\sup \{ \|\Delta K(t,s)\| \mid (t,s) \in [-\tau,\infty)^2 \setminus [-\tau,0]^2 \}, & t \in (0, \infty), \\
\|\Delta K\|_{\mathcal{K}_{\alpha,b}([-\tau,0]^2, \mathbb{R})} & t = 0.
\end{cases} \)

Lemma 2.14
Let \( K_1, K_2 \) be solutions of system (2.18) with the initial function \( k_1 \) and \( k_2 \), respectively,
\( k_2 - k_2 \in \mathcal{K}_{\alpha,b}([-\tau,0]^2, \mathbb{R}) \).
Then it holds:
\[
\forall t, s \in [-\tau, \infty) : \left| K_2(t,s) - K_1(t,s) \right| \leq \|k_2 - k_1\|_{\mathcal{K}_{\alpha,b}([-\tau,0]^2, \mathbb{R})} e^{C(a,\beta,\gamma)(t+s+\gamma)}.
\]

Proof:
(1) For the following estimations we use the set
\[
A_r := \{ (t,u) \in \mathbb{R}^2 | t \in [-\tau, r], u \in [-t \wedge -\tau, -t] \} .
\]
\( A_r \) describes tuples \( (t,u) \) for which \( t + u \in [-\tau,0] \).
Furthermore, for denotational convenience define
\[
S_0^r := \sup_{t,s \in [-\tau,0], \forall \sigma = 0} \left\{ \|\Delta K(t,s)\| \right\} ,
\]
\[
S_1^r := \sup_{s \in [-\tau,0]} \left\{ \int \int k(r - r, s) |d\sigma| dr \right\} ,
\]
\[
S_2^r := \int \int \int k(r - r, r + v) |d\sigma d\sigma d\sigma| dr .
\]
(2) For all \( t \in (0, \infty) \) it holds:
\[
\Delta K(0,0) \leq S_0^r ,
\]
\[ | \int_0^t \int_0^r \Delta K(r + u, r) \, da(u) \, dr | \leq \int_0^t \int_0^r \Delta K(r + u, r) | \, d[a](u) \, dr \]
\[ \leq \int_0^t \int_0^r \delta(r) \, d[a](u) \, dr \leq \int_0^t d[a](u) \int_0^r \delta(r) \, dr , \]
\[ | \int_0^t \int_0^r \Delta K(r + u, r + v) \, db(u) \, db(v) \, dr | \]
\[ \leq \int_0^t \int_0^r | \Delta K(r + u, r + v) | \, d[b](u) \, d[b](v) \, dr \]
\[ \leq S_r^2 + \int_0^t \int_0^r \delta(r) \, d[b](u) \, d[b](v) \, dr \]
\[ \leq S_r^2 + ( \int_0^r d[b](u) )^2 \int_0^r \delta(r) \, dr . \]

By 2.10 (ii) then it follows:
\[ | \Delta K(t, \ldots, t) | \leq S_r^0 + S_r^2 + C(a, b, \tau) \int_0^t \delta(r) \, dr \leq || \Delta K ||_{K_{\alpha, \beta}([-r, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^t \delta(r) \, dr . \]

(3) Let \( \infty > s \geq -r \). Then for all \( t \in (0 \cup s, \infty) \) it holds:
\[ | \Delta K(t, s) | \leq | \Delta K(t, s, s) | + | \int_0^t \int_0^s \Delta K(r + u, s) \, da(u) \, dr | \]
\[ \leq ( || \Delta K ||_{K_{\alpha, \beta}([-s, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^s \delta(r) \, dr \int_0^s d[a](u) + \int_0^s \delta(r) | \, da(u) \, dr | )_{1[0, \infty)} (s) \]
\[ + ( S_r^0 + \int_0^t \int_0^s \delta(r) \, d[a](u) \, d[b](u) (1 - 1_{A_{\mathbb{R}}}((t, u))) \, dr + S_r^2 ) \int_0^s 1_{[-r, 0]}(s) \]
\[ \leq || \Delta K ||_{K_{\alpha, \beta}([-s, 0]^2, \mathbb{R})} \int_0^s 1_{[0, \infty)}(s) + ( S_r^0 + S_r^2 ) \int_0^s 1_{[-r, 0]}(s) \]
\[ + C(a, b, \tau) \int_0^s \delta(r) \, dr + \int_0^s d[a](u) \int_0^s \delta(r) \, dr \]
\[ \leq || \Delta K ||_{K_{\alpha, \beta}([-s, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^s \delta(r) \, dr . \]

(4) Let \( t \in (0, \infty) \). Due to (2), (3), \( \forall \delta \geq -r, t > \delta > 0, t > s \geq -r \) it holds:
\[ | \Delta K(\delta, \bar{s}) | \leq | \Delta K(t, t, \bar{s}) | + C(a, b, \tau) \int_0^s \delta(r) \, dr \]
\[ \leq | \Delta K ||_{K_{\alpha, \beta}([-t, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^s \delta(r) \, dr , \]
\[ | \Delta K(t, t) | \leq | \Delta K ||_{K_{\alpha, \beta}([-t, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^s \delta(r) \, dr , \]
\[ | \Delta K(t, s) | \leq | \Delta K ||_{K_{\alpha, \beta}([-t, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^s \delta(r) \, dr . \]

Hence, using the definition of \( \delta(0) \), for all \( t \in [0, \infty) \) it holds:
\[ \delta(t) = \sup \{ | \Delta K(\bar{s}, \bar{s}) | : (\bar{s}, \bar{s}) \in [-r, 0]^2 \} \]
\[ \leq || \Delta K ||_{K_{\alpha, \beta}([-r, 0]^2, \mathbb{R})} + C(a, b, \tau) \int_0^t \delta(r) \, dr . \]

Applying the lemma of Gronwall to \( \delta \) on \([0, \infty)\) we get the desired result. \( \square \)

This growth bound for the difference of solutions is used now to show a criterion for uniqueness of solutions.

**Corollary 2.15**

Let \( K_1, K_2 \) be two solutions of the amplified system (2.18) with the same initial function \( k \in K_{\alpha, \beta}([-r, 0]^2, \mathbb{R}) \). Then it holds:
\[ K_1 = K_2. \]

**Proof:**
\( \forall t, s \in K_2^2 (-r) \setminus [-r, 0]^2 \) it holds:
\[ | K_2(t, s) - K_1(t, s) | \leq || k - k ||_{K_{\alpha, \beta}([-r, 0]^2, \mathbb{R})} e^{C(a, b, \tau) (t \vee s / 0) \tau} = 0. \]
As \( K_1 [-r, 0] = k = K_2 [-r, 0] \), the proof is complete.

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Furthermore we obtain trivially:

**Corollary 2.16**
Let \( K \) be a solution of the amplified system (2.20) with the initial function \( k = 0 \).
Then it holds:
\[ K \equiv 0. \]

**Proof:**
\( K_0 \equiv 0 \) is a solution of the amplified system with the zero initial function.
Corollary 2.15 shows that \( K_0 \) is the unique solution.

Combining lemma 2.12 and corollary 2.16 we obtain a growth bound for solutions of (2.18)

**Corollary 2.17**
Let \( K \) be a solution of the amplified system (2.18) with the initial function \( k \in K_{a,b}([-\tau,0]^2, \mathbb{R}) \).
Then it holds:
\[ \forall t, s \in [-\tau, \infty) : |K(t,s)| \leq \|k\|_{K_{a,b}([-\tau,0]^2, \mathbb{R})} e^{C(a,b,\tau)(t\vee s)\tau}. \]

**Proof:**
Let \( K_2 = K, K_1 = 0 \) and apply lemma 2.12.

Finally we consider the existence of solutions of the amplified system. As a tool we use the well-known method of Picard-iterates.

**Definition 2.18**
\( \forall (t,s) \in [-\tau, \infty)^2; \)
\[ K_0(t,s) := k(0 \wedge t, 0 \wedge s). \]

\( \forall n \in \mathbb{N} \) define:
\[ 0 \geq t, s \geq -\tau: \quad K_{n+1}(t,s) = k(t,s), \]
\[ t \geq 0: \quad K_{n+1}(t,t) = K_n(0,0) + 2 \int_0^t \int_0^{t-\tau} K_n(r+s,r) \, da(u) \, dr \]
\[ + \int_0^t \int_0^{t-\tau} K_n(r+s,r+v) \, db(u) \, dv \, dr, \]
\[ s \geq -\tau, t \geq 0 \vee s: \quad K_{n+1}(t,s) = K_n(0 \vee t, 0 \vee s) + \int_0^t \int_0^{s-\tau} K_n(r+s, r+v) \, da(u) \, dv \, dr. \]

The sequence \( \{K_n\}_{n \in \mathbb{N}} \) is called **Picard-iterations** and the \( K_n, n \in \mathbb{N} \), are called the **Picard-iterates**.

**Theorem 2.19**
Let \( K \) be a solution of (2.18) with the initial function \( k \in K_{a,b}([-\tau,0]^2, \mathbb{R}) \).
Then it holds:
(i) \( \forall T \in [0, \infty): \{K_n\}_{n \in \mathbb{N}} \) forms a Cauchy-sequence on \([-\tau, T]^2\) with norm \( \|\cdot\|_{C([-\tau,T]^2, \mathbb{R})} \).
(ii) The system (2.18) has a solution \( K \).
(iii) \( \forall t,s \in [-\tau,T]: \quad \|K(t,s) - K_n(t,s)\| \leq \|k\|_{K_{a,b}([-\tau,0]^2, \mathbb{R})} 2 e^{C(a,b,\tau)T} \sum_{k=0}^{n+1} \frac{(C(a,b,\tau)T)^k}{k!}. \)

**Proof:** (of theorem 2.22)
Deferred to section 5.5.

**Corollary 2.20**
Let \( k \in K_{a,b} \).
Then it holds:
(i) \( \|k\|_{K_{a,b}([-\tau,0]^2, \mathbb{R})} \leq (1 + \tau C(a,b,\tau)) \|k\|_{C([-\tau,0]^2, \mathbb{R})} \),
(ii) \( k \in K_{a,b}([-\tau,0]^2, \mathbb{R}) \).
Proof:
(i) \[ ||k||_{C([-\tau,0]^2, \mathbb{R})} \leq ||k||_{C([-\tau,0]^2, \mathbb{R})} + 2\tau \int_{-\tau}^{0} |a(s)||k||_{C([-\tau,0]^2, \mathbb{R})} + \tau \left( \int_{-\tau}^{0} |b(s)||k||_{C([-\tau,0]^2, \mathbb{R})} \right)^2 \]
\[ \leq \left( 1 + \tau C(a, b, \tau) \right) ||k||_{C([-\tau,0]^2, \mathbb{R})} . \]
(ii) As \[ ||k||_{C([-\tau,0]^2, \mathbb{R})} < \infty \] and due to (i), \[ ||k||_{C_{a,b}([-\tau,0]^2, \mathbb{R})} < \infty. \]

We have characterized conditions for the unique existence of solutions of the system (2.18). We also know that comoment functions \( K_2 \) of solutions \( X \) of linear SDDE (2.1) are solutions of the system (2.18). This raises the question whether all solutions \( K \) of the system (2.18) are a comoment function \( K_2 \) of a linear SDDE (2.1). This cannot be confirmed.

**Lemma 2.21**

Let \( k \in C([-\tau,0]^2, \mathbb{R}) \) with \( k(t,t) < 0 \) for all \( t \in [-\tau,0] \).

Then it holds:

The solution \( K \) of system (2.18) with initial function \( k \) is not a comoment function of a solution of the linear SDDE (2.1).

**Proof:**

\( k \) is negative on \([-\tau,0]^2 \cap \Delta(\mathbb{R}^2) \) whereas comoment functions are non-negative on \([-\tau,0]^2 \cap \Delta(\mathbb{R}^2) \).

\( \square \)

### 2.4 Some properties of comoment functions satisfying the amplified system

In this subsection we consider some properties of comoment functions satisfying the amplified system. We start with the continuity.

**Theorem 2.22**

Let \( K \) be a solution of (2.18) with the initial function \( k \in C([-\tau,0]^2, \mathbb{R}) \).

Then it holds:

The system (2.18) has a continuous solution.

**Proof:**
(1) As in theorem 2.14 we consider the Picard-iterates introduced in (2.18).
    Then \( \forall (t, s) \in [-\tau, \infty)^2 \) it holds:
    \[
    K_{n+1}(t, s) = K_n(0 \wedge t, 0 \wedge s) + 2 \int_0^t \int_{-\tau}^0 K_n(r + u, r) \, du \, dr + \int_0^t \int_{-\tau}^t K_n(r + u, r + v) \, dv \, dr \\
    + \int_0^t \int_{-\tau}^0 K_n(r + u, s) \, du \, dr.
    \]

(2) By the assumption on \( k, K_0 \) is continuous on \([-\tau, \infty)^2\).

(3) Fix \( n \in \mathbb{N} \) and assume, that \( K_n \) is continuous. Let \( t \geq s \geq -\tau, \bar{t} \geq \bar{s} \geq -\tau \) w.o.l.g. Then it holds:
    \[
    K_{n+1}(t, s) - K_{n+1} (\bar{t}, \bar{s}) \\
    = K_n(0 \wedge t, 0 \wedge s) - K_n(0 \wedge \bar{t}, 0 \wedge \bar{s}) \\
    + 2 \int_{0 \wedge s}^{0 \wedge t} \int_{0 \wedge \bar{s}}^{0 \wedge \bar{t}} K_n(r + u, r) \, du \, dr - \int_{0 \wedge s}^{0 \wedge t} \int_{0 \wedge \bar{s}}^{0 \wedge \bar{t}} K_n(r + u, r) \, du \, dr \\
    + \int_{0 \wedge s}^{0 \wedge t} \int_{0 \wedge \bar{s}}^{0 \wedge \bar{t}} K_n(r + u, r + v) \, dv \, dr - \int_{0 \wedge s}^{0 \wedge t} \int_{0 \wedge \bar{s}}^{0 \wedge \bar{t}} K_n(r + u, r + v) \, dv \, dr \\
    + \int_{0 \wedge s}^{0 \wedge t} \int_{0 \wedge \bar{s}}^{0 \wedge \bar{t}} K_n(r + u, s) \, du \, dr - \int_{0 \wedge s}^{0 \wedge t} \int_{0 \wedge \bar{s}}^{0 \wedge \bar{t}} K_n(r + u, s) \, du \, dr
    \]
    Hence:
    \[
    |K_{n+1}(t, s) - K_{n+1} (\bar{t}, \bar{s})| \\
    \leq |K_n(0 \wedge t, 0 \wedge s) - K_n(0 \wedge \bar{t}, 0 \wedge \bar{s})| \\
    + C(a, b, r) \sup_{r \in [0 \wedge s, 0 \wedge \bar{s}], u \in [-\tau, 0]} \{|K_n(r + u, r)|\} \sup_{u \in [0 \wedge s, 0 \wedge \bar{s}], \nu \in [-\tau, 0]} \{|K_n(u, s)| - K_n(u, \bar{s})|\} \\
    + 2 \sup_{r \in [0 \wedge s, 0 \wedge \bar{s}], u \in [-\tau, 0]} \{|K_n(r + u, s)|\} \sup_{u \in [0 \wedge s, 0 \wedge \bar{s}], \nu \in [-\tau, 0]} \{|K_n(r + u, s)|\}
    \]
    \[
    \rightarrow 0 \quad \text{as} \ n \rightarrow \infty
    \]
    as \( K_n \) is continuous.

    But then \( K_{n+1} \) is also continuous. By complete induction one obtains that \( \forall n \in \mathbb{N} : K_n \) is continuous.

(4) As \( \|k\|_{K_0([-\tau, 0]^2, \mathbb{R})} \leq C(a, b, r) \|C_{1([-\tau, 0]^2, \mathbb{R})} \leq \infty \), we know from lemma 2.22, that for all \( T \in [0, \infty) \) \{ \( K_n \) \} converges in the \( C([-\tau, 0]^2, \mathbb{R}) \) to a limit \( K \) on each compact set \([-\tau, T]\). In (11) of the proof of lemma 2.22 we derived, that for all \( T \in [0, \infty) \), \( t, s \in [-\tau, T]^2 - [-\tau, 0]^2 \), \( n, m \in \mathbb{N} \):
    \[
    |K_{n+1}(t, s) - K_{n+m+1}(t, s)| \\
    \leq n \sum_{k=0}^{m} |K_{n+1}(t, s)| \\
    \leq \|k\|_{K_0([-\tau, 0]^2, \mathbb{R})} 2 (e^{C(a, b, r) T} - \sum_{k=0}^{n} \frac{(C(a, b, r) T)^k}{k!})
    \]
    Hence for all \( T \in [0, \infty) \) \{ \( K_n \) \} is a Cauchy-sequence in \( C([-\tau, 0]^2, \mathbb{R}) \). As \( C([-\tau, 0]^2, \mathbb{R}) \) is complete, \( K \) is continuous.

In the previous theorem the continuity of a solution \( K \) of the amplified system (2.18) could be proved by relying merely on the properties of the initial functions and on the amplified system. For more general than continuous initial functions this is not so easy to achieve anymore. The reason is that the amplified system (2.18) is a system of Gateaux-derivatives and not of Frechet-derivatives. Hence it provides information about the solutions \( K \) on rays. However, local properties like continuity of \( K \) depend not only on values of \( K \) on rays but on full-dimensional environments.

As we here are rather interested in comoment functions \( K^X \) of solutions of SDDE (2.1) than in general solutions of the amplified system (2.18), we limit in the following to these functions. The comoment functions \( K^X \) however have additional properties which are a consequence of the SDDE the process X satisfies.
Theorem 2.23
Let \( \forall t_1, t_2 \in \mathbb{K}_+^2 (0) \) : \( A(t_1, t_2) := \{ h = (h_1, h_2) \mid (t_1 + h_1, t_2 + h_2) \in \mathbb{K}_+^2 (0) \} \),

\( X \) be a solution of (2.1) with conomoment function \( K_2^X \),

\[ \forall T \in [0, \infty) : \sup_{t \in [-T, T]} \{ E[|X(t)|^2] \} \leq C_T < \infty. \]

Then it holds:

(i) \( \forall \gamma \in (0, \frac{1}{2}), \forall t_1, t_2 \in \mathbb{K}_+^2 (0) : \lim_{h \in A(t_1, t_2), |h| \to 0} \frac{1}{|h|^2} |K_2^X (t_1 + h_1, t_2 + h_2) - K_2^X (t_1, t_2)| = 0, \)

(ii) For \( \gamma = \frac{1}{2}, \forall t_1, t_2 \in \mathbb{K}_+^2 (0) : \lim_{h \in A(t_1, t_2), |h| \to 0} \frac{1}{|h|^2} |K_2^X (t_1 + h_1, t_2 + h_2) - K_2^X (t_1, t_2)| \leq \int d|b(u)| C_T, \)

(iii) \( K_2^X \) is continuous on \( \mathbb{K}_+^2 (0) \).

Proof: (of theorem 2.23)

(1) Let \( t_1, t_2 \in [0, \infty), h = (h_1, h_2) \in A(t_1, t_2) \). For any \( t, h \in \mathbb{R} \) define \( M(t, h) := (t + h) \land t \) and \( m(t, h) := (t + h) \land t \). Then it holds:

\[ |K_2^X (t_1 + h_1, t_2 + h_2) - K_2^X (t_1, t_2)| \]

\[ \leq |K_2^X (t_1 + h_1, t_2 + h_2) - K_2^X (t_1, t_2 + h_2)| + |K_2^X (t_1, t_2 + h_2) - K_2^X (t_1, t_2)| \]

\[ \leq |E(\{X(t_1 + h_1) - X(t_1)\} X(t_2 + h_2)| + |E[X(t_1) (X(t_2 + h_2) - X(t_2))]| \]

\[ \leq |E[\int_{m(t, h) \land t}^{M(t, h)} X(u + r) da(u) dr X(t_2 + h_2)]| + |E[\int_{m(t, h) \land t}^{M(t, h)} X(u + r) db(u) dW(r) X(t_2 + h_2)]| \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} |E[X(t_1) X(u + r) X(t_2 + h_2)]| da(u) dr \]

\[ + |E[X(t_1) X(u + r) X(t_2 + h_2)]| db(u) \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} |E[X(t_1) X(u + r) X(t_2 + h_2)]| |db(u)| \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]

\[ \leq \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |da(u)| dr \]

\[ + \int_{m(t, h) \land t}^{M(t, h)} E[|X(u + r)|^2] \frac{1}{2} E[|X(t_2 + h_2)|^2] \frac{1}{2} |db(u)| dr \]
(2) For $\gamma \in (0, \frac{1}{2})$, $t_i \in [0, \infty)$, $i = 1, 2$, it holds:

$$\lim_{h \to 0} \frac{1}{M(t_i, h)} \left( \int_{m(t_i, h)} E\|X(u + r)^2\| dr \right)^{\frac{1}{2}} \leq C_2^{\frac{1}{2}},$$

$$\lim_{h \to 0} \frac{1}{M(t_i, h)} \left( \int_{m(t_i, h)} E\|X(u + r)^2\| dr \right)^{\frac{1}{2}} \leq C_2^{\frac{1}{2}},$$

(3) Hence it follows ($i = 1, 2$)

$$\lim_{h \to 0} \frac{1}{M(t_i, h)} \left( \int_{m(t_i, h)} E\|X(u + r)^2\| dr \right)^{\frac{1}{2}} = 0,$$

for $\gamma \in (0, \frac{1}{2})$:

$$\lim_{h \to 0} \frac{1}{M(t_i, h)} \left( \int_{m(t_i, h)} E\|X(u + r)^2\| dr \right)^{\frac{1}{2}} \leq C_2^{\frac{1}{2}}.$$

for $\gamma \in (0, \frac{1}{2}]$:

$$\lim_{h \to 0} \frac{1}{M(t_i, h)} \left( \int_{m(t_i, h)} E\|X(u + r)^2\| dr \right)^{\frac{1}{2}} = 0.$$

This proves the theorem.

In the previous theorem we have proven the continuity of a compound function $K_2^X$ of a solution $X$ of (2.1). We have proven even more: a Hölder-continuity. The Hölder-continuity indicates that the compound functions $K_2^X$ have a certain smoothness. So next we want to investigate the Fréchet-differentiability of compound functions $K_2^X$.

In a first step we investigate whether the trajectories of the SDDE possesses already a smoothness property. The motivation is that for a deterministic linear DDE

$$x'(t) = \int_{-\tau}^{0} x(t + u) du, \quad t \in [0, \infty),$$

it can be shown that the trajectories become increasingly smoother with the time evolving ([2]). However, for solutions $X$ of (2.1) this is not true anymore as the following example of the scalar geometric Brownian Motion shows:

$$dX(t) = \alpha X(t) dt + \beta X(t) dW(t), \quad t \in [0, \infty)$$

$$X(0) = \xi \in \mathbb{R}$$

It is a well-known fact that the trajectories of solutions $X(t) = \xi e^{(\alpha - \frac{\beta^2}{2})t + \beta W(t)}$ are almost nowhere differentiable on $[0, \infty)$.

On the other hand it is known that the expectations of solutions $X$ of (2.1) satisfy the DDE (2.27). Hence the expectations of the trajectories $E\{X(t)\}$ possess the smoothing property on $[0, \infty)$. So it is reasonable to expect more regularity of compound functions $K_2^X$ than just continuity.

The following theorem confirms this expectation except a minor limitation. It can be shown that the compound function $K_2^X$ is Fréchet-differentiable on $K_2^X, \Delta(t)$.

An essential part of the proof consists in using the linearity of expectations, integrals and of showing that occurring expectations satisfy inhomogeneous affine DDE related to (2.27). The next auxiliary lemma illustrates this. But first we introduce:

**Definition 2.24**

Let $x_0$ be the solution of the scalar DDE

$$x'(\hat{t}) = \int_{-\tau}^{0} x(\hat{t} + u) du, \quad \hat{t} > 0,$$

$$x(\hat{t}) = \delta(\hat{t}), \quad \hat{t} \leq 0.$$

Then $x_0$ is called the **fundamental solution** of (2.26).
Lemma 2.25

Let \( s \in [t_0, \infty) \), \( v \in [-\tau, 0] \), \( h \in \mathbb{R}_+ \),

\[ Y \text{ a } \mathcal{F}_t\text{-measurable random variable,} \]

\[ G(t; s, h, v) = \mathbb{E}[X(t) \int_s^{s+h} X(r + v) \, dW(r) \, Y], \quad t \in [t_0, \infty), \]

\[ \mathcal{G}(t; s, h) = \mathbb{E}[X(t) \int_s^{s+h} \int_s^{s+h} X(r + v) \, db(u) \, dW(r) \, Y], \quad t \in [t_0, \infty), \]

Then it holds for all \( t \in [t_0, \infty) \):

\[ G(t; s, h, v) = \int_s^{t \land (s+h)} x_0(t - r) \int_r^0 \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr, \]

\[ \mathcal{G}(t; s, h) = \int_s^{t \land (s+h)} x_0(t - r) \int_r^0 \int_{-\tau-r}^0 \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, db(v) \, dr. \]

Proof:
(1) For all $t \in (s, \infty)$ it holds:
\[
G(t; s, h, v) = \mathbb{E}[X(t) \int_s^{s+h} X(r + v) \, dW(r) \, Y]
\]
\[
= \mathbb{E}[X(s) \int_s^{s+h} X(r + v) \, dW(r) \, Y]
\]
\[
+ \mathbb{E}\left[ \int_t^s X(r + u) \, da(u) \int_s^{s+h} X(r + v) \, dW(r) \, Y \right]
\]
\[
+ \mathbb{E}\left[ \int_s^{s+h} X(r + u) \, db(u) \int_t^s X(r + v) \, dW(r) \, Y \right]
\]
\[
= \int_t^s Fubini \left[ \int_s^{s+h} X(r + v) \, dW(r) \left| \mathcal{F}_s \right. \right] X(s) \, Y
\]
\[
+ \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, da(u) \, dr
\]
\[
+ \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr
\]
\[
= \int_t^s \int_s^{s+h} G(r + u; s, h, v) \, da(u) \, dr + \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr
\]
\[
= \int_t^s \int_s^{s+h} G(r + u; s, v, h) \, da(u) \, dr + \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr
\]
\[
= \int_t^s \int_s^{s+h} Fubini \left[ \int_s^{s+h} X(r + v) \, dW(r) \left| \mathcal{F}_s \right. \right] X(s) \, Y
\]
\[
+ \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, da(u) \, dr
\]
\[
+ \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr
\]

For all $t \in [t_0, s]$ it holds:
\[
G(t; s, h, v) = \mathbb{E}[X(t) \int_s^{s+h} X(r + v) \, dW(r) \, Y]
\]
\[
= \mathbb{E}[X(s) \int_s^{s+h} X(r + v) \, dW(r) \, Y]
\]
\[
= \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, da(u) \, dr + \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr
\]
\[
= \int_t^s \int_s^{s+h} Fubini \left[ \int_s^{s+h} X(r + v) \, dW(r) \left| \mathcal{F}_s \right. \right] X(s) \, Y
\]
\[
+ \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, da(u) \, dr
\]
\[
+ \int_t^s \int_s^{s+h} \mathbb{E}[X(r + u) \, X(r + v) \, Y] \, db(u) \, dr
\]

(2) From (1) follows that $G(\hat{t} + s; s, h, v)$ satisfies the affine DDE:
\[
x'\left(\hat{t}\right) = \int_0^{\hat{t}} x(\hat{t} + u) \, da(u) + g(\hat{t}), \quad \hat{t} > 0,
\]
\[
x(\hat{t}) = 0, \quad \hat{t} \leq 0,
\]
where
\[
g(\hat{t}) = \int_0^{\hat{t}} \mathbb{E}[X(\hat{t} + s + u) \, X(\hat{t} + s + v) \, Y] \, db(u) \, 1_{[0, h]}(\hat{t}), \quad \hat{t} \geq 0.
\]

From theory of affine DDE it is known ([2], pp.174), that a solution $x$ of (2.27) can be represented by a variation-of-parameters formula. Using the initial function $0$ we conclude:
\[
x(\hat{t}) = \int_0^{\hat{t}} x_0(\hat{t} - \tilde{r}) g(\tilde{r}) \, d\tilde{r}, \quad \hat{t} \geq 0.
\]
\[
x_0 \text{ is the so-called fundamental solution of (2.27) and is defined above.}
(3) From (2) it follows for all \( t \in (s, \infty) \):
\[
G(t; s, h, v) = \int_s^t x_0(t - r) \int_0^r \mathbb{E}[X(r + u) X(r + v) Y] \, db(u) \, 1_{[s, s+\delta]}(t) \, dr
\]
\[
= \int_s^t x_0(t - r) \int_0^r \mathbb{E}[X(r + u) X(r + v) Y] \, db(u) \, dr
\]
\[
= \frac{t \wedge (s+\delta)}{s} \int_s^t x_0(t - r) \int_0^r \mathbb{E}[X(r + u) X(r + v) Y] \, db(u) \, dr.
\]
Hence it follows for all \( t \in (s, \infty) \):
\[
\mathbb{G}(t; s, h) = \left. \int_0^t G(t; s, h, v) \, db(v) \right|_{t \wedge (s+\delta)} = \int_0^t \int_0^r x_0(t - r) \int_0^r \mathbb{E}[X(r + u) X(r + v) Y] \, db(u) \, dr \, db(v)
\]
\[
\mathbb{G}(t; s, h) = \left. \int_0^t G(t; s, h, v) \, db(v) \right|_{t \wedge (s+\delta)} = \int_0^t \int_0^r x_0(t - r) \int_0^r \mathbb{E}[X(r + u) X(r + v) Y] \, db(u) \, b(v) \, dr.
\]
\[
(2.29)
\]
\[
(2.30)
\]
Now we are prepared to prove:

**Theorem 2.26**
The comoment function \( K^X_2 \) of a solution \( X \) of (2.15) is Frechet-differentiable on \( K^2_{+, \delta} (\tau) \).
The Frechet-derivative of \( K^X_2 \) is:
\[
(2.26)
\]
\[
(2.27)
\]
and is continuous on \( K^2_{+, \delta} (\tau) \).
The proof:
Deferred to Section 5.5.

2.5 An example for amplified systems corresponding to comoment functions of order \( p = 2 \)
Consider the geometric Brownian motion defined by
\[
\begin{align*}
\frac{dX(t)}{X(0)} &= \alpha X(t) \, dt + \beta X(t) \, dW(t), \quad t \in (0, \infty),
\end{align*}
\]
\[
(2.31)
\]
It can be easily verified that the exact solution of (2.31) is
\[
X(t) = \xi e^{(\alpha - \frac{1}{2} \beta^2) t + \beta W(t)}, \quad t \in (0, \infty).
\]
\[
(2.32)
\]
\[
(2.33)
\]
\[X \text{ has the comoment function}
\]
\[
K^X_2(t, s) = c^2 e^{\alpha t + (\alpha + \beta^2) s}, \quad \infty > t \geq s \geq 0, \quad c^2 = \mathbb{E}\left[ \xi^2 \right].
\]
It holds:
\[
\frac{d}{d\nu} K^X_2(x, \nu) = \frac{d}{d\nu} e^{(2\alpha + \beta^2) \nu} = 2\alpha c^2 e^{(2\alpha + \beta^2) \nu} + \beta^2 c^2 e^{(2\alpha + \beta^2) \nu} = 2\alpha K^X_2(t, s) + \beta^2 K^X_2(t, s),
\]
\[
(2.32)
\]
For any \( \tau \in [0, \infty) \), the SDE (2.31) can be considered as an SDDE
\[
\begin{align*}
dX(t) &= \int_0^t X(t + u) \, da(u) \, dt + \int_0^t X(t + u) \, db(u) \, dW(t), \quad t \in (0, \infty),
\end{align*}
\]
\[
(2.33)
\]
\[X(t) = \xi \delta_0(t), \quad t \in [-\tau, 0] \]
where $a = \alpha \delta_0$ and $b = \beta \delta_0$.

Then it holds:

\[
\begin{align*}
0 & \int K^X_2(t, s) \, da(u) = \alpha K^X_2(t, s), & t \in [0, \infty), \ s \in [0, \tau], \\
0 & \int_0^t K^X_2(t, v) \, db(v) = \beta^2 K^X_2(t, t), & t \in [0, \infty).
\end{align*}
\]  

(2.34)

Hence (2.32) and (2.34) show that the comoment function $K$ satisfies the amplified system. Furthermore, the Frechet-derivative of $K^X_2$ does exist for all $t > s \geq 0$ and is

\[
(K^X_2)'(t, s) = (c^2 \alpha e^{\alpha t + (\alpha + \beta)^2 s}, c^2 (\alpha + \beta) e^{\alpha t + (\alpha + \beta)^2 s})^T
\]

On the other hand, the fundamental solution of

\[
x'(t) = \int_0^t x(u + t) \, da(u) \, dt
\]

is $x_0 = e^{\alpha (t - s)}$. Furthermore, for $t > s \geq 0$ it holds:

\[
\begin{align*}
0 & \int_0^t K^X_2(t, u) \, da(u) = c^2 \alpha e^{\alpha t + (\alpha + \beta^2) s}, \\
0 & \int_0^t K^X_2(t, u + v) \, da(v) + x_0(t - s) \int_0^t K^X_2(s + u, s + v) \, db(u) b(v) = c^2 \alpha e^{\alpha t + (\alpha + \beta^2) s} + e^{\alpha (t - s)} c^2 \beta^2 e^{(2\alpha + \beta) s} \\
& = c^2 (\alpha + \beta^2) e^{\alpha t + (\alpha + \beta^2) s}.
\end{align*}
\]

3 An approach to evaluate the evolution of quadratic moments

In the previous section we have shown that for a solution $X$ of a SDDE (2.1) the second moments function $m^X$ depends on and is embedded into the comoment function $K^X_2$ of $X$. Furthermore we showed that $K^X_2$ satisfies the amplified system (2.3). In this section we want to consider the asymptotic behaviour of $K^X_2$ and derive a criterion for the asymptotic behaviour of $m^X$.

First we start with some preliminary considerations. From (2.20) we know that for all $s \in [0, \infty)$, $t \in [s - \tau, \infty)$ $K^X_2(t, s)$ satisfies the DDE

\[
\begin{align*}
x'(t) &= \int_0^t x(t + u) \, da(u), & t \in [0, \infty) \\
x(t) &= \xi, & t \in [-\tau, 0].
\end{align*}
\]  

(3.1)

where $\xi(t) = \xi_\sigma(t) := K^X_2(t + s, s)$. From the theory about DDE ([5], [2]) it is known that the asymptotic behaviour of $K^X_2$ for $t \to \infty$ depends on $\xi$ and on the set

\[
\sigma(\alpha) := \{ \lambda \in \mathbb{C} \ | \ \lambda = \text{arg} \int_0^t e^{\lambda u} \, da(u) \}.
\]  

(3.2)

In particular for $s = 0$ one obtains:

**Lemma 3.1**

Let $X$ be solution of (2.18) and $K^X_2$ its comoment function.

Then it holds:

(i) $\max(Re(\sigma(\alpha))) > 0 \Rightarrow \exists \xi$ such that from $K^X_2(\omega, 0) = \xi$ it follows $\lim_{t \to \infty} |K^X_2(t, 0)| = \infty$.

(ii) $\max(Re(\sigma(\alpha))) \leq 0 \Rightarrow \forall k \in K_{sup}$ it follows $\lim_{t \to \infty} |K^X_2(t, 0)| < \infty$.

For $s > 0$ the start function $\xi$ (in (3.1)) depends on $K^X_2$. It is not obvious whether for any $\xi$ there exists $k \in K_{sup}$ such that $K^X_2(s + \omega, s) = \xi$. So if $Re(\sigma(\alpha)) \subset (-\infty, 0)$, for all $s > 0$ $K^X_2(\omega, s)$ is bounded. However, if $Re(\sigma(\alpha)) \cap (0, \infty)$ it is not clear whether for all $s > 0$ there exist $k \in K_{sup}$ such that
\[ \lim_{t \to \infty} |K_{2X}(t,0)| = \infty. \]

We have seen so far that we can characterize partially the asymptotic behaviour of \( K_{2X} \) along rays \( \mathcal{R}(s) := \{(t,s) \mid t \in [s,\infty)\} \) by \( \sigma(a) \). However, in the focus of our interest is \( \mu_2^X \). So a natural question is whether it is sufficient to know the behaviour of \( K_{2X} \) along rays \( \mathcal{R}(s) \). The Hölder-inequality leads to the criterion:

**Lemma 3.2**

Let \( X \) be solution of (2.18) and \( K_{2X} \) its comoment function,

\[ \exists s \in [0,\infty) : \mathbb{E}[|X(s)|^2] \neq 0. \]

Then it holds:

\[ \lim_{t \to \infty} |K_{2X}(t,s)| = \infty \implies \lim_{t \to \infty} |K_{2X}(t,t)| = \infty. \]

**Proof:**

\[ \lim_{t \to \infty} K_{2X}(t,s) \leq \lim_{t \to \infty} K_{2X}(t,t)^\frac{1}{b} K_{2X}(s,s)^\frac{1}{b}. \]

On the other hand, from the boundedness of \( K_{2X} \) on rays \( \mathcal{R}(s), s \in [0,\infty), \lim_{t \to \infty} |K_{2X}(t,t)| < \infty \) cannot be concluded. This proves the following example:

**Example 3.3**

We continue the example from section 2.5. The comoment function of the geometric Brownian motions is \( K(t,s) = e^{\alpha t + \frac{1}{2} \beta^2 s}. \) Hence if \( \alpha < 0 \) and \( 2\alpha + \beta^2 > 0 \) it follows: \( \lim_{t \to \infty} K(t,t) = \infty, \forall s \in [0,\infty): \lim_{t \to \infty} K(t,s) = 0. \)

This makes it necessary to investigate \( K_{2X} \) on \( \Delta \) directly.

In standard references to DDE ([2], [5]) one can find two approaches to derive stability criteria for solutions of the DDE (3.1) The first is to show via integration by parts that a solution \( x \) of solves a renewal equation. The change into the frequency space then leads to criteria in terms of characteristic equations (spectral theory). The second is to reformulate (3.1) as a linear ODE in a suitable Banach space, to consider the solution operator of this equation, to show its semigroup properties and to use the theory of semigroups to analyze the asymptotic behaviour (properties of the generator). Here we follow the second approach for the equation (2.1). We start with a helpful observation.

**Lemma 3.4**

Define \( D_t^2 = \{ (u,v) \mid u,v \in [-\tau,\infty), |v - u| \leq \tau \}. \)

Any solution \( K \) of (2.18) is uniquely determined by its values on \( D_t^2. \)

**Proof:**

(1) The function \( z \) satisfies the equation (2.22). But this shows that for any \( s \in [0,\infty) \) the function value \( z(s+\tau) \) is uniquely determined by the function values \( z(s) \) and \( \{ y_u(v) \mid u,v \in [s,s+\tau]^2 \} \subset D_t^2. \)

(2) For any \( s \in [0,\infty) \) the function \( y_s \) satisfies the equation (2.21), a linear DDE. This shows that for any \( s \in [0,\infty) \) the function \( y_s \) is uniquely determined by the function values \( \{ y_u(v) \mid u \in [s,s+\tau]^2 \} \cup \{ y_u(v) \mid u = s, v \in [s,s+\tau]\} \) on \( [s,\infty) \).

(3) By (1), (2) follows the objective.

Hence, although the function \( K \) has two dimensions, its decisive dynamics occurs in \( D_t^2 \), one-dimensional along the axis \( \Delta_t. \) This one-dimensionality allows us to introduce a time-continuous solution group in an suitable Banach space which describes the dynamics of \( K_{2X} \) in \( D_t^2. \) The aim is then to show that the solution operator is a strong continuous semigroup and to apply well-known semigroup theory to describe the asymptotic behaviour of the \( K_{2X} \) along \( \Delta_t. \)

The Banach-space we consider is the Banach space \( (\mathcal{K}_{mp}([-\tau,0]^2,\mathbb{R}), \| \cdot \|_{\mathcal{C}([-\tau,0]^2,\mathbb{R})}). \)
Definition 3.5
Let $T_{sup} := \{ T(t) \}_{t \in [0, \infty)}$ be an operator family such that:

(i) $\forall t \in [0, \infty), \beta, \gamma \in \mathbb{R}^+$ : $T(t) : K_{sup}([-\tau, 0]^2, \mathbb{R}) \rightarrow K_{sup}([-\tau, 0]^2, \mathbb{R})$,
(ii) $T(t) = \beta \implies \forall u, v \in [-\tau, 0]^2 : \gamma(u,v) = K_k(t+u,t+v)$,

The following theorem states the main property of $T$.

Theorem 3.6
$T_{sup}$ is a strong continuous semigroup.

Proof:
(1) By theorem 2.22, for any $k \in K_{sup}([-\tau, 0], \mathbb{R})$ we know that the solution $K$ of (2.18) with initial function $f_k$ is continuous on $[0, \infty)$.

(2) For any $k \in K_{sup}([-\tau, 0]^2, \mathbb{R})$ and $u, v \in [-\tau, 0]$ it holds: $(T(0))k(u,v) = K(0+u,0+v) = k(u,v)$.

(3) Let $s \in [0, \infty), \beta, \gamma \in \mathbb{R}^+$ define $T(t) := T(s)k$ and $K_k := T(t-s)K_k$. Let $K_k$ and $K_k^*$ be the solution of (2.18) with initial function $k$ and $K_k$, respectively. Define $K_k^*(t,s) := \forall T, s \in [-\tau, \infty)$ $K_k(k(t,s)) = K_k^*(t,s)$.

(4) As $K_k$ solves (2.18), by definition $K_k^*$ solves (2.18) as well. Furthermore, $\forall u, v \in [-\tau, 0]$ it holds: $K_k^*(u,v) = K_k(s+u,s+v) = k(u,v)$. Hence $K_k^* = K_k$.

(5) For all $u, v \in [-\tau, 0]$ it holds: $(T(t-s))k(u,v) = K_k(t-s+u,t-s+v)$.

(6) By definition $T_{sup}$ is a strong continuous semigroup.

The time evolution of strong continuous semigroups can be characterized by the properties of its generator. So next we investigate the generator of $T_{sup}$.

Definition 3.7
Let $k \in K_{sup}([-\tau, 0]^2, \mathbb{R})$. Define $\epsilon_r(t,s) := (-t \land s) \land (r + (t \land s))$. For all $(t,s) \in [-\tau, 0]$ define:

$$D_G(k)(t,s) := \left\{ \begin{array}{ll} \lim_{h \rightarrow 0, \epsilon_r(t,s)} \frac{k(t+h,s)-k(t,s)}{h}, & \exists \lim_{h \rightarrow 0, \epsilon_r(t,s)} \frac{k(t+h,s+h)-k(t,s)}{h}, \\ \infty, & \text{otherwise} \end{array} \right.$$  

Lemma 3.8
Let $A$ be the generator of $T_{sup}$.

Then it holds:

(i) $\forall k \in K_{sup}([-\tau, 0]^2, \mathbb{R}) : A \circ k = D_G(k)$.

(ii) $D(A) = \{ k \in K_{sup}([-\tau, 0]^2, \mathbb{R}) | D_G(k) \in K_{sup}([-\tau, 0]^2, \mathbb{R}), \forall t \in [-\tau, 0] \exists \frac{\partial}{\partial t} k(t,0) \}

\forall t \in [-\tau, 0]: D_G(k)(t,0) = \frac{\partial}{\partial t} k(t,0) + \int k(t,u) da(u),

D_G(k)(0,0) = 2 \int k(u,0) da(u) + \int \int k(u,v) db(u) db(v) \}.$
Proof:
(1) For any \( k \in K(\mathbb{R}) \), let \( K_k \) be the solution of (2.18) with the initial function \( k \).
(2) Let \( k \in D(A) \subset C([-\tau,0]^2,\mathbb{R}) \). Then it holds:
\[
0 = \lim_{h \to 0} \left\| \frac{1}{h} (T(h)k - k) \right\|_{C([-\tau,0]^2,\mathbb{R})} = \lim_{h \to 0} \sup_{t \in [-\tau,0]} \left\| \frac{1}{h} (K(t+h,s+h) - K(t,s)) - (A_k)(t,s) \right\|.
\]
(3) From (2) it follows that for all \( t,s \in [-\tau,0] \):
\[
(A_k)(t,s) = \lim_{h \to 0, h \leq \tau} \left( \frac{1}{h} (k(t+h,s+h) - k(t,s)) \right).
\]
(4) As \( k \in D(A) \), \( A_k \in K_{sup}([-\tau,0]^2,\mathbb{R}) \). But then for all \( s \in [-\tau,0] \) \( A_k \) is uniformly continuous on \( \mathcal{R}(s) = \{(u,v) \mid (u,v) = (-\tau,s) + \alpha (1,1), \alpha \in [0,1] \} \).
(5) For all \( t,s \in [-\tau,0] \) and \( h \in (0, -\tau - (t \land s)) \):
\[
\left\| \frac{1}{h} \int_{-\tau}^{0} (k(t+h,s+h) - k(t,s)) - A(t,s) \right\| \leq \frac{1}{h} \left( k(t,s) - k(t-h,s-h) \right) + |A(t-h,s-h) - A(t,s)|.
\]
By the uniform convergence in (2) and (5) it follows: \( \lim_{h \to 0} \frac{1}{h} \left( k(t+h,s+h) - k(t,s) \right) = A(t,s) \).
(6) (3) and (5) show (i).
(7) For \( t \in [-\tau,0] \): \( -\tau - t \geq h \geq -t \) it follows:
\[
(A_k)(t,0) = \lim_{h \to 0, h \leq \tau} \left( \frac{1}{h} (K(t+h,0) - K(t,0)) \right)
\]
It follows:
\[
(A_k)(0,0) = \lim_{h \to 0} \left( \frac{1}{h} (K(h,0) - K(0,0)) \right)
\]
(8) (2), (3) show (ii).

We now consider the spectrum \( \sigma(A) \) of the semigroup \( T_{sup} \). It turns out that it can be determined and characterized by a linear DDE with linear constraints.

**Definition 3.9**
The equation
\[
\begin{align*}
    f'(t) &= \lambda f(t) - \int_{-\tau}^{0} f(-|u-t|) e^{\lambda(u\vee t)} \, du(t), \quad t \in [-\tau,0], \\
    \lambda f(0) &= 2 \int_{-\tau}^{0} f(u) \, du + \int_{-\tau}^{0} \int_{-\tau}^{0} f(-|u-v|) e^{\lambda(u\vee v)} \, du \, dv.
\end{align*}
\]
with \( \lambda \in \mathbb{C} \) is called characteristic equation of order 2. Any \( \lambda \in \mathbb{C} \), for which (3.4) has a non-zero solution \( f \in C([-\tau,0],\mathbb{R}) \cap C^1([-\tau,0],\mathbb{R}) \), is called an eigenvalue of order 2.

**Theorem 3.10**
Let \( X \) be a solution of (2.1) and \( K^X_0 \) its common function.
Then it holds:
\[
\sigma(A) = \sigma_p(A) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ is eigenvalue of order 2} \}.
\]

Proof:
(1) As \( k \in \mathcal{K}([-\tau,0],\mathbb{R}) \), \( K^X \) is continuous. Let \( t \geq \tau \). By corollary \((2.17)\) \( K^X \) satisfies an exponential growth condition. Hence \( T(t) \) maps bounded sets \( T(t) S \in \mathcal{C}([-\tau,0]^2,\mathbb{R}) \) into bounded sets \( T(t) S \in \mathcal{C}([-\tau,0]^2,\mathbb{R}) \). By theorem \((2.23)\) \( K^X \) is Hölder-continuous for all \( \gamma \in (0,\frac{1}{2}) \) and has a uniform finite bound of the Hölder-derivative for \( \gamma = \frac{1}{2} \). Hence \( T(t) S \) is precompact in \( \mathcal{C}([-\tau,0]^2,\mathbb{R}) \) by the theorem of Arzela-Ascoli. Hence \( \{T(t)\}_{t \in [0,\infty)} \) is eventually compact. But then it follows \( \sigma(A) = \sigma_f(A) \) (see \([2]\)).

(2) From the definition of the point spectrum we recall that \( \lambda \in \sigma(A) \iff \exists \lambda \in \mathcal{D}(A) : (\lambda - A) k = 0 \).

(3) From (2) we conclude that \( \forall \lambda \in [-\tau,0], s \in (t,0) : \lim_{h \to 0^+} \frac{1}{h} (k(t + h, s + h) - k(t, s)) = \lambda k(t, s) \). That is for any \( u \in [-\tau,0] \) \( x(r) := k(r + u, r) \) satisfies the linear ODE \( x'(r) = \lambda x(r) \) for \( r \in [-\tau - u, 0] \).

(4) Define \( \forall u \in [-\tau,0] : f(u) := \lambda(u, 0) \).

By lemma \(3.8(ii)\) we know that \( f \) is differentiable on \([-\tau,0]\).

Due to (3) and continuity of \( k \) it follows \( \forall s \in [-\tau,0] : k(t, s) = f(-|t|) e^{\lambda(t/s)} \).

(5) Let \( t \in [-\tau,0] \). Then it holds:

\[
0 = (\lambda k - Ak) (t,0) = \lambda f(t) - f'(t) - \int_{-\tau}^{0} f(-|t| - u) e^{\lambda(tu)} da(u).
\]

Furthermore it holds:

\[
0 = (\lambda k - Ak) (0,0) = \lambda f(0) - 2 \int_{-\tau}^{0} f(u) da(u) - \int_{-\tau}^{0} \int_{-\tau}^{0} f(-|u|) e^{\lambda(\lambda u t) - \lambda u t} da(u) db(v).
\]

(6) The claim follows from the definition of the \( \mathcal{D}(A) \).

Now as the theorem is proven we try to simplify the characteristic equation of order 2. Consider \( g(t) = f(t) e^{-\lambda t} \). Then it holds

\[
g'(t) = -\lambda f(t) e^{-\lambda t} + f'(t) e^{-\lambda t}
\]

\[
= -\lambda f(t) e^{-\lambda t} + \lambda f(t) e^{-\lambda t} - \int_{-\tau}^{0} f(-|u|) e^{\lambda(\lambda u t) - \lambda u t} da(u),
\]

As \( f, g \in \mathcal{C}([-\tau,0],\mathbb{R}) \), this differential equation can be restated as an integral equation

\[
g(t) = g(0) + \int_{-\tau}^{0} \int_{-\tau}^{0} g(-|u|) e^{\lambda((u-s)\wedge 0)} da(u) ds.
\]

Furthermore it holds:

\[
\int_{-\tau}^{0} \int_{-\tau}^{0} g(-|u|) e^{\lambda((u-s)\wedge 0)} da(u) ds = \int_{-\tau}^{0} \int_{-\tau}^{0} g(-|u|) e^{\lambda((v-s)\wedge 0)} ds da(u).
\]

Hence:

\[
\int_{-\tau}^{0} \int_{-\tau}^{0} g(-|u|) e^{\lambda((u-s)\wedge 0)} da(u) ds
\]

\[
= \int_{-\tau}^{0} g(v) (a(v) - a(-\tau \vee (v + t))) e^{\lambda v} dv + \int_{-\tau}^{0} g(-v) (a(0) - a(v + t)) dv
\]

\[
= \int_{-\tau}^{0} g(v) (a(v) - a(-\tau \vee (v + t))) e^{\lambda v} dv + \int_{-\tau}^{0} g(v) (a(0) - a(t - v)) dv
\]

\[
= \int_{-\tau}^{0} g(v) (a(v) - a(-\tau \vee (v + t))) e^{\lambda v} + (a(0) - a(t - v)) 1_{\beta^0}(v) g(v) dv
\]

\[
= \int_{-\tau}^{0} k_\lambda(t,v) g(v) dv,
\]

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where \( k_\lambda(t,v) := (a(v) - a(-\tau \vee (v + t))) e^{\lambda v} + (a(0) - a(t - v)) \) \( 1_{[0,\theta]}(v) \).

This allows us to characterize the spectrum of the semigroup \( T \) as solutions of a Volterra-integral equation with constraints.

**Theorem 3.11**

Let \( X \) be a solution of (2.1) and \( K_\lambda^N \) its component function,

\[
k_\lambda(t,v) = (a(v) - a(-\tau \vee (v + t))) e^{\lambda v} + (a(0) - a(t - v)) \) \( 1_{[0,\theta]}(v) \), \quad t, v \in [-\tau, 0],
\[
g(t) = g(0) + \int_{-\tau}^{t} k_\lambda(t,v) g(v) \, dv, \quad t \in [-\tau, 0), \tag{3.5}
\]

\[
\lambda g(0) = 2 \int_{-\tau}^{0} g(u) e^{\lambda u} \, du + \int_{-\tau}^{0} \int_{-\tau}^{0} \int_{-\tau}^{0} g(-|u - v|) e^{\lambda(u \wedge v)} \, db(u) \, db(v).
\]

Then it holds:

\[
\sigma(A) = \{ \lambda \in \mathbb{C} \mid \exists g \in C([-\tau, 0], \mathbb{R}) \cap C([-\tau, 0], \mathbb{R}): \text{g satisfies (3.5) and } g \neq 0 \}. 
\]

**Proof:**

Follows immediately from lemma 3.4, the above computations and the fact that \( e^{\lambda u} \) is strictly positive (implying \( f \equiv 0 \Leftrightarrow g \equiv 0 \)).

Finally we observe that the \( \tau \)-dependent systems (3.4) and (3.5) can be rescaled to a standard problem.

**Lemma 3.12**

Let \( \tau \in (0, \infty) \),

\[
\tilde{a}, \tilde{b} \in ([-1, 0], \mathbb{R}) \text{ defined by } \forall u \in [-1, 0]: \tilde{a} = r a(\tau u), \tilde{b} = \sqrt{\tau} b(\tau u).
\]

Consider the systems

\[
f'(t) = \lambda f(t) - \int_{-1}^{0} f(-|u - t|) e^{\lambda(u \wedge t)} \, da(u), \quad t \in [-1, 0], \tag{3.6}
\]

\[
\lambda f(0) = 2 \int_{-1}^{0} f(u) \, da(u) + \int_{-1}^{0} \int_{-1}^{0} f(-|u - v|) e^{\lambda(u \wedge v)} \, db(u) \, db(v).
\]

and

\[
k_\lambda(t,v) = (\tilde{a}(v) - \tilde{a}(-1 \vee (v + t))) e^{\lambda v} + (\tilde{a}(0) - \tilde{a}(t - v)) \) \( 1_{[0,\theta]}(v) \), \quad t, v \in [-1, 0],
\]

\[
g(t) = g(0) + \int_{-1}^{t} k_\lambda(t,v) g(v) \, dv, \quad t \in [-1, 0), \tag{3.7}
\]

\[
\lambda g(0) = 2 \int_{-1}^{0} g(u) e^{\lambda u} \, da(u) + \int_{-1}^{0} \int_{-1}^{0} g(-|u - v|) e^{\lambda(u \wedge v)} \, db(u) \, db(v).
\]

Define \( \sigma_f(A) = \{ \lambda \in \mathbb{C} \mid \exists f \in C([-1, 0], \mathbb{R}) \cap C([-1, 0], \mathbb{R}): \text{f satisfies (3.6) and } f \neq 0 \} \),

\[
\sigma_g(A) = \{ \lambda \in \mathbb{C} \mid \exists g \in C([-1, 0], \mathbb{R}) \cap C([-1, 0], \mathbb{R}): \text{g satisfies (3.7) and } g \neq 0 \}. 
\]

Then it holds:

\[
\tau \sigma(A) = \sigma_f(A) = \sigma_g(A) \tag{3.8}
\]

**Proof:**

(1) Let \( \lambda \in \sigma(A) \) and \( f \neq 0 \) an eigenfunction corresponding to \( \lambda \) satisfying (3.4).

Define \( \tilde{f} \in C([-1, 0], \mathbb{R}) \) defined by: \( \forall u \in [-1, 0]: \tilde{f}(\tilde{u}) = f(\tau \tilde{u}), \quad \lambda \tilde{f} = \lambda f(\tau \tilde{u}), \quad \tilde{a} = r a, \tilde{b} = \sqrt{\tau} b, t = \tau \tilde{t}. \)

Then for all \( \tilde{u} \in [-1, 0] \) it holds:

\[
\tilde{f}'(\tilde{u}) = \tau f'(\tilde{u}) = \lambda f(\tilde{u}) - \int_{-\tau}^{0} f(-|\tau \tilde{u} - \tilde{v}|) e^{\lambda(\tau \tilde{v})} (\tau \tilde{v}) d(\tau a(\tilde{v}))
\]

\[
= \lambda \tilde{f}(\tilde{u}) - \int_{-\tau}^{0} f(-|\tilde{u} - \tilde{v}|) e^{\lambda(\tau \tilde{v})} (\tau \tilde{v}) d(\tilde{a}(\tilde{v}))
\]

\[
\lambda \tilde{f}(0) = \lambda \tilde{f}(0) = 2 \int_{-1}^{0} f(\tau \tilde{u}) (\tau a(\tilde{u})) + \int_{-1}^{0} \int_{-1}^{0} f(-|\tau \tilde{u} - \tau \tilde{v}|) e^{\lambda(\tau \tilde{u} + \tau \tilde{v})} \tilde{a} d(\tilde{a}) \, db, \quad \tilde{a} \in [-1, 0], \tilde{v} \in [-1, 0], \tilde{a} \neq 0, \tilde{b} \neq 0.
\]

As \( \tilde{f} \neq 0, \tilde{\lambda} \) belongs to \( \sigma_f \). Hence \( \tau \sigma(A) \subseteq \sigma_f(A) \).
(2) Let $\tilde{\lambda} \in \sigma_f$ and $\tilde{f} \not= 0$ an eigenfunction corresponding to $\tilde{\lambda}$ satisfying (3.6). Define $f \in C([-\tau, 0], \mathbb{R})$ by: $\forall u \in [-\tau, 0]: f(u) = \tilde{f}(\frac{u}{\tau})$. 
$\lambda = \frac{\lambda}{\tau}, \mu = u, \nu = \frac{\nu}{\tau}, \tau = \frac{\tau}{\tau}$,
Then for all $t \in [-\tau, 0]$ it holds:

$$f'(t) = \frac{1}{\tau} f'(\frac{t}{\tau}) = \frac{1}{\tau} \lambda \tau f(\frac{t}{\tau}) - \int_{-\tau}^{0} f(-\frac{t}{\tau} - \xi \tau) e^{\lambda \tau} (-\frac{1}{\tau} ; -\tau) \, d\lambda(\frac{\xi \tau}{\tau})$$
$$= \lambda f(u) - \int_{-\tau}^{0} f(-|u| - \tau) \, e^{\lambda (u - \tau \nu)} \, da(u)$$

$$f(0) = \frac{1}{\tau} f(0) = \int_{0}^{0} f(0) \, da(0) + \int_{-\tau}^{0} f(-|u| - \tau) \, e^{\lambda (u - \tau \nu)} \, db(u)$$

As $f \not= 0$, $\lambda$ belongs to $\sigma(A)$. Hence $\sigma_f(A) \subseteq \tau \sigma(A)$, and with (1) it follows $\sigma_f(A) = \tau \sigma(A)$.

(3) From lemma 3.11 it follows that $\sigma_f(A) = \sigma_g(A).

Finally we conclude:

**Corollary 3.13**

Let $X$ be a solution of (2.1) and $K_x$ be its common moment with initial function $k \in C([-\tau, 0]^2, \mathbb{R})$. Then it holds:

(i) If $\exists \lambda \in \sigma(A)$ with $\text{Re}(\lambda) > 0$, then $\exists k \in C([-\tau, 0]^2, \mathbb{R})$ such that $\lim_{t \to \infty} \mathbb{E} |X(t)|^2 = \infty$.

(ii) If $\forall \lambda \in \sigma(A) : \text{Re}(\lambda) < 0$, then $\lim_{t \to \infty} \mathbb{E} |X(t)|^2 = 0$.

(iii) If $\forall \lambda \in \sigma(A) : \text{Re}(\lambda) \leq 0$, then $\lim_{t \to \infty} \mathbb{E} |X(t)|^2 < \infty$.

**Proof:**

(1) Let $(t_n, s_n) \in \mathbb{N} \subset D_2^2$ be a sequence such that $\lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = \infty, \forall n \in \mathbb{N}, |t_n - s_n| \leq \tau \text{ and } \lim_{n \to \infty} |K(t_n, s_n)| = \infty$. By the Hölder-inequality: $|K(t_n, s_n)| \leq |K(t_n, t_n)|^{\frac{2}{3}} |K(s_n, s_n)|^{\frac{1}{3}}$. It follows:

$$\lim_{n \to \infty} |K(t_n, s_n)| = \infty.$$

(2) Let $k \in K_{sup}([-\tau, 0]^2, \mathbb{R})$.

If $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} = \infty$, by (1) it follows $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})}$.

If $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} = 0$, by (1) it follows $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} = 0$.

If $\sup_{t \in [0, \infty)} \{ \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} \} \leq \sup_{t \in [0, \infty)} \{ \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} \}$.

(3) The semigroup theory relates $\sigma(A)$ to $\|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})}$.

Define $m(A) = \max \{ \text{Re}(\lambda) | \lambda \in \sigma(A) \}$.

If $m(A) > 0$ then there exists $k \in K_{sup}([-\tau, 0]^2, \mathbb{R})$ such that $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} = \infty$.

If $m(A) < 0$ then for all $k \in K_{sup}([-\tau, 0]^2, \mathbb{R})$, $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} = 0$.

If $m(A) = 0$ then for all $k \in K_{sup}([-\tau, 0]^2, \mathbb{R})$, $\lim_{t \to \infty} \|T(t)k\|_{C([-\tau, 0]^2, \mathbb{R})} < \infty$.

(4) From (1)-(3) follows the claim.

**Example 3.14**

Consider the SDE describing a geometric Brownian motion:

$$dX(t) = a_0 X(t) \, dt + b_0 X(t) \, dW(t), \quad t \in [0, \infty).$$

Then the system (3.4) is

$$f'(t) = (\lambda - a_0 f(t), \quad t \in [-\tau, 0],$$

$$\lambda f(0) = 2a_0 f(0) + b_0 f(0).$$

(3.10)
and has the non-zero solutions
\begin{align*}
f(0) &= c_f, \\ f(t) &= c_f e^{\lambda - a_0} t, \\ \lambda &= 2a_0 + b_0^2.
\end{align*}

On the other hand system (3.5) is
\begin{align*}
k_\lambda(t,v) &= a_0 \delta_0 (v) (1 - \delta_t (0)) e^{\lambda t} + a_0 (1 - \delta_t (0)) 1_{[0,0]} (v), \\ g(t) &= g(0) + \int_0^t a_0 g(v) d(v), \\ \lambda g(0) &= 2a_0 g(0) + b_0^2 g(0).
\end{align*}
and has the non-zero solutions
\begin{align*}
g(0) &= c_g, \\ g(t) &= c_g e^{-a_0 t}, \\ \lambda &= 2a_0 + b_0^2.
\end{align*}

Obviously, if \( c_f = c_g \) one confirms \( g(t) = f(t) e^{-\lambda t} \).

Corollary 3.13 yields then that
\begin{enumerate}[(i)]
\item \( \lim_{t \to \infty} E[X(t)^2] = \infty \quad 2a_0 + b_0^2 > 0 \),
\item \( \lim_{t \to \infty} E[X(t)^2] = 0 \quad 2a_0 + b_0^2 < 0 \),
\item \( \forall t \in [0, \infty) : E[X(t)^2] \leq \infty \quad 2a_0 + b_0^2 = 0 \).
\end{enumerate}

**Example 3.15**

Consider the equation
\[ dX(t) = (a_0 X(t) + a_1 X(t - \tau)) dt + (b_0 X(t) + b_1 X(t - \tau)) dW(t), \quad t \in [0, \infty) \]

Then \( X \) is a SDDE (2.1) where the parameter functions are \( a(u) = a_1 \delta_{[-1, \infty)} (t) + a_0 \delta_{[0, \infty)} (t) \) and \( b(u) = b_1 \delta_{[-1, \infty)} (t) + b_0 \delta_{[0, \infty)} (t) \). For equation (14.1) the system (3.4) is
\begin{align*}
f' (t) &= (\lambda - a_0) f(t) - a_1 e^{\lambda t} f(-\tau - t), \\ \lambda f(0) &= 2a_0 f(0) + 2a_1 f(-\tau) + (b_0^2 + b_1^2 e^{-\lambda t}) f(0) + 2b_0 b_1 f(-\tau).
\end{align*}

The system (3.5) is
\begin{align*}
k_\lambda(t,v) &= a_0 1_{1,[0]} (v) + a_1 1_{[-1,-1,0]} (v), \\ g(t) &= g(0) + \int_0^t a_0 g(v) dv + a_1 \int_{-\tau}^t g(v) e^{\lambda t} dv, \\ \lambda g(0) &= 2a_0 g(0) + 2a_1 e^{\lambda t} g(-\tau) + (b_0^2 + b_1^2 e^{-\lambda t}) g(0) + 2b_0 b_1 e^{\lambda t} g(-\tau).
\end{align*}

Let \( \tilde a_0 = a_0 \tau, \tilde a_1 = a_1 \tau \). Then the normalized coefficient functions \( \tilde a, \tilde b \) are defined by \( \tilde a(u) = \tilde a_1 \delta_{[-1, \infty)} (t) + \tilde a_0 \delta_{[0, \infty)} (t) \) and \( \tilde b(u) = \tilde b_1 \delta_{[-1, \infty)} (t) + \tilde b_0 \delta_{[0, \infty)} (t) \). Hence the normalized characteristic systems are
\begin{align*}
f' (t) &= (\lambda - \tilde a_0) f(t) - \tilde a_1 e^{\lambda t} f(-1 - t), \\ \lambda f(0) &= 2\tilde a_0 f(0) + 2\tilde a_1 f(-1) + (\tilde b_0^2 + \tilde b_1^2 e^{-\lambda}) f(0) + 2\tilde b_0 \tilde b_1 f(-1).
\end{align*}

and
\begin{align*}
k_\lambda(t,v) &= \tilde a_0 1_{1,[0]} (v) + \tilde a_1 1_{[-1, -1,0]} (v), \\ g(t) &= g(0) + \tilde a_0 \int_0^t g(v) dv + \tilde a_1 \int_{-1}^t g(v) e^{\lambda t} dv, \\ \lambda g(0) &= 2\tilde a_0 g(0) + 2\tilde a_1 e^{\lambda t} g(-1) + (\tilde b_0^2 + \tilde b_1^2 e^{-\lambda}) g(0) + 2\tilde b_0 \tilde b_1 e^{\lambda t} g(-1).
\end{align*}
4 Weak stationarity

A side effect of considering the amplified system (2.18) is a characterization of weak stationary solutions of (2.1).

**Definition 4.1**

Let \((Y, \mathbb{T}, \mathbb{T} \subset \mathbb{R})\) be a stochastic process with finite first order moments. The function \(m^X_1: \mathbb{T} \rightarrow \mathbb{R}

\[ t \mapsto m^X_1(t) := \mathbb{E}[Y(t)] \]

is called the **first order moment or mean function**. If \(m^X_1\) is constant in \(\mathbb{T}\) then \(m^X_1\) is called stationary **mean function**.

**Definition 4.2**

Let \((Y, \mathbb{T}, \mathbb{T} \subset \mathbb{R})\) be a stochastic process with finite second order moments. The function \(K^X_2: \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}

\[ (t, s) \mapsto K^X_2(t, s) := \mathbb{E}[Y(t)Y(s)] \]

is called the **second order moment or covariogram function**. If \(\exists K \in (\mathbb{R}, \mathbb{R})\) such that \(\forall t, s \in \mathbb{T}: K^X_2(t, s) = K(t - s)\), then \(K^X_2\) is called stationary **covariogram function**.

**Definition 4.3**

Let \((Y, [0, \infty))\) be a stochastic process with finite first and second order moments. If \(m^X_1\) and \(K^X_2\) are stationary, then \(Y\) is called a **weakly stationary process**.

We consider now solutions \(X\) of (2.1). Then \(X\) has the following representation

\[
X(t) = X(0) + \int_0^t X(u + r) \, da(u) \, dr + \int_0^t X(u + r) \, db(u) \, dW(r) , \quad t \in (0, \infty) , \tag{4.1}
X(t) = \xi(t) , \quad t \in [-\tau, 0] .
\]

We use this representation to derive a criterion for a constant first moments function of solutions \(X\) of (2.1).

**Theorem 4.4**

Let \(X\) be a solution of (2.1) with the first moments function \(m^X_1\).

Then it holds: \(m^X_1\) is stationary on \([-\tau, \infty) \iff (i) \forall t \in [-\tau, 0]: m^X_1(t) = m^X_1(0) , (ii) 0 = m^X_1(0) a([-\tau, 0]) .

Proof:

(1) Let \(\forall t \in [-\tau, \infty) : m^X_1(t) = m^X_1(0)\). Then (i) is obvious. For \(t \in [0, \infty)\) it follows:

\[
m^X_1(t) = \mathbb{E}[X(t)] = \mathbb{E}[X(0)] + \int_0^t \mathbb{E}[X(u + r)] \, da(u) \, dr
\]

\[
= m^X_1(0) + \int_0^t m^X_1(u + r) \, da(u) \, dr
\]

\[
m^X_1(0)
= m^X_1(0) + \int_0^t m^X_1(0) \, da(u) \, dr
\]

\[
0
= m^X_1(0) a([-\tau, 0]) .
\]

This shows (ii).

(2) Let (i) and (ii) be satisfied. By (4.1) for \(t \in [0, \infty)\) it follows that \(m^X_1\) satisfies the DDE (here in the equivalent integral representation):

\[
x(t) = x(0) + \int_0^t x(u + r) \, da(u) \, dr , \quad t \in [0, \infty) ,
\]

\[
x(t) = m^X_1(0) , \quad t \in [-\tau, 0] .
\]

The above DDE has a unique solution. However as \(a([-\tau, 0]) = 0, x(t) \equiv m^X_1(0)\) solves the above DDE. Hence \(m^X_1 \equiv m^X_1(0)\) on \([-\tau, \infty)\).

Next we use the representation (4.1) to derive a criterion for stationary covariogram functions \(K^X_2\) of solutions \(X\) of (2.1).
Theorem 4.5

Let $X$ be a solution of $(2.1)$ with the cocoyne function $K^X_2$,

$K^X_2$ satisfy $(2.18)$ with the initial condition $k \in K_{n,b}([-\tau,0]^2, \mathbb{R})$.

Then it holds:

(i) $K^X_2$ is stationary on $[-\tau, \infty)$ if $\forall u, v \in [-\tau,0]^2 : k(u,v) = f(-|u-v|)$ where $f \in (-\tau, 0], \mathbb{R}$ satisfies

\[ 0 = 2 \int_{-\tau}^{0} f(u) \, da(u) + \int_{-\tau}^{0} \int_{-\tau}^{0} f(-|u-v|) \, db(u) \, db(v), \]

\[ f(t) = f(0) + \int_{t}^{0} \int_{-\tau}^{0} f(-|u-v|) \, da(u) \, dr, \quad t \in [-\tau, 0]. \]

(ii) If $K^X_2$ is stationary on $[-\tau, \infty)$, then $\forall s \in [-\tau, \infty), \bar{t} \in [0, \infty): K^X_2(t+s, \bar{t}) = x_f(\bar{t})$ where $x_f$ solves $(3.1)$ with $\xi = k(\omega, 0)$.

Proof:
(1) If $K_2^X$ is stationary, then there exists $\tilde{K}_2^X$ such that $\forall t, s \in [-\tau, 0]: K_2^X(t, s) = \tilde{K}_2^X(t - s)$.

Hence:
$$
\forall t, s \in [-\tau, 0]: \quad K_2^X(t, t) = \tilde{K}_2^X(t - t) = \tilde{K}_2^X(0) = \tilde{K}_2^X(s - s) = K_2^X(t, s).
$$

$$
\forall t \geq s \geq -\tau: \quad K_2^X(t, s) = K_2^X(t - s) = K_2^X(t - s, 0).
$$

(2) Define $\forall u \in [-\tau, 0]: f(u) = K_2^X(u, 0)$ and $\forall t \in [-\tau, 0]: x_f(t) = K_2^X(t, 0)$. (1) implies:

$$
\forall t \in [0, \infty): \quad 0 = \left. \frac{d}{dt} K_2^X(x, x) \right|_{x = t} = \left. \frac{d}{dt} K_2^X(x, x) \right|_{x = u = 0} = \left(2 \int_{-\tau}^{0} f(u, 0) du + \int_{-\tau}^{0} K_2^X(u, v) dv \right) .
$$

$$
\forall t \in [0, \infty): \quad 0 = \left. \frac{d}{dt} K_2^X(x, 0) \right|_{x = t} = \left. \frac{d}{dt} K_2^X(x, 0) \right|_{x = u = 0} = \left(2 \int_{-\tau}^{0} f(u) du + \int_{-\tau}^{0} f(u - u \vee v) dv \right) .
$$

$$
\forall t \in [-\tau, 0]: \quad x_f'(t) = \left(2 \int_{-\tau}^{t} f(u) du \right) + \int_{-\tau}^{t} x_f(u) du dr.
$$

$$
\forall t \in [0, \infty): \quad f(t) = f(0) + \int_{-\tau}^{0} f(u) du dr.
$$

This shows $\Rightarrow$ in (i) and (ii).

(3) Let $k$ and $f$ be given by (iii) and define $x_f$ to be the solution of (3.1) with $\xi = f$.

Define $\forall s \in [-\tau, \infty), \tilde{t} \in [0, \infty): \tilde{K}(\tilde{t} + s, s) := \tilde{K}(s, \tilde{t} + s) := x_f(\tilde{t})$.

(4) Let $u \in [-\tau, 0], v = 0$. Then $k(u, 0) = f(u \vee u \vee v) = f(u)$.

(5) $\tilde{K}$ defined in (3) satisfies by definition $\tilde{K}|_{[-\tau, 0]} = k$.

(6) Let $\tilde{t} \in [0, \infty), t \in [-\tau, \infty)$. Define $\tilde{t} := t - s$. Then $\tilde{t} \geq 0$ and $t = \tilde{t} + s$.

By definition of $\tilde{K}$ it follows $\tilde{K}(t, s) = \tilde{K}(\tilde{t} + s, s) = x_f(\tilde{t})$.

Hence: $\lim_{\tilde{t} \to 0, \tilde{t} \geq 0} \frac{1}{\tilde{t}} (\tilde{K}(t + \tilde{t}, s) - \tilde{K}(\tilde{t}, s)) = \lim_{\tilde{t} \to 0, \tilde{t} \geq 0} \frac{1}{\tilde{t}} (x_f(\tilde{t} + \tilde{t}) - x_f(\tilde{t})) = x_f'(\tilde{t})$.

As $x_f$ satisfies (3.1), it follows: $x_f'(\tilde{t}) = \int_{-\tau}^{\tilde{t}} f(u) du = \int_{-\tau}^{\tilde{t}} f(u) du dr$.

By definition of $\tilde{K}$, it follows $\forall u \in [-\tau, 0]: x_f(u) = \tilde{K}(\tilde{u} + u - s, s) = \tilde{K}(t + u, s)$.

Hence: $\lim_{\tilde{t} \to 0, \tilde{t} \geq 0} \frac{1}{\tilde{t}} (\tilde{K}(t + \tilde{t}, s) - \tilde{K}(\tilde{t}, s)) = \int_{-\tau}^{t} \tilde{K}(t + u, s) du$.

(7) By definition of $\tilde{K}$, $\forall t \in [0, \infty)$ it holds: $\tilde{K}(t, t) = x_f(0)$.

Hence: $\lim_{\tilde{t} \to 0, \tilde{t} \geq 0} \frac{1}{\tilde{t}} (\tilde{K}(t + \tilde{t}, t + \tilde{t}) - \tilde{K}(t, t)) = 0$.

By definition of $f, x_f$ and $\tilde{K}$ it follows that:
$$
\forall u \in [-\tau, 0]: \quad f(u) = x_f(u) = \tilde{K}(t + u, t) = K(t + u, t).
$$

Hence: $\int_{-\tau}^{t} f(u) du dr = \int_{-\tau}^{t} \tilde{K}(t + u, t) du dr$.

But then it follows by (iii):
$$
\lim_{\tilde{t} \to 0, \tilde{t} \geq 0} \frac{1}{\tilde{t}} (\tilde{K}(t + \tilde{t}, t + \tilde{t}) - \tilde{K}(t, t)) = 0
$$

$$
= 2 \int_{-\tau}^{t} f(u) du dr + \int_{-\tau}^{t} f(u) du dr
$$

$$
= 2 \int_{-\tau}^{t} \tilde{K}(t + u, t) du dr + \int_{-\tau}^{t} \tilde{K}(t + u, t) du dr
$$

(8) By (3), (5), (6), (7) $\tilde{K}$ solves the amplified system (2.18) with the initial function $k$.

As $K_2^X$ solves (2.18) with the initial function $k$, too. By corollary 2.15 it follows $K_2^X = \tilde{K}$.

This proves $\Leftrightarrow$ in (i).

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Now we can state necessary conditions on the existence of weakly stationary solutions of (2.1).

**Theorem 4.6**

Let \( X \) be a solution of (2.1) with the mean function \( m^X_1 \) and comoment function \( K^X_2 \),
\[ k \in \mathcal{K}_{a,b}([-\tau,0],\mathbb{R}) \text{ defined by: } \forall u,v \in [-\tau,0]: \mathbb{E}[X(u)X(v)], \]
\[ f \in ([-\tau,0],\mathbb{R}) \text{ defined by } \forall u \in [-\tau,0]: f(u) = k(u,0), \]
\( x_f \) solution of (3.1) with \( \xi = f \).

If \( X \) is weakly stationary on \([-\tau,\infty)\), then it holds:
\( i \) \( 0 = m^X_2(0) a([-\tau,0]) \),
\( ii \) \( \forall u,v \in [-\tau,0]: k(u,v) = f(-|u-v|) \),
\( iii \) \( 0 = 2 \int_{-\tau}^0 f(u)da(u) + \int_{-\tau}^0 \int_{-\tau}^0 f(-|u-v|)db(u)db(v) \),
\[ f(t) = f(0) + \int_{-\tau}^t \int_{-\tau}^0 f(-|u-r|)da(u)dr, \quad t \in [-\tau,0], \]
\( iv \) \( K^X_2(t,s) = x_f(|t-s|) \).

**Proof:**

Combine the theorems 4.4 and 4.5. □

**Corollary 4.7**

If \( X \) is weakly stationary and \( k \in \mathcal{K}_{sup}([-\tau,0]^2,\mathbb{R}) \) then 0 is an eigenvalue of order 2 and \( f = k(0) \) is an eigenfunction of order 2 of the generator of \( \mathcal{T}_{sup} \). This is as the condition 4.6.(iii) is (3.4) for \( \lambda = 0 \).

**Example 4.8**

We continue the example from section 2.5 and consider the geometric Brownian Motion.

If \( X \) has a stationary mean function by theorem 4.4 it follows:
\[ 0 = \mathbb{E}[X(0)]^\alpha. \]

If \( X \) has a stationary comoment function by theorem 4.5 it follows:
\[ 0 = 2 \int_{-\tau}^0 f(u+r)da(u) + \int_{-\tau}^0 \int_{-\tau}^0 f(u,v)-u \vee v db(u)db(v), \quad 0 = (2\alpha + \beta^2)f(0) \]
\[ f(t) = f(0) + \int_{-\tau}^t \int_{-\tau}^0 f(-|u-r|)da(u)dr \quad \iff \quad f(t) = f(0) + \alpha \int_{-\tau}^t f(r)dr \]

hence
\[ f(0) = \mathbb{E}[X(0)^2], \quad f(t) = f(0) e^{-\alpha t} \quad \text{and} \quad x_f(t) = f(0) e^{\beta t}, \]
\[ K^X_2(t,s) = \mathbb{E}[X(0)^2] e^{\alpha |t-s|} \quad \text{and} \quad 0 = (2\alpha + \beta^2)\mathbb{E}[X(0)^2]. \]

If \( X \) is weakly stationary by theorem 4.6 it follows (using the Jensen- and Cauchy-Schwarz-inequality)
\[ 0 \neq 2\alpha + \beta^2, \quad \forall t \in [0,\infty) \mathbb{E}[X(t)^2] = \mathbb{E}[X(0)^2] = 0, \quad \text{or} \]
\[ 0 = 2\alpha + \beta^2, \quad 0 \neq \alpha, \quad \forall t \in [0,\infty) \mathbb{E}[X(t)] = 0 \quad \text{or} \]
\[ 0 = \alpha = \beta. \]

The mean function of \( X \) is \( m^X_1 = \mathbb{E}[\xi] e^{\alpha t}. \)

Hence \( X \) has a stationary mean function if and only if \( \mathbb{E}[\xi] = 0 \) or \( \alpha = 0. \)

The comoment function of \( X \) is \( K^X_2(t,s) = \mathbb{E}[\xi^2] e^{\alpha |t-s| + (\alpha + \beta^2) |t\vee s|} = \mathbb{E}[\xi^2] e^{\alpha |t-s| + (2\alpha + \beta^2) |t\wedge s|}. \)

Hence \( X \) has a stationary comoment function if \( \mathbb{E}[\xi^2] = 0 \) or \( 0 = 2\alpha + \beta^2. \)

Finally, using the Jensen- and Cauchy-Schwarz-inequality, \( X \) is weakly stationary if and only if
\[ 0 \neq 2\alpha + \beta^2, \quad \mathbb{E}[\xi^2] = \mathbb{E}[\xi] = 0, \quad \text{or} \]
\[ 0 = 2\alpha + \beta^2, \quad \alpha \neq 0, \quad \mathbb{E}[\xi] = 0, \quad \text{or} \]
\[ 0 = 2\alpha + \beta^2, \quad \alpha = 0. \]

In this case the necessary conditions are identical with the exact conditions.

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5 Appendix

5.1 The theorem of Fubini

Theorem 5.1
Let $(X, A), (Y, B)$ measurable spaces, 
$\mu$ a $\sigma$-finite measure on $(X, A)$, $\nu$ a $\sigma$-finite measure on $(Y, B)$.

(i) Let $\forall$ non-negative $f \in (X \times Y, \mathbb{R}), (A_X \times A_Y, B)$ -measurable,

$$f_X(x) := \int f(x,y) \, d\nu(y), \quad f_Y(y) := \int f(x,y) \, d\nu(y),$$

Then it holds:
(1) $f_X$ is non-negative, $A$-measurable, $f_Y$ is non-negative, $B$-measurable
(2) $\int \int f(x,y) \, d\mu \otimes \nu(x,y) = \int \int f_X(x) \, d\mu(x) = \int f_Y(y) \, d\nu(y)$.

(ii) Let $\forall f \in (X \times Y, \mathbb{R}), (A_X \times A_Y, B)$ -measurable and $\mu \otimes \nu$-integrable.

$$f_X(x) := \int f(x,y) \, d\nu(y), \quad f_Y(y) := \int f(x,y) \, d\nu(y),$$

$$A_X = \{ x \in X \mid f(x, \_ ) is \nu - integrable \}, \quad A_Y = \{ y \in Y \mid f(\_ , y) is \mu - integrable \}.$$

Then it holds:
(1) $\forall x \in X: f(x, \_ )$ is $A_Y$-measurable, $\nu$-integrable on $A_Y$ and $A_Y \in A_Y$,
(2) $\forall y \in Y: f(\_ , y)$ is $A_X$-measurable, $\mu$-integrable on $A_X$ and $A_X \in A_X$,
(3) $\int \int f(x,y) \, d\mu \otimes \nu(x,y) = \int \int f_X(x) \, d\mu(x) = \int f_Y(y) \, d\nu(y)$.

(iii) Let $\forall f \in (X \times Y, \mathbb{R}), (A_X \times A_Y, B)$ -measurable,

$$\int \int |f| \, d\mu \otimes \nu < \infty \text{ or } \int \int |f| \, d\mu \, d\nu < \infty \text{ or } \int \int |f| \, d\mu \, d\nu < \infty.$$

Then it holds:
(1) $\int \int |f| \, d\mu \otimes \nu = \int \int |f| \, d\nu \, d\mu = \int \int |f| \, d\mu \, d\nu < \infty,$
(2) $f$ is $\mu \otimes \nu$-integrable.

Proof:
See [9], pp.173.

5.2 Weakly compact sets on $C(X)$

Theorem 5.2 (Arzelà-Ascoli)
Let $(X, d)$ be a metric space and $S \subset C(X, R)$.

Let (i) $S$ be bounded,
(ii) $S$ be a set of uniformly continuous functions, that is

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall f \in S, x_1, x_2 \in X: \quad d(x_1, x_2) < \delta \implies |f(x_1) - f(x_2)| \leq \epsilon.$$

Then it holds:
$S$ is weakly compact.

If $S$ is in addition to (i) and (ii) closed, then it holds:
$S$ is compact.

Proof:
See [7], pp.68-70.

Lemma 5.3
Let $S \subset C(X, R), S$ bounded,

$$\forall f \in S, x_1, x_2 \in X: |f(x_2) - f(x_1)| \leq L_f |x_2 - x_1|^{\frac{1}{\gamma}},$$

$$\gamma := \inf \{ \gamma | f \in S \} > 0, \quad L := \sup \{ L_f | f \in S \} < \infty.$$

Then it holds:
$S$ is weakly compact.

Proof:
Let $\epsilon \in (0, \infty)$. Choose $\delta = \left( \frac{\epsilon^{0.5}}{L \gamma} \right)^{\frac{1}{\gamma}}.$
Then for any \( f \in S \) and \( x_2, x_1 \in X \) with \( |x_1 - x_2| < \delta \) it holds:
\[
|f(x_2) - f(x_1)| < L_f |x_2 - x_1| \leq L_f \delta \implies L_f \delta \leq \varepsilon \land \delta \leq \varepsilon.
\]
Hence, \( S \) is uniformly continuous.
As \( S \) is bounded by assumption, by the theorem of Arzela-Ascoli \( S \) is precompact. \( \square \)

### 5.3 Spectral theory of linear operators

The spectral theory of linear operators is the generalization of the theory of eigenvalues of linear, finite dimensional operators. We recall some definitions and results.

Let \( (X, \| \cdot \|_X) \) be a Banach space.

**Definition 5.4**

Let \( L \in (D(L), X) \) be a linear operator with \( D(L) \subseteq X \).

1. Then \( \lambda \in \mathbb{C} \) is called **resolvent of** \( L \) if and only if \( (\lambda I - K)^{-1} \) exists and is bounded, that is if
   (i) \( \lambda I - L \) is injective
   (ii) \( \text{Im}(\lambda I - L) = X \)
   (iii) \( (\lambda I - L)^{-1} \) is bounded

2. The set \( \rho(L) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ resolvent of } L \} \) is called **resolvent set of** \( L \).

3. The set \( \sigma(L) = \mathbb{C} \setminus \rho(L) \) is called **spectrum of** \( L \).

The spectrum of infinite linear operators can be more complicated as for finite dimensional linear operators and can be divided into different types classifying how the above conditions (i) – (iii) are not satisfied.

**Definition 5.5**

Let \( L \in (D(L), X) \) be a linear operator with \( D(L) \subseteq X \).

1. The set \( \sigma_p(L) = \{ \lambda \in \mathbb{C} \mid \exists x \in X \setminus \{0\} : (\lambda I - L)x = 0 \} \) is called **point spectrum of** \( L \) and its elements **eigenvalues of** \( L \).

2. Let \( \lambda \in \sigma_p(L) \). Any \( x \in X \setminus \{0\} \) with \( (\lambda I - L)x = 0 \) is called an **eigenvector of** \( L \).
   Let \( N \subseteq X \) be the kernel (or nullspace) of \( (\lambda I - L) \).
   Then \( N \) is called **eigenspace of** \( \lambda \) and \( \text{dim}(N) \) **geometric multiplicity of** \( \lambda \).
   Let \( \tilde{N} \subseteq X \) be the smallest closed, linear subspace including \( \bigcup_{k=1}^\infty N(\lambda I - L)^k \).

3. Then \( \tilde{N} \) is called **generalized eigenspace of** \( L \) and \( \text{dim}(\tilde{N}) \) **algebraic multiplicity of** \( \lambda \).

4. The set \( \sigma_c(L) = \{ \lambda \in \mathbb{C} \mid (\lambda I - L) \text{ injective, } \text{Im}(\lambda I - L) \subseteq X, \text{Im}(\lambda I - L) = X \} \) is called **continuous spectrum of** \( L \).

5. The set \( \sigma_r(L) = \{ \lambda \in \mathbb{C} \mid (\lambda I - L) \text{ injective, } \text{Im}(\lambda I - L) \subseteq X \} \) is called **residual spectrum of** \( L \).

The spectrum has some associated characteristic values.

**Definition 5.6**

Let \( L \in (D(L), X) \) be a bounded linear operator with \( D(L) \subseteq X \).
Then \( r(L) := \sup \{ |\lambda| \mid \lambda \in \sigma(L) \} \) is called the **spectral radius of** \( L \).

Let \( L \in (D(L), X) \) be a closed linear operator with \( D(L) \subseteq X \).
\( s(L) := \sup \{ \text{Re}(\lambda) \mid \lambda \in \sigma(L) \} \) is called the **spectral bound of** \( L \).

### 5.4 Semigroups

Semigroups are related to abstract differential equations. We recall some definitions and results.
Let \( (X, \|\cdot\|_X) \) be a Banach space.

**Definition 5.7**

Let \( T = \{ T(t) \}_{t \in [0, \infty)} \) be a family of bounded linear operators on \( X \).
(1) If
   (i) \( T(0) = id \),
   (ii) for all \( t, s \in [0, \infty) \): \( T(t)T(s) = T(s)T(t) \),
then \( T \) is called a **semigroup of operators**.

(2) If in addition to (1).(i) and (1).(ii) it holds
   (iii) for all \( x \in X \): \( \lim_{h \to 0} |T(h)x - x|_X = 0 \),
then \( T \) is called a **strongly continuous semigroup of operators**.

With a strongly continuous semigroup one can associate an abstract differential equation

\[
\frac{d}{dt}(T(t)x) = A(T(t)x), \quad x \in D(A),
\]

where we define:

**Definition 5.8**

Let \( A : D(A) \rightarrow X \) be an operator, with

(i) \( D(A) := \{ x \in X \mid \exists \lim_{h \to 0} \frac{1}{h}(T(h)x - x) \in X \} \),

(ii) \( \forall x \in D(A) : Ax := \lim_{h \to 0} \frac{1}{h}(T(h)x - x) \).

Then \( A \) is called **infinitesimal generator of \( T \)** and \( D \) is called **domain of \( A \)**.

Strongly continuous semigroups have the following properties:

**Lemma 5.9**

(i) \( D(A) \) is dense in \( X \),

(ii) \( A \) is closed,

(iii) \( \exists x \in \mathbb{R}, M \in [1, \infty) : |T(t)| \leq Me^{ct} \).

**Proof:**

See [2], pp.453-454. \( \square \)

The property 5.9.(iii) is an important property as it states that there exists an exponential growth bound for the norm of \( T(t) \) and \( t \rightarrow \infty \). It is natural to ask for the smallest one and to use it as a characteristic of \( T(t) \).

**Definition 5.10**

Let \( \alpha_0 = \inf \{ c \in \mathbb{R} \mid \exists M \in [1, \infty) : |T(t)| \leq Me^{ct} \} \).

Then \( \alpha_0 \) is called the **growth bound of \( T \)**.

Reversely it holds:

**Theorem 5.11** (Hille-Yosida)

Let \( A \in (D(A), X) \) be a linear operator.

Then the following two statements are equivalent:

(1) \( A \) is an infinitesimal generator of a strongly continuous semigroup \( T = \{ T(t) \}_{t \in [0, \infty)} \) with \( |T(t)| \leq Me^{ct} \),

(2) (i) \( A \) is closed, \( D(A) \) is dense in \( X \),
   (ii) \( (c, \infty) \subseteq \rho(A) \),
   (iii) \( \forall \lambda > \alpha, k \in \mathbb{N} \setminus \{0\} : |(\lambda I - A)^{-k}| \leq \frac{M}{(\lambda - \sigma)^k} \).

**Proof:**

See [2], p.455. \( \square \)

From finite dimensional operators it is known, that there is a relation between the growth of \( \{ T(t) \}_{t \in [0, \infty)} \) and the eigenvalues of the generator \( A \) of \( T \). The relation is described in the so-called **spectral mapping theorem** stating that \( \sigma(T(t)) = e^{\sigma(A)} \) where \( \sigma(T(t)) \) are the eigenvalues of \( T(t) \) and \( \sigma(A) \) are the eigenvalues of the infinitesimal generator \( A \) of \( T \). For strongly continuous semigroups there are several relations known.
Lemma 5.12
(i) \( \sigma(P(L)) = P(\sigma(L)) \), \( P \) polynomial,
(ii) \( \sigma_p(T(t)) \setminus \{0\} = e^{t \sigma_p(A)} \),
(iii) \( r(T(t)) = e^{t \alpha_1} \).
(iv) \( s(A) \leq c_0 \).

Proof:

The relation 5.12. (iii) interpretes the growth bound in terms of the linear operators \( T(t) \).
The relation 5.12. (iv) provides a characterization of the asymptotic behaviour of \( T \) in terms of its
infinitesimal generator; the spectrum of \( A \) provides a lower bound for the growth bound of the generated
semigroup. This criterion is in general unsharp. An example for \( s(A) < c_0 \) can be found in [2], p.470.
However, in special cases the characterization of the asymptotic behaviour of \( T \) in terms of its infinitesimal
generator is possible. One of the special cases is the following.

Definition 5.13
Let \( T = \{ T(t) \}_{t \in [0, \infty)} \) be a strongly continuous semigroup.
\( T \) is called eventually compact if and only if \( \exists t_0 \in [0, \infty) \) \( \forall t \in [t_0, \infty): T(t) \) is compact.

For eventually compact strongly continuous semigroups the following is known.

Theorem 5.14
Let \( T = \{ T(t) \}_{t \in [0, \infty)} \) be a strongly continuous, eventually compact semigroup,
\( A \) be the infinitesimal generator of \( T \).

Then it holds:
(i) \( \sigma(A) = \sigma_p(A) \),
(ii) \( s(A) = c_0 \).

Proof:
See [2], p.97.

Example 5.15
Consider affine DDE.
Then \( T = \{ T(t) \}_{t \in [0, \infty)} \) defined by \( T(t)\phi := x_t \), \( t \in [0, \infty) \) is eventually compact ([5], p.194).

5.5 Proofs of selected theorems and lemmas

Proof (Lemma 2.4)
(i) and (iii) are already proven. So (ii) remains to be shown.
First we represent the difference quotient as a sum. It holds:
\[
\frac{1}{h}(E[X(t-h)^2 - X(t)^2])
\]
\[
= \frac{1}{h}E[X(t+h)(X(t-h))]
\]
\[
= \frac{1}{h}E[(2X(t-h) + \int_{t-h}^{t-h} X(s) ds + \int_{t-h}^{t} X(s) ds)]
\]
\[
= \frac{1}{h}E[X(t-h)\int_{t-h}^{t} X(s) ds]
\]
\[
+ 2E[X(t-h)\int_{t-h}^{t} X(s) ds]
\]
(5.2)
Now we determine the limits of each of the five summands in (5.2):

(i) \( \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(s + u) da(u) \, ds \right] \)

\[
= \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(t - h)X(s + u) da(u) \, ds \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(t - h)X(s + u) \, da(u) \, ds \right]
\]

\[
= \mathbb{E} \left[ \int_{t}^{0} X(t)X(t + u) \, da(u) \right]
\]

(ii) \( \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(s + u) \, db(u) \, dW(s) \right] \)

\[
= \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(s + u) \, db(u) \, dW(s) \right] = 0
\]

(iii) \( \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(s + u) \, da(u) \, ds \right] \)

\[
\leq \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} \int_{-t}^{0} \left| \mathbb{E} \left[ \int_{0}^{X(s + u) da(u) \, ds} \right] \right| \, dr
\]

\[
\leq \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} \int_{-t}^{0} \mathbb{E} \left[ \left| \int_{0}^{X(s + u) da(u) \, ds} \right|^{1/2} \mathbb{E} \left[ \left| \int_{0}^{X(s + u) da(u) \, ds} \right|^{2} \right]^{1/2} \, dr
\]

(iv) \( \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^{t} \int_{-t}^{0} X(s + u) \, db(u) \, dW(s) \right] \)

\[
= \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} \int_{-t}^{0} \mathbb{E} \left[ \int_{0}^{X(s + u) \, db(u) \, dW(s)} \right] \]

\[
\leq \lim_{h \to 0} \frac{1}{h} \int_{t-h}^{t} \int_{-t}^{0} \mathbb{E} \left[ X(s + u)^{2} \right]^{1/2} \left( \int_{t-h}^{t} \mathbb{E} \left[ X(r + v)^{2} \right] \, dr \right)^{1/2} \, db(u) \, dW(s),
\]

\[
\leq 0,
\]
\[
(v) \ F \left[ \frac{f}{t-h_r} \int_{h-t}^{0} X(s + u) \, db(u) \, dW(s) \right] \left[ \frac{f}{t-h_r} \int_{h-t}^{0} X(u) \, db(u) \, dW(s) \right]
\]
\[
= \ F_{\text{LR integral}} \left[ \frac{f}{t-h_r} \int_{h-t}^{0} \mathbb{E}[X(s + u) X(u) | \mathcal{F}_{t-h_r}] \, db(u) | \mathcal{F}_{t-h_r} \right] \, dW(s)
\]
\[
= \lim_{h \to 0} \ F_{\text{LR integral}} \left[ \frac{f}{t-h_r} \int_{h-t}^{0} \mathbb{E}[X(s + u) X(u) | \mathcal{F}_{t-h_r}] \, db(u) | \mathcal{F}_{t-h_r} \right] \, dW(s)
\]

Plugging (i)-(v) into (5.2) yields then the limit to be shown. \( \Box \)

**Proof:** (lemma 2.6)  
As in the previous lemma we use the integral representation of \( X(t) - X(t-h) \) for \( h \geq 0 \) resulting from the SDDE to find a suitable representation of the difference quotient and take then the limit \( h \downarrow 0 \).

\[
-\frac{1}{h} \mathbb{E}[X(t-h) - X(t) | X(r)]
\]
\[
= \ -\frac{1}{h} \mathbb{E}[\int_{t-h}^{t} X(s + u) \, da(u) \, dW(s)]
\]
\[
= \ F_{\text{LR integral}} \left[ \frac{f}{t-h_r} \int_{h-t}^{0} \mathbb{E}[X(s + u) X(u) | \mathcal{F}_{t-h_r}] \, db(u) | \mathcal{F}_{t-h_r} \right] \, dW(s)
\]
\[
= \lim_{h \to 0} \ F_{\text{LR integral}} \left[ \frac{f}{t-h_r} \int_{h-t}^{0} \mathbb{E}[X(s + u) X(u) | \mathcal{F}_{t-h_r}] \, db(u) | \mathcal{F}_{t-h_r} \right] \, dW(s)
\]
\[
\lim_{h \to 0} \ -\frac{1}{h} \mathbb{E}[X(t-h) - X(t)] = \frac{f}{t} \mathbb{E}[X(t) X(r) | \mathcal{F}_{t-h}] \, da(u).
\]

\( \Box \)

**Proof:** (theorem 2.22)
(1) We investigate the differences of consecutive Picard-iterates. \( \forall n \in \mathbb{N}, \forall t, s \in [-\tau, \infty) \) define:
\[
\begin{align*}
\Delta_{n+1}K(t, s) &:= K_{n+1}(t, s) - K_n(t, s) \\
n_{n+1}(t) &:= \sup\{ |\Delta_{n+1}K(s_1, s_2) | \mid t \geq s_1 \geq s_2 \geq -\tau \}.
\end{align*}
\]
By (1) and the previous definition, for all \( n \in \mathbb{N} \) and for all \( 0 \geq t \geq s \geq -\tau \) it holds:
\[
\Delta_{n+1}K(t, s) = 0
\]
and for all \( n \in \mathbb{N}, t \in (0, \infty), s \in [-\tau, t) \) it holds:
\[
\Delta_{n+2}K(t, t) = \Delta_{n+2}K(0, 0) + 2 \int_0^t \int_{-\tau}^{t} \Delta_{n+1}K(r + u, r) \, da(u) \, dr
\]

\[
+ \int_0^t \int_{-\tau}^{t} \Delta_{n+1}K(r + u, r + v) \, db(u) \, db(v) \, dr,
\]
\[
\Delta_{n+2}K(t, s) = \Delta_{n+2}K(s, s) + \int_s^t \Delta_{n+1}K(r + u, s) \, da(u) \, dr.
\]

(2) By the definition of the Picard iterates \( \forall n \in \mathbb{N}, s \in [-\tau, \infty), t \in [0 \vee s, \infty) \) it holds:
\[
\Delta_{n+1}K(0, 0) = 0,
\]
\[
|\Delta_{n+2}K(t, t)| \leq 2 \int d[a](u) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr + \frac{1}{2} \int d[b](u) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr \leq C(a, b, \tau) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr,
\]
\[
|\Delta_{n+2}K(t, s)| \leq |\Delta_{n+2}K(0 \vee s, s)| + \int_0^{t \vee s} \int_{-\tau}^{t} \Delta_{n+1}(r) \, da(u) \, dr,
\]
\[
\leq C(a, b, \tau) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr + \int_{-\tau}^{t} d[a](u) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr \leq C(a, b, \tau) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr.
\]

(3) Fix \( t \in (0, \infty) \). By (2) for all \( \bar{s}, \tilde{s} \) with \( t \geq \bar{s} \geq \tilde{s} \geq -\tau \) it holds:
\[
|\Delta_{n+2}K(\bar{s}, \tilde{s})| \leq C(a, b, \tau) \int_{0 \vee \bar{s}}^{\tilde{s}} \Delta_{n+1}(r) \, dr \leq C(a, b, \tau) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr.
\]

(4) Using (2),(3) \( \forall n \in \mathbb{N} \) and \( \forall t \in [0, \infty) \) it holds:
\[
\delta_{n+2}(t) = \sup\{ |\Delta_{n+2}K(\bar{s}, \tilde{s}) | \mid t \geq \bar{s} \geq \tilde{s} \geq -\tau \} \leq C(a, b, \tau) \int_{-\tau}^{t} \Delta_{n+1}(r) \, dr.
\]

(5) As in (3) of the proof of lemma 2.14, we introduce two index sets and a constant:

\[
\begin{align*}
A_r := & \{ (t, u) \in \mathbb{R}^2 \mid t \in [-\tau, \tau], u \in [-t \wedge -\tau, -t] \} \\
B_r := & \{ (t, u, v) \in \mathbb{R}^3 \mid t \in [-\tau, \tau], u, v \in [-t \wedge -\tau, -t] \},
\end{align*}
\]
\[
S_0 := \{ \delta(t, s) \mid t, s \in [0, \infty) \},
\]
\[
S_1 := \sup_{s \in [0, \infty]} \{ \int_s^t |k(r + u, s)| \, da(u) \, dr \},
\]
\[
S_2 := \int_0^t \int_{-\tau}^{t} |k(r + u, r + v)| \, da(u) \, db(v) \, dr.
\]
(6) Let \( t \in [0, \infty) \). Then it holds:

\[
\begin{align*}
|\Delta_1 K(t, t)| & \leq 2 \left( \int_{0}^{t} \int_{0}^{t} |K_0(r + u, r)\, da(u)\, dr | + \int_{0}^{t} \int_{0}^{t} |K_0(r + u, r + v)\, db(u)\, db(v)\, dv | \right) \\
& \leq 2 \left( \int_{0}^{t} \int_{0}^{t} |K_0(r + u, r)| \, 1_{A_{+}}((r, u))\, da(u)\, dr \\
& + 2 \left( \int_{0}^{t} \int_{0}^{t} |K_0(r + u, r, \ldots, r)| \, (1 - 1_{A_{+}}((r, u)))\, da(u)\, dr \\
& + \int_{0}^{t} \int_{0}^{t} |K_0(r + u, r + v)| \, 1_{B_{+}}((r, u, v))\, db(u)\, db(v)\, dv | \\
& + \int_{0}^{t} \int_{0}^{t} |K_0(r + u, r + v)| \, (1 - 1_{B_{+}}((r, u, v)))\, db(u)\, db(v)\, dv | \right) \\
& \leq 2 S_{1} + 2 \int_{0}^{t} d|a|(u) \, S_{0}^{2} \, t + \int_{0}^{t} d|b|(u) \, S_{0}^{2} \, t \\
& \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) + t C(a,b,\tau) \, S_{0}^{2}.
\end{align*}
\]

(7) For all \( s \in [-\tau, \infty) \), \( t \in (0 \vee s, \infty) \) it holds:

\[
\begin{align*}
|\Delta_1 K(t, s)| & \leq \left( \int_{0}^{s} \int_{0}^{s} |K_0(r + u, s)\, da(u)\, dr | \right) \\
& \leq \int_{0}^{s} \int_{0}^{s} |K_0(r + u, s)|\, da(u)\, dr \, 1_{[-\tau,0]}(s) \\
& + \int_{s}^{t} \int_{s}^{t} |K_0(r + u, s)|\, da(u)\, dr \, 1_{(0,\infty)}(s) \\
& + \int_{s}^{t} \int_{s}^{t} |K_0(r + u, s)|\, da(u)\, dr \, 1_{(0,\infty)}(s) \\
& \leq S_{1} + \int_{0}^{t} d|a|(u) \, S_{0}^{2} \, t \\
& \leq S_{1} + t \, C(a,b,\tau) \, S_{0}^{2}.
\end{align*}
\]

(8) Due to (6)-(7) for all \( s \in [-\tau, \infty) \), \( t \in [0 \vee s, \infty) \) it holds:

\[
|\Delta_1 K(t, s)| \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) \, (1 + t \, C(a,b,\tau)) .
\]

(9) By definition of \( K_0 \) for all \( s \in [-\tau,0] \), \( t \in [0 \vee s, \infty) \) it holds:

\[
|K_0(t, s)| \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) .
\]

(10) From (8) and (10) we can conclude via complete induction that for all \( t \in [0, \infty) \):

\[
\begin{align*}
\delta_{n+1}(T) & \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) \left( \frac{1}{n!}(C(a,b,\tau)T)^n + \frac{1}{(n+1)!}(C(a,b,\tau)T)^{n+1} \right) , \\
\sum_{n=0}^{\infty} \delta_{n+1}(T) & \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) \left( 2 \, e^{C(a,b,\tau)T} - 1 \right) < \infty.
\end{align*}
\]

With (9) it follows then that for all \( n \in \mathbb{N} \), \( m \in \mathbb{N} \setminus \{0\} \), \( T \geq 0 \) and \( (t,s) \in [-\tau,T]^2 \setminus [-\tau,0]^2 \):

\[
\begin{align*}
|K_{n+1}(t,s)| & \leq |K_0(t,s)| + \sum_{k=1}^{n+1} |\Delta K_k(t,s)| \\
& \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) \, 2 \, e^{C(a,b,\tau) T} , \\
|K_{n+1}(t,s) - K_{n+m+1}(t,s)| & \leq \sum_{k=n+1}^{n+m} |\Delta K_k(t,s)| \\
& \leq \|k\|_{\mathcal{K}_{a,b}}([-\tau,0]^{\mathbb{P}}, \mathbb{R}) \, 2 \, (e^{C(a,b,\tau) T} - 1) .
\end{align*}
\]

Hence \( \{K_n\}_{n \in \mathbb{N}} \) is uniformly bounded and a Cauchy-sequence on \([\tau,T]^2 \setminus [-\tau,0]^2\). But then the pointwise limit \( K := \lim_{n \to \infty} K_n \) does exist on \([\tau,T]^2\).
Now we show, that $K$ found in (11) satisfies (2.4). Let $n$ be in $N \setminus \{0\}$. Then for all $t \in (0, \infty)$ it holds:

$$
| K(t,t) - K(0,0) - 2 \int_{t}^{0} \int_{0}^{0} K(r+u,r) \, du \, dr - \int_{t}^{0} \int_{0}^{0} K(r+u,r+v) \, db(u) \, db(v) \, dr |
$$

$$
\leq | K(t,t) - K_{n+1}(t,t) | + | K(0,0) - K_{n}(0,0) | 
+ | 2 \int_{t}^{0} \int_{0}^{0} K(r+u,r) - K_{n}(r+u,r) \, du \, dr |
$$

$$
+ | \int_{t}^{0} \int_{0}^{0} K(r+u,r+v) - K_{n}(r+u,r+v) \, db(u) \, db(v) \, dr |
$$

$$
\leq \| K - K_{n+1} \|_{\mathcal{C}([0,T]^2,\mathbb{R})} + C(a,b,\tau) \| K - K_{n} \|_{\mathcal{C}([0,T]^2,\mathbb{R})} 
$$

Furthermore, for any $s \geq -\tau$, $t \in (0 \vee s, \infty)$:

$$
| K(t,s) - K(0 \vee s,s) - \int_{0 \vee s}^{t} \int_{0}^{0} K(r+u,s) \, du \, dr |
$$

$$
\leq | K(t,s) - K_{n+1}(t,s) | + | K(0 \vee s) - K_{n}(0 \vee s) | 
+ | \int_{0 \vee s}^{t} \int_{0}^{0} K(r+u,s) - K_{n}(r+u,s) \, du \, dr |
$$

$$
\leq \| K - K_{n+1} \|_{\mathcal{C}([0,T]^2,\mathbb{R})} + C(a,b,\tau) \| K - K_{n} \|_{\mathcal{C}([0,T]^2,\mathbb{R})} 
$$

As \( \lim_{n \to \infty} \| K - K_{n} \|_{\mathcal{C}([0,T]^2,\mathbb{R})} = 0 \) by (10) the theorem is proved. \( \square \)

**Proof:** (theorem 2.26)
(1) We want to partition the total difference into suitable summands. First we introduce some necessary variables. Let \( t > s \geq 0, h = (h_1, h_2) \in \mathbb{R}^2 \) such that \( ||h||_2 \leq \frac{1}{4}[t - s] \).
Define \( M(a, b) = \max\{a + b, a\}, m(a, b) = \min\{a + b, a\} \) for all \( a, b \in \mathbb{R} \).
Now we derive partition the total difference. It holds:
\[
\begin{align*}
K(t + h_1, s + h_2) - K(t, s) &= \mathbb{E}[X(t + h_1) - X(t)] (X(s + h_2) - X(s)) + \mathbb{E}[X(t + h_1) - X(t)] X(s) \\
&+ \mathbb{E}[X(t)(X(s + h_2) - X(s))] \\
&= (-1)^{\delta_{[0,\infty)}(h_1) + \delta_{[0,\infty)}(h_2)} \left( \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r + v) \, dv \, dr \right] - h_1 \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(t + v) \, dv \, dr \right) \\
&+ \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r + v) \, dv \, dr \right] - \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} X(r) \, dr \right]
\end{align*}
\]

(2) We deal now with one particular summand of (1) and can show that it is of order \( o(||h||_2) \).
It holds:
\[
\begin{align*}
&\mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r + v) \, dv \, dr \right] = \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r_1 + v_1)X(r_2 + v_2) \right] \, dv_2 \, dr_2 \\
&\leq \int_{h_1, h_2} \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r_1 + v_1)X(r_2 + v_2) \right] \, dv_2 \, dr_2
\end{align*}
\]
As \( \lim_{h \to 0} \frac{||h||_2}{||h||_2} = 0 \), the above term converges to 0 for \( h \to 0 \).

(3) As in (2) we deal with particular summands of (1) and show that they vanish. Due to the choice of \( h, [m(h_1), M(h_1)] \cap [m(h_2), M(h_2)] = \emptyset \). But then, using the properties of the Ito-integral, it holds:
\[
\begin{align*}
\mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r + v) \, dv \, dr \right] = 0,
\end{align*}
\]

(4) Now we deal with a particular summand from (1) and filter out its \( h \)-linear fraction.
It holds:
\[
\begin{align*}
&\frac{1}{||h||_2} \left( (-1)^{\delta_{[0,\infty)}(h_1)} \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r + v) \, dv \, dr \right] - h_1 \mathbb{E} \left[ X(t + v)X(s) \right] \, dv \right) \\
&= \frac{1}{||h||_2} \left( (-1)^{\delta_{[0,\infty)}(h_1)} \mathbb{E} \left[ \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} X(r + v) \, dv \, dr \right] - \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} \mathbb{E} \left[ X(t + v)X(s) \right] \, dv \, dr \right) \\
&= \frac{1}{||h||_2} \left( \int_{m(h_1)}^{M(h_1)} \int_{m(h_2)}^{M(h_2)} \mathbb{E} \left[ X(r + v)X(s) \right] - \mathbb{E} \left[ X(t + v)X(s) \right] \, dv \, dr \right)
\end{align*}
\]
\[ \leq \frac{|h_2|}{||h||_2} \left( \frac{M(t, h_1)}{m(t, h_1)} \int_{m(t, h_1)-\tau}^{M(s, h_2)} \int_0^M |K(r+v, s) - K(t+v, s)| \, d|a|(v) \, dr \right) \]
\[ \leq \frac{|h_2|}{||h||_2} \sup_{r \in [m(t, h_1), M(t, h_1)]} \left\{ \int_0^M |K(r+v, s) - K(t+v, s)| \, d|a|(v) \right\} \]

It holds \( 0 \leq \frac{|h_2|}{||h||_2} \leq 1 \). As \( K \) is continuous on \([0, \infty)^2\), it is uniformly continuous on compact sets and the above term tends to 0 for \( h \to 0 \).

(5) As in (4) we deal with a particular summand from (1) and filter out its \( h \)-linear fraction.

It holds:
\[ \frac{1}{||h||_2} \left| (-1) \delta_{[0, \infty)}(h_2) \right| E\left[ X(t) \int_{m(s, h_2)}^{M(s, h_2)} \int_0^M X(r+v) \, da(v) \, dr \right] - \frac{1}{||h||_2} \left| (-1) \delta_{[0, \infty)}(h_2) \right| E\left[ X(t) \int_{m(s, h_2)}^{M(s, h_2)} \int_0^M X(r+v) \, da(v) \, dr \right] \]
\[ \leq \frac{|h_2|}{||h||_2} \sup_{r \in [m(s, h_2), M(s, h_2)]} \left\{ \int_0^M |K(t+v) - K(t+s, v)| \, d|a|(v) \right\} \]

It holds \( 0 \leq \frac{|h_2|}{||h||_2} \leq 1 \). As \( K \) is continuous on \([0, \infty)^2\), it is uniformly continuous on compact sets and the above term tends to 0 for \( h \to 0 \).

(6) Defining first for all \( \tilde{t}, \tilde{s} \in [t_0, \infty), \tilde{h} \in \mathbb{R}^+ \):
\[ \mathcal{G}(\tilde{t}; \tilde{s}, \tilde{h}) = E\left[ X(\tilde{t}) \int_{\tilde{s}}^{\tilde{t}} \int_0^M X(\tilde{r}+\tilde{\nu}) \, d\tilde{\nu}(\tilde{\nu}) \, dW(\tilde{\nu}) \right] , \]
we continue to evaluate the terms from (1).

Due to Lemma (2.25) it holds:
\[ \leq \frac{|h_2|}{||h||_2} \left| \int_{m(s, h_2)}^{M(s, h_2)} \int_0^M X(r+v) \, da(v) \, dr \right| \]
\[ \leq \frac{|h_2|}{||h||_2} \left| \int_{m(s, h_2)}^{M(s, h_2)} \int_0^M X(r+v) \, da(v) \, dr \right| \]
\[ \leq \frac{|h_2|}{||h||_2} \left| \int_{m(s, h_2)}^{M(s, h_2)} \int_0^M |K(t+v) - K(t+s, v)| \, d|a|(v) \right| \]
\[ \leq \frac{|h_2|}{||h||_2} \sup_{r \in [m(s, h_2), M(s, h_2)], r+s} \left\{ \int_0^M |K(t+v) - K(t+s, v)| \, d|a|(v) \right\} \]

As \( 0 \leq \lim_{h \to \infty} \frac{|h_2|}{||h||_2} \leq \lim_{h \to \infty} \frac{1}{||h||_2} = 0 \) and the suprema are bounded, the above term converges to 0.

Furthermore, it holds:
\[ \leq \frac{|h_2|}{||h||_2} \left| \int_{m(s, h_2)}^{M(s, h_2)} \int_0^M |X(r+v) - X(r)| \, da(v) \, dr \right| \]
\[ \leq \frac{|h_2|}{||h||_2} \sup_{r \in [m(s, h_2), M(s, h_2)]} \left\{ \int_0^M |K(t+v) - K(t+s, v)| \, d|a|(v) \right\} \]

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\[
\int_{\mathcal{M}(s,h_2)} x_0(t - r) \int_{m(s,h_2)}^0 \int_{-r}^0 K(r + u, r + v) dB(u) dB(v) \, dr
\]
\[
+ \int_{m(s,h_2)}^0 \int_{-r}^0 K(s + u, s + v) dB(u) dB(v) \, dr
\]
\[
\leq \frac{|h_2|}{\|h\|_2} \left( |x_0(t - s)| \sup_{r \in [m(s,h_2), M(s,h_2)]} \left\{ |\int_{-r}^0 K(r + u, r + v) - K(s + u, s + v) dB(u) dB(v) \, dr| \right\} \right)
\]

(7) Collecting the \(O(\|h\|_2)\) terms from (4), (5), (6), we get:
\[
\lim_{h \to 0} \frac{1}{\|h\|_2} \left| K(t + h_1, s + h_2) - K(t, s) - h_1 \int_{-r}^0 K(t + u, s) da(u) \right.
\]
\[
- h_2 \left( \int_{-r}^0 K(t, s + v) da(v) + x_0(t - s) \int_{-r}^0 K(s + u, s + v) dB(u) dB(v) \right) = 0.
\]

This proves the differentiability of \(K\) on \(K^2_{\gamma,>}(\tau)\).

As \(K\) is continuous on \(K^2_{\gamma}(0)\), symmetric on \(K^2_{\gamma,\Delta}(-\tau)\), it has continuous derivatives on \(K^2_{\gamma,\Delta}(\tau)\). 

\[\square\]

**Acknowledge**

I thank Prof. Küchler (Humboldt University Berlin), Prof. C.T.H. Baker (University of Manchester), Dr. Markus Reiβ and Dr. Markus Riedle (Humboldt University Berlin) for fruitful discussions and helpful comments.

**References**


