Exponential Stability in p-th Mean of Solutions, and of Convergent Euler-type Solutions, of Stochastic Delay Differential Equations

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Dedicated to Pieter van der Houwen to mark his retirement.

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Abstract

Results are presented on the stability of solutions of *stochastic delay differential equations with multiplicative noise*, and of convergent numerical solutions obtained by a a method of *Euler-Maruyama* type. An attempt is made to provide a fairly self-contained presentation.

A basic concept of the stability of a solution of an evolutionary stochastic delay differential equation is concerned with the sensitivity of the solution to perturbations in the initial function. We recall the stability definitions considered herein and show that an inequality of Halanay type (derivable via comparison theory), and deterministic results, can be employed to derive stability conditions for solutions of suitable equations.

In practice, closed form solutions of stochastic delay differential equations are unlikely to be available. In the second part of the paper a stability theory for numerical solutions (solutions of Euler type) is considered. A convergence result is recalled, for completeness, and new stability results are obtained using a discrete analogue of the continuous Halanay-type inequality and results for a deterministic recurrence relation.

Various results for stochastic (ordinary) differential equations, with no time lag, or for deterministic delay differential equations, can be deduced from the results given here.

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1 Introduction

This paper divides naturally into two halves, the first concerning stochastic delay differential equations (SDDEs) with an initial condition and the second concerning the numerical solution of such a problem. In the first part we present the class of problems addressed, define our stability concepts, and obtain a stability criterion. In the second part we recall a numerical method defined by reference to a uniform mesh and a convergence result for the approximate solution as the mesh-width tends to zero; we present the stability concepts for the discretized case, and obtain a stability criterion for the numerical solution. Extensions and open problems are indicated.

The development of numerical methods for SDDEs is relatively new, compared with that for deterministic delay differential equations (DDEs) or for stochastic ordinary differential equations (SODEs) – see, *e.g.*, [1] and [15] respectively. The main practical conclusions to be drawn from this paper are that, when using the Euler-Maruyama formula for an SDDE with a fixed stepsize, the choice of stepsize is limited both by accuracy and by stability — but that under precisely stated conditions the numerical solution simulates that of the analytic problem. The main theoretical conclusion is that continuous and discrete inequalities of Halanay type can be given a rôle, as an alternative to other approaches, in the stability analysis. The authors attempt to provide a fairly self-contained presentation (with references for further reading) on the grounds that the subtleties arising in stochastic analysis or numerical analysis or problems with time lag are often peculiar to the respective specialties. This approach results, of course, in a longer paper than would otherwise be the case; the *cognoscenti* may wish to concentrate upon the Lemmas and Theorems.

2 Stochastic Delay Differential Equations

Mao [23] and Mohammed [26] examine in some detail SDDEs of the type analyzed here; Kolmanovskii and Myshkis [18, Chapter 5] set out the formulation of the problem and some key results in a succinct fashion (see also [19, Chapter 10] and Kolmanovskii and Nosov [17]). Mohammed's article [25] gives a very good introduction to several aspects of stochastic functional differential equations.

We consider for $t \geq t_0$ the Itô equation

$$X(t) = X(t_0) + \int_{t_0}^{t} F(s, X(s), X(s-\tau))\, ds + \int_{t_0}^{t} G(s, X(s), X(s-\tau)) dW(s) \quad (2.1a)$$

(where, see [18, Chapter 8], $W(t)$ is a standard *Wiener process*, or *Brownian motion process*) with the stochastic integral defined in the Itô sense, and where

$$X(t) = \Phi(t) \quad \text{for } t \in J := [t_0 - \tau, t_0]. \quad (2.1b)$$

A reference to (2.1) should be interpreted as a reference to (2.1a)–(2.1b). To indicate the dependence upon the initial function one may write

$$X(t) \equiv X(t;\ t_0, \Phi). \quad (2.2)$$

Eqn. (2.1a) is a *stochastic delay differential equation* (SDDE) of Itô type, with "fixed lag" $\tau > 0$, and is often written in the compact form

$$dX(t) = F(t, X(t), X(t-\tau))\, dt + G(t, X(t), X(t-\tau))\, dW(t) \ (t \geq t_0), \quad (2.3)$$

subject to the "initial condition" $X(t) = \Phi(t)$ for $t \in J$ in (2.1b).

As regards $\Phi(\cdot)$, we request (see, *e.g.*, [18, Chapter 8], [26, Chapter 2]) the following condition.

Condition C0 $\Phi(t)$ satisfies $\mathcal{E}(\sup_{t\in J}|\Phi(t)|^2) < \infty$ where the notation $\mathcal{E}$ denotes the expectation; almost all sample paths are continuous and $\Phi(t)$ is independent of the σ-algebra generated by $W(t)$.

Remarks:

• For a useful summary of the properties of expectation and conditional expectation, refer to Mao [23, pp. 8–9]. Williams [34] is a good source for background stochastic analysis. It seems to be increasingly apparent that the Malliavin stochastic analysis (*cf.* [12]) will be used for future advances in the numerics of SDDEs.

• Additional conditions on $\Phi(\cdot)$ are imposed below, for the analysis of the order of convergence recalled here from [2].

• To be precise in the formulation of the problem, let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $(\mathcal{A}_t)$ satisfying the usual conditions, *i.e.*, the filtration $(\mathcal{A}_t)_{t\geq t_0}$ is right-continuous, and each $\mathcal{A}_t$, $t \geq t_0$, contains all P-null sets in $\mathcal{A}$. With $\mathcal{E}(X) = \int_\Omega X\, dP$, we say, if $\mathcal{E}\ (|X|^p) < \infty$, that

$$X \in \mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{A}, P) \text{ for } 1 \leq p \leq \infty \tag{2.4}$$

and we then define $\|X\|_p = (\mathcal{E}\ (|X|^p))^{\frac{1}{p}}$.

• In the problem *without* time lag, which may be written

$$dX(t) = \mathrm{F}(t, X(t))\, dt + \mathrm{G}(t, X(t))\, dW(t), \ (t \geq t_0), \tag{2.5a}$$

$$X(t_0) = X_0, \tag{2.5b}$$

eqn. (2.5a) is termed an *Itô stochastic (ordinary) differential equation* (SODE). The two equations (2.5a)–(2.5b) are to be interpreted as

$$X(t) = X(t_0) + \int_{t_0}^t \mathrm{F}(s, X(s))\, ds + \int_{t_0}^t \mathrm{G}(s, X(s))\, dW(s) \ (t \geq t_0), \tag{2.5c}$$

where the second integral is an Itô integral.

The solution of (2.1) is often considered for $t \in [t_0, T]$ with some finite T, but stability in the sense that it is discussed here (see [17, 23], etc.) concerns the effect on a solution for $t \in [t_0, \infty)$ (*in particular, as* $t \to \infty$), of "admissible" changes in the initial function $\Phi(\cdot)$. Other definitions of stability may involve the effect of persistent or "steady-acting" perturbations in (2.1) for $t \in [t_0 - \tau, \infty)$.

Remark:

• When $t_0 \leq t' < t''$ we have, by virtue of (2.1a),

$$\begin{aligned} X(t'') \ = \ X(t') \ + \int_{t'}^{t''} F(s, X(s), X(s-\tau))\ ds + \\ + \int_{t'}^{t''} G(s, X(s), X(s-\tau)) dW(s). \end{aligned} \tag{2.6}$$

On the interval $[t_0+n\tau, t_0+(n+1)\tau]$, where $n \in \mathbb{N}$, we therefore have $X(t) = X_n(t)$ where

$$\begin{aligned} X_n(t) = \ X_n(t_0+n\tau) + \int_{t_0+n\tau}^t F(s, X_n(s), X_{n-1}(s-\tau))\ ds + \\ \int_{t_0+n\tau}^t G(s, X_n(s), X_{n-1}(s-\tau)) dW(s) \quad (t \geq t_0 + n\tau), \end{aligned} \tag{2.7a}$$

with an initial condition

$$X_n(t) = \Phi_n(t) := X_{n-1}(t) \qquad \text{for } t \in [t_0 + (n-1)\tau, t_0 + n\tau], \tag{2.7b}$$

for $n \geq 0$, and with $X_{-1}(t) := \Phi(t)$. For deterministic problems, such a relation is the basis of Bellman's *method of steps*; for stochastic problems, a similar approach was commented upon in [2] and exploited earlier by (for example) Mao [23, p. 155]; *cf.* [16]. In the case of a "fixed lag", $\Phi_n(\cdot)$ satisfies the appropriate "shifted" version of Condition C0.

• The *numerical* solution of SDDEs has received rather little attention. For the details of a convergence result for the numerical scheme to be discussed in Section 3.1 we refer to [2]. The reader may compare the results obtained in [2] with results in [16, 30, 31, 32]. See also Mao [23] for a discussion of "Caratheodory" and "Cauchy-Maruyama" approximations. We have recently had our attention drawn to [12]; that substantial work contains important developments and further relevant citations.

2.1 Conditions on the functions F, G, and Φ.

We have $F, G : [t_0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $G : [t_0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [-\tau, 0] \to \mathbb{R}$. We will suppose that T may be taken arbitrarily large. Baker and Buckwar [2], who gave the details of their mathematics in the case of an *autonomous* equation

$$X(t) = X(t_0) + \int_{t_0}^{t} f(X(s), X(s-\tau))\ ds + \int_{t_0}^{t} g(X(s), X(s-\tau)) dW(s), \tag{2.8}$$

imposed assumptions related to Conditions C1–C5 here following.

Condition C1: The functions F and G are continuous on their domains of definition.

Condition C2: The functions F and G satisfy, on their domains of definition, a uniform Lipschitz condition, *i.e.*, there exist positive constants L_1, L_2, L_3 and L_4 such that for all $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathbb{R}$ and $t \in [t_0, T]$

$$|F(t, \phi_1, \psi_1) - F(t, \phi_2, \psi_2)| \leq L_1|\phi_1 - \phi_2| \ + \ L_2|\psi_1 - \psi_2|, \tag{2.9}$$

and

$$|G(t, \phi_1, \psi_1) - G(t, \phi_2, \psi_2)| \leq L_3|\phi_1 - \phi_2| \ + \ L_4|\psi_1 - \psi_2|. \tag{2.10}$$

Condition C3: The functions F and G satisfy a linear growth condition, *i.e.*, there exist positive constants K_1 and K_2 (that may depend on T) such that for all $\phi, \phi_1, \psi, \psi_1 \in \mathbb{R}$ and $t \in [0, T]$

$$|F(t, \phi, \phi_1)|^2 \leq K_1(1 + |\phi|^2 + |\phi_1|^2), \tag{2.11}$$

and

$$|G(t, \psi, \psi_1)|^2 \leq K_2(1 + |\psi|^2 + |\psi_1|^2). \tag{2.12}$$

Condition C4: The function $\Phi(\cdot)$ is Hölder-continuous with exponent γ, *i.e.*, there exists a positive constant L_5 such that for $t, s \in [t_0 - \tau, t_0]$

$$\mathcal{E}(|\Phi(t) - \Phi(s)|^{\varrho}) \ \leq \ L_5\ |t - s|^{\varrho\gamma}, \quad \text{for } \varrho = 1, 2. \tag{2.13}$$

Condition C5: The partial derivatives

$$\frac{\partial F}{\partial t}, \quad \frac{\partial F}{\partial \phi}, \quad \frac{\partial F}{\partial \psi}, \quad \frac{\partial^2 F}{\partial \phi^2}, \quad \frac{\partial^2 F}{\partial \psi^2}, \quad \text{and} \quad \frac{\partial^2 F}{\partial \phi\, \partial \psi},$$

of $F(t, \phi, \psi)$ exist and are uniformly bounded on the domain of definition of F.

Remarks:

- Some writers appear to be less formal in their statement of assumptions.
- Conditions C1 – C5 are employed for the convergence theory recalled in Section 3.1. Condition C5 is the natural extension of an assumption made by Milstein [28, p. 20] in his discussion of the Euler-Maruyama method for SODEs.

Definition 2.1 *An $\mathbb{R}$-valued stochastic process $X(t) : [t_0 - \tau, T] \times \Omega \to \mathbb{R}$ is called a* strong solution *of (2.1a), if it is a measurable, sample-continuous process such that $X\Big|[t_0, T]$ is $(\mathcal{A}_t)_{t_0 \le t \le T}$-adapted, F and G are continuous functions and X satisfies, almost surely, (2.1a) and the initial condition $X(t) = \Phi(t)$ ($t \in [t_0 - \tau, t_0]$). A solution $X(t)$ is said to be path-wise unique if any other solution $\widehat{X}(t)$ is indistinguishable from it, i.e.,*

$$P\Big(X(t) \;=\; \widehat{X}(t) \;\; \textit{for all } t_0 - \tau \le t \le T\Big) \;=\; 1.$$

We need to know that there exists a path-wise unique solution $X(t; t_0, \Phi)$ of (2.1), and we have the following result.

Theorem 2.1 *Assume that the functions F and G satisfy the assumptions C1 to C3 above. Then there exists a unique strong solution to equations (2.1).*

Proof: A proof of Theorem 2.1 can be found in [23]. □

We suppose that the equation admits the *null solution* (or trivial solution) $X(t) \equiv 0$ (for $t \ge t_0 - \tau$), and to this end we request that the following condition holds.
Condition C6: The functions F and G satisfy

$$F(t, 0, 0) = 0 \text{ and } G(t, 0, 0) = 0, \text{ for } t \ge t_0.$$

2.2 Stability in p-th mean

There are at least three different types of stability in a stochastic sense; *cf.* [23, p. 109]. We will be concerned, for $1 \le p < \infty$, with basic ideas of *pth-mean stability* (in particular with *mean-square stability* obtained on setting $p = 2$) of the null solution of eqn. (2.1), with respect to perturbations in $\Phi(\cdot)$. We here recall various definitions and obtain a condition for a certain type of stability of a solution of a stochastic delay differential equation. Recall that $J = [t_0 - \tau, t_0]$.

Definition 2.1 *For some $p > 0$, the null solution of the SDDE (2.1) is termed (1) locally* stable in the p-th mean, *if for each $\epsilon > 0$, there exists a $\delta \ge 0$ such that*

$$\mathcal{E}(|X(t; t_0, \Phi)|^p) < \epsilon \tag{2.14a}$$

whenever $t \ge t_0$ and $\mathcal{E}(\sup_{t \in J} |\Phi(t)|^p) < \delta$;

(2) locally asymptotically stable in the p-th mean, *if it is stable in the p-th mean and if there exists a $\delta \ge 0$ such that whenever $\mathcal{E}(\sup_{t \in J} |\Phi(t)|^p) < \delta$*

$$\mathcal{E}(|X(t; t_0, \Phi)|^p) \to 0 \textit{ for } t \to \infty; \tag{2.14b}$$

(3) locally exponentially stable in the p-th mean *if it is stable in the p-th mean and if there exists a* $\delta \geq 0$ *such that whenever* $\mathcal{E}(\sup_{t\in J}|\Phi(t)|^p) < \delta$ *there exists some finite constant* C *and a* $\nu^\star > 0$ *such that*

$$\mathcal{E}(|X(t;t_0,\Phi)|^p) \leq C\ \mathcal{E}(\sup_{s\in J}|\Phi(s)|^p)\ \exp(-\nu^\star(t-t_0)) \quad (t_0 \leq t < \infty). \tag{2.14c}$$

If, in the above, δ *may be taken arbitrarily large then the stability is in each case* global *rather than local.*

Remarks:

- If $\hat{\nu}$ is the largest of the (positive) values $\nu^\star$ for which (2.14c) holds, it may be termed the *rate constant* of exponential stability in the p-th mean.
- Mao's terminology *p-th moment exponential stability* (see [23, p. 126]) is synonymous with our terminology *global exponential stability in the p-th mean.*
- For other concepts of stability see, for example, [17, 23] and compare the definitions given in [11] and in [23] in the case of SODEs.

2.3 Stability of a non-null solution

Observe that the condition $G(t,0,0) = 0$ in Condition C6 excludes from consideration equations that have purely additive noise. The following comment indicates that our stability definitions, applying as they do to the null solution, are less restricted than appears at first sight. Suppose that $X(t)$ is a given *non-null* solution of (2.1). Then the associated equation (to be solved for $U(t)$),

$$\begin{aligned} dU(t) &= F_\star(t,U(t),U(t-\tau))\ dt \\ &\qquad + G_\star(t,U(t),U(t-\tau))\ dW(t) \quad (t \geq t_0), \end{aligned} \tag{2.15a}$$

$$U(t) = 0 \quad (t \in J := [t_0-\tau, t_0]), \tag{2.15b}$$

where

$$F_\star(t,U(t),U(t-\tau)) = F(t, X(t)+U(t), X(t-\tau)+U(t-\tau)), \tag{2.16a}$$

$$G_\star(t,U(t),U(t-\tau)) = G(t, X(t)+U(t), X(t-\tau)+U(t-\tau)), \tag{2.16b}$$

has the null solution $U(t) \equiv 0$ for $t \geq t_0-\tau$. The sensitivity of $X(t)$ to perturbations is now reflected in the sensitivity of the null solution $U(t)$ to perturbations. Our local stability definitions for the null solution were formulated in terms of the boundedness or decay of $\mathcal{E}(|X(t;t_0,\Phi)|^p)$ for suitably small $\mathcal{E}(\sup_{t\in J}|\Phi(t)|^p)$. This should not be allowed to obscure the fact that stability of a general solution $X(t)$ relates to the effect $\delta X(t)$ of making a *change* from $\Phi(t)$ to $\Phi(t)+\delta\Phi(t)$ (the effect being measured by $\mathcal{E}(|\delta X(t)|^p)$). For a nonlinear SDDE, it is possible for a non-null solution $X(t)$, having the property that $\mathcal{E}(X(t;t_0,\Phi)) \to 0$, to be unstable. Again for a nonlinear SDDE, when $\Phi_1(t)$ and $\Phi_2(t)$ are distinguishable, $X(t;t_0,\Phi_1)$ and $X(t;t_0,\Phi_2)$ may have different stability properties; however, for a linear SODE all solutions simultaneously have the same stability properties.

2.4 Insight from deterministic equations

It is useful to compare the definitions given above with the corresponding stability definitions (see, for example, [1, 17]) for a deterministic problem

$$x'(t) = f(t,x(t),x(t-\tau)) \quad (t \geq t_0), \tag{2.17a}$$

$$x(t) = \varphi(t) \quad (t \in [t_0-\tau, t_0]), \tag{2.17b}$$

where $\varphi(\cdot) \in C[t_0 - \tau, t_0]$ or, alternatively, where $\varphi(\cdot)$ may be supposed to be bounded and piecewise continuous. We recall that the usual right-hand and two-sided derivatives are,

$$x'_+(t) = \lim_{\delta \searrow 0} \frac{y(t+\delta) - y(t)}{\delta} \quad \text{and} \quad x'(t) = \lim_{\delta \to 0} \frac{y(t+\delta) - y(t)}{\delta},$$

and when the two-sided derivative $x'(t)$ does not exist in (2.17a) it is to be interpreted as the right-hand derivative[1]. We may more properly re-write (2.17a) as

$$x'_+(t) = f(t, x(t), x(t-\tau)) \quad (t \geq t_0). \tag{2.17c}$$

When $f(t, 0, 0) = 0$, (2.17) possesses a null solution $x(t) \equiv 0$. The "deterministic" stability definitions for the null solution can be deduced from those given for the stochastic problem.

The equation

$$x'_+(t) = -\alpha x(t) + \beta x(t - \tau) \quad \text{for } t \geq t_0 \tag{2.18}$$

(where $\tau \geq 0$) is often taken as a test equation to develop insight. One can show that the null solution of (2.18) is globally stable when, in particular,

$$0 \leq |\beta| \leq \alpha. \tag{2.19}$$

This is a sufficient but not necessary condition. For arbitrary $\alpha, \beta \in \mathbb{R}$ and $\tau \geq 0$, a *necessary and sufficient condition* for global stability of the null solution of (2.18) under continuous bounded perturbations of the initial function on $[t_0 - \tau, t_0]$, is that all of the (at most countable) zeros $\{\nu_\ell\}$ of the stability function

$$\mathcal{Q}(\nu) \equiv \mathcal{Q}(\nu;\ \alpha, \beta, \tau) := \nu + \alpha - \beta \exp\{-\nu\tau\} \tag{2.20}$$

have *non-positive real part* (any with vanishing real part being simple zeros). $\mathcal{Q}$ is an example of a *quasi-polynomial* in ν.

The zeros of $\mathcal{Q}$ have a rôle later; we shall give a simple analytical result. For $\tau = 0$, $\mathcal{Q}(\nu) = \nu - (\alpha - \beta)$ and everything is obvious. Suppose, therefore, that $\tau > 0$ and we convey the spirit if we first scale the independent variable and set $\nu_\tau = \tau\nu$ in order to consider, with $\alpha_\tau = \alpha\tau$ and $\beta_\tau = \beta\tau$, the function $\mathcal{Q}_\tau(\cdot)$ where

$$\mathcal{Q}_\tau(\nu_\tau) := \nu_\tau + \alpha_\tau - \beta_\tau \exp(-\nu_\tau) \quad (0 < \beta_\tau < \alpha_\tau).$$

This function has a single real zero, which we denote $-\nu_\tau^*$, lying in $(-\alpha_\tau, 0)$. The existence of such a zero follows because $\mathcal{Q}_\tau(-\alpha_\tau) = -\beta_\tau \exp(\alpha_\tau) < 0$ and $\mathcal{Q}_\tau(0) = \alpha_\tau - \beta_\tau > 0$. Its uniqueness follows because the derivative $\mathcal{Q}'_\tau(\nu_\tau)$ is $1 + \beta_\tau \exp(-\nu_\tau) > 1$. We can now refine the result $\nu_\tau \in (-\alpha_\tau, 0)$. Since $\mathcal{Q}_\tau(\beta_\tau - \alpha_\tau) = \beta_\tau(1 - \exp(\alpha_\tau - \beta_\tau)) < 0$ we have $\nu_\tau \in (\beta_\tau - \alpha_\tau, 0)$. If we introduce $\nu_*^{[\tau]}$ in order to write $-\nu_\tau^* = \nu_*^{[\tau]} - \alpha_\tau$ then $\nu_*^{[\tau]} \exp(\nu_*^{[\tau]}) = \beta_\tau \exp(\alpha_\tau)$. If we retain *explicit* dependence upon τ we reach the following conclusion.

Lemma 2.1 *Suppose that $\tau \geq 0$ and $0 < \beta < \alpha$. The function $\mathcal{Q}(\nu;\ \alpha, \beta, \tau) = \nu + \alpha - \beta \exp\{-\nu\tau\}$ in (2.20) has a single real zero $-\nu^\star$ which is negative. If $\tau = 0$ then $\nu^\star = \alpha - \beta$; if $\tau > 0$ then $\beta - \alpha < -\nu^\star < 0$, and we can write*

$$-\nu^\star = \nu_\star - \alpha \tag{2.21a}$$

where

$$\nu_\star \exp\{\nu_\star \tau\} = \beta \exp\{\alpha\tau\} \ (\text{so that } 0 < \nu_\star \in (\beta, \alpha)). \tag{2.21b}$$

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[1]This interpretation is a standard convention at a left end point such as t_0. If $\varphi(\cdot)$ is piecewise continuous with a finite jump at $t^\star \in (t_0 - \tau, t_0)$, the right-hand derivative is invoked at $t^\star + \tau$.

Numerical approximations to $\nu^\star$ for various parameter values provide additional insight. (They may be obtained by Newton's iteration.)

Remark:

• There is a variety of different approaches to the study of stability. Since the equation (2.18) is both linear and has constant coefficients it is appropriate to indicate very briefly some of the many alternative approaches which can sometimes be applied not only to (2.18) (which we use to illustrate technique) but also to linear equations with variable coefficients or non-linear equations. For alternatives to those followed here we refer to the relevant literature cited (where the following observations may also be discovered).

• First consider the map

$$X \mapsto |X| \tag{2.22a}$$

for $X \in \mathbb{R}$, which, given the function $x : [t_0 - \tau, \infty) \to \mathbb{R}$ gives rise to

$$u(t) = |x(t)| \equiv \text{sign}\,\{x(t)\}x(t). \tag{2.22b}$$

Now $u(t) \geq 0$ with equality only if $x(t) = 0$ and $u'_+(t) = \text{sign}\,\{x(t)\}x'_+(t)$. If $x'_+(t) = -\alpha x(t) + \beta x(t-\tau)$, then $u'_+(t) = -\alpha\,\text{sign}\,\{x(t)\}x(t) + \beta\,\text{sign}\,\{x(t)\}x(t-\tau)$ and hence $u'_+(t) \leq -\alpha|x(t)| + |\beta||x(t-\tau)|$ from which we deduce for the non-negative function $u(t)$ that

$$u'_+(t) \leq -\alpha u(t) + |\beta| \sup_{s\in[0,1]} u(t - s\tau). \tag{2.23}$$

If we show that (when $\sup_{s\in[0,1]} u(t_0 - s\tau)$ is small) that $u(t)$ is (i) bounded, (ii) tends to zero, or (iii) decays exponentially, we can deduce the stability, asymptotic stability, or exponential stability of the null solution of (2.18).

• Alternatively, consider the function $w(t) = \frac{1}{2}|x(t)|^2$ derived from the map

$$X \mapsto \frac{1}{2}|X|^2. \tag{2.24}$$

We have $w'_+(t) = x(t)x'_+(t)$. If $x(t)$ is a solution of (2.18) then $w'_+(t) = -\alpha|x(t)|^2 + \beta x(t)x(t-\tau)$. It follows immediately that

$$w'_+(t) \leq -\alpha|x(t)|^2 + \beta \sup_{s\in[0,1]} |x(t - s\tau)|^2,$$

so $w_+(t) = |x(t)|^2$ satisfies

$$w'_+(t) \leq -\alpha w(t) + |\beta| \sup_{s\in[0,1]} w(t - s\tau). \tag{2.25}$$

If we establish (when $\sup_{s\in[0,1]} w(t_0 - s\tau)$ is small) that $w(t)$ is (i) bounded, (ii) tends to zero, or (iii) decays exponentially, we can deduce, respectively, the stability, asymptotic stability, or exponential stability of the null solution.

• Consider now the equation $x'_+(t) = -\alpha x(t) + \beta_*(t)x(t-\tau)$ in which $\beta_*(t)$ is right-continuous for $t \in [t_0, \infty)$ and is bounded with $\sup_{t\geq t_0}|\beta(t)| = \beta < \infty$. Then the preceding inequalities (2.23) and (2.25) are still valid and can be used to establish stability of the null solution.

• A different tactic to that employed above is indicated in Section 4.7.

2.5 An inequality.

Motivated by the need, in our study of stability in a stochastic sense, to examine inequalities of the type (2.23), (2.25), we shall employ the following results.

Lemma 2.2 *Suppose that $\tau \geq 0$, that $0 < \beta < \alpha$, and that the function $\mathcal{Q}$ is defined by (2.20) and suppose $\nu^\star \in (0, \alpha - \beta]$ satisfies $\mathcal{Q}(-\nu^\star;\ \alpha, \beta, \tau) = 0$. Then, for arbitrary $K > 0$, the positive monotonic-decreasing function*

$$\widehat{w}_K(t) = K \exp\{-\nu^\star(t - t_0)\}, \tag{2.26}$$

satisfies

$$\widehat{w}_K'(t) = -\alpha \widehat{w}_K(t) + \beta \sup_{s \in [0,1]} \widehat{w}_K(t - s\tau) \quad \text{for } t \geq t_0. \tag{2.27}$$

Proof: By the preceding lemma, Lemma 2.1, for $0 < \beta < \alpha$ the unique real zero of $\mathcal{Q}(\nu)$ is a value $-\nu^\star \in [\beta - \alpha, 0)$. We verify by substitution, employing the property $\mathcal{Q}(-\nu^\star;\ \alpha, \beta, \tau) = 0$, that for any $K > 0$

$$\widehat{w}_K'(t) = -\alpha \widehat{w}_K(t) + \beta \widehat{w}_K(t - \tau) \tag{2.28}$$

for $t \geq t_0$. The result follows since $\widehat{w}_K(t)$ is monotonic decreasing. □

Recall that, for a continuous real-valued function $y(\cdot)$ of a real variable, the Dini-derivative $D^+y(t)$ is defined as

$$D^+y(t) = \limsup_{\delta \searrow 0} \frac{y(t+\delta) - y(t)}{\delta}.$$

We shall not here require the other Dini-derivatives $D^-y(t)$, $D_+y(t)$, $D_-y(t)$. From the respective definitions of the differing types of derivatives, we have the following result.

Lemma 2.3 *(a) Suppose that $u(\cdot)$ is real-valued and continuous on $[t_0, T)$ and $D^+u(t) \leq 0$ for $t \in [t_0, T)$ (with the possible exception of a countable set of points in $[t_0, T)$). Then $u(t)$ is a non-increasing function of t for $t \in [t_0, T)$. (b) If $y(\cdot)$ has a right-hand derivative $y'_+(t)$ at t, then $D^+y(t) = y'_+(t)$ and if $y'(t)$ exists then $D^+y(t) = y'(t)$. (c) Assume $u(\cdot)$ and $w(\cdot)$ are continuous on an interval (t_0, T) and in addition the derivative $w'(\cdot)$ exists on (t_0, T). Then if $y(t) = u(t) + w(t)$ we have $D^+y(t) = D^+u(t) + w'(t)$ on (t_0, T),*

Part (a) of the above result is attributed to Zygmund by Lakshmikantham and Leela [21, Volume 1, p. 9] and part (c) is noted by Walter [33] (observe that D^+ is not a linear operator on the space of continuous functions). The following theorem is related to a result due to Halanay; it employs the Dini derivative D^+. For Halanay's theorem and some related results see, *e.g.*, [4, 6, 7, 22].

Theorem 2.1 *Suppose that the positive-valued function $v : [t_0 - \tau, \infty) \to \mathbb{R}^+$, is continuous (on $[t_0 - \tau, \infty)$), for given $\tau \geq 0$. Suppose, further, that*

$$0 < \beta < \alpha, \tag{2.29a}$$

$$D^+v(t) \leq -\alpha\ v(t)\ +\ \beta \sup_{s \in [0,1]} v(t - s\tau),\ t \in [t_0, \infty). \tag{2.29b}$$

Then

$$v(t) \leq \{\sup_{t_0 - \tau \leq s \leq t_0} v(s)\}\ \exp\{-\nu^\star(t - t_0)\} \quad \text{for } t_0 \leq t < \infty, \tag{2.30}$$

where $\nu^\star \in (0, \alpha - \beta]$ is the value in Lemmas 2.1 and 2.2.

Proof: The case where $\tau = 0$ is immediate, so let $\tau > 0$. Set $K = \sup_{s \in J} v(s)$, where $J = [t_0 - \tau, t_0]$ (by assumption, $K > 0$) in the definition of $\widehat{w}_K(t)$ above. Choose any $T > t_0$ and let $\ell > 1$ be arbitrary. The monotonic decreasing function $\widehat{w}_K(\cdot)$ is differentiable on $[t_0 - \tau, T]$. We now establish that

$$v(t) < \ell \widehat{w}_K(t) \text{ for } t_0 - \tau \le t < T. \tag{2.31}$$

We give a proof by contradiction. Obviously

$$v(t) = \widehat{w}_K(t) < \ell \widehat{w}_K(t) \text{ when } t_0 - \tau \le t < t_0. \tag{2.32a}$$

In contradiction of (2.31), suppose that $v(t_1) = \ell \widehat{w}_K(t_1)$, for some argument $t_1 \in (t_0, T)$. Since $v(\cdot)$ and $\widehat{w}_K(\cdot)$ are continuous functions, there must exist some *least* value $t_1 \in (t_0, T)$ such that

$$[v(t) - \ell \widehat{w}_K(t)] < 0 \text{ for } t_0 - \tau \le t < t_1 \quad \text{and} \quad [v(t_1) - \ell \widehat{w}_K(t_1)] = 0. \tag{2.32b}$$

It follows, firstly, that

$$\sup_{s \in [0,1]} v(t - s\tau) < \ell \sup_{s \in [0,1]} \widehat{w}_K(t - s\tau) \text{ for } t \in [t_0, t_1], \tag{2.32c}$$

and, secondly, that $[v(t) - \ell \widehat{w}_K(t)]$ must be increasing on some subinterval of $[t_0, t_1]$. But (appealing to Lemma 2.3) no such subinterval exists, since, on $[t_0, t_1]$, $D^+[v(t) - \ell \widehat{w}_K(t)] = D^+ v(t) - \ell \widehat{w}_K'(t) \le -\alpha\, [v(t) - \ell \widehat{w}_K(t)] + \beta\, [\sup_{s \in [0,1]} v(t - s\tau) - \sup_{s \in [0,1]} \ell \widehat{w}_K(t - s\tau)]$, and hence

$$\begin{aligned} D^+[v(t) - \ell \widehat{w}_K(t)] &\le (\beta - \alpha)[\sup_{s \in [0,1]} v(t - s\tau) - \sup_{s \in [0,1]} \ell \widehat{w}_K(t - s\tau)] \\ &< 0, \qquad \text{on } [t_0, t_1]. \end{aligned} \tag{2.32d}$$

Hence, appealing again to Lemma 2.3, we know that $[v(t) - \ell \widehat{w}_K(t)]$ is non-increasing on $[t_0, t_1]$. Thus, the premise that $v(t) = \ell \widehat{w}_K(t)$ for some $t = t_1 \in (t_0, T)$ is false. Eqn. (2.31) now follows. Now letting $\ell \to 1$ we conclude that $v(t) \le \widehat{w}_K(t)$; finally, since T is arbitrary, (2.30) follows. □

2.6 Some stochastic analysis

For completeness, we recall the following definition.

Definition 2.2 *(a) We denote by* $\mathrm{L}^\varrho[t_0, T]$ *the family of* $\mathcal{A}_t$*-adapted processes* $\{Y(t) : t \ge t_0\}$ *such that* $\int_{t_0}^T |Y(t)|^\varrho dt < \infty$ *and by* $\mathrm{L}^\varrho_{\mathrm{loc}}[t_0, \infty)$ *the family of processes* $\{Y(t) : t \ge t_0\}$ *such that, for every* $T > t_0$, $\{Y(t) : t \ge t_0\} \in \mathrm{L}^\varrho[t_0, T]$. *(b) We say that a stochastic process* $X(t)$ *on* $[t_0, T]$, *given, with* $f \in \mathrm{L}^1[t_0, T]$, $g \in \mathrm{L}^2[t_0, T]$, *by* $X(t) = X(t_0) + \int_{t_0}^t f(s)ds + \int_{t_0}^t g(s)dW(s)$ *(where the second integral above is to be interpreted in the Itô sense), has a stochastic differential* $dX(t)$ *given by* $dX(t) = f(t)dt + g(t)dW(t)$.

We denote by U_t, U_x and U_{xx}, respectively, the partial derivatives $\dfrac{\partial U}{\partial t}, \dfrac{\partial U}{\partial x}$ and $\dfrac{\partial^2 U}{\partial x^2}$ of a function $U \equiv U(t, x)$. We recall the following result (see Mao [23]).

Theorem 2.2 (The Itô Formula) *Let* $U : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ *have continuous partial derivatives* U_t, U_x *and* U_{xx} *and let* $X(t)$ *be a stochastic process with stochastic differential* $dX(t) = f(t)dt + g(t)dW(t)$ *where* $f \in \mathrm{L}^1_{\mathrm{loc}}[t_0, \infty)$, $g \in \mathrm{L}^2_{\mathrm{loc}}[t_0, \infty)$. *Then the process* $U(t, X(t))$, *defined on* $0 \le t < \infty$, *with initial value* $U(t_0, X(t_0))$, *also* *has a stochastic differential with respect to the same Wiener process* $W(t)$*, and we have for any* $0 \le t < \infty$

$$dU(t,X(t)) = \left\{ U_t(t,X(t)) + U_x(t,X(t))\ f(t) + \frac{1}{2}U_{xx}(t,X(t))\ g^2(t) \right\}\ dt \\ + U_x(t,X(t))\ g(t)\ dW(t) \tag{2.33}$$

almost surely. In this formula, f *and* g *can be state-dependent (e.g.,* $f(t) = F(t,X(t),X(t-\tau))$*,* $g(t) = G(t,X(t),X(t-\tau))$*).*

The Itô formula can be considered to be a "stochastic chain rule". One can restrict the above theorem to the interval $[t_0,T]$ in an obvious manner. In the special case that $X(t)$ is a solution of (2.1), the stochastic differential (2.33) $dU(t,X(t))$ takes, when U_t, U_x, U_{xx} are continuous, the form

$$dU(t,X(t)) = \{U_t(t,X(t)) + U_x(t,X(t))\ F(t,X(t),X(t-\tau)) \\ + \frac{1}{2}U_{xx}(t,X(t))\ G^2(t,X(t),X(t-\tau))\}\ dt \\ + U_x(t,X(t))\ G(t,X(t),X(t-\tau))\ dW(t). \tag{2.34}$$

We shall use the equivalent integral form of (2.34), below.

2.7 A Lyapunov-type theory

We shall prove the following result.

Theorem 2.3 *Assume (a) conditions C1–C6 hold and that* $X(t) \equiv X(t;t_0,\Phi)$ *is a solution of (2.1). Assume, further, (b) that there exists a positive, continuous function* $V(t,x)$ *(for* $t \ge t_0 - \tau$ *and* $x \in \mathbb{R}$*); (c) that there exist positive constants* c_1*,* c_2*, and* $p > 1$*, such that*

$$c_1\ |x|^p \le V(t,x)\ \le\ c_2|x|^p, \tag{2.35a}$$

when $t \ge t_0 - \tau$ *and* $x \in \mathbb{R}$*. Finally, suppose (d) that, when* $t \ge t_0$*,*

$$D^+\mathcal{E}\big(V(t,X(t))\big) \le -\alpha\mathcal{E}\big(V(t,X(t))\big) + \beta\mathcal{E}\big(V(t-\tau,X(t-\tau))\big), \tag{2.35b}$$

for some values $0 < \beta < \alpha$*. Then*

$$\mathcal{E}(|X(t;t_0,\Phi)|^p) \le \frac{c_2}{c_1}\mathcal{E}(\sup_{s\in[t_0-\tau,t_0]}|\Phi(s)|^p)\ \exp(-\nu^\star(t-t_0)). \tag{2.36}$$

with $\nu^\star$ *given in terms of* α*,* β *by (2.21), and the null solution of eqn. (2.1) is therefore globally exponentially stable in the p-th mean.*

Proof: The Dini derivative $D^+\mathcal{E}\big(V(t,X(t))\big)$ in (2.35b) is $D^+v(t)$ where

$$v(t) := \mathcal{E}(V(t,X(t))) \equiv \mathcal{E}(V(t,X(t;t_0,\Phi))).$$

It can be shown (see, *e.g.*, [24]) that $v(t)$ exists for $t \ge t_0 - \tau$ and is continuous (and non-negative); it is therefore Dini-differentiable for $t \in [t_0-\tau,\infty)$. Eqn. (2.35b) yields, expressed in terms of $v(t)$,

$$D^+v(t)\ \le\ -\alpha v(t) + \beta \sup_{s\in[0,1]} v(t-s\tau), \quad t \ge t_0,$$

and we obtain from Theorem 2.1 the exponentially decreasing bound on $v(t)$:

$$v(t) \le \big\{\sup_{s\in J} v(s)\big\}\ \exp\{-\nu^\star(t-t_0)\} \quad \text{for } t_0 \le t < \infty. \tag{2.37}$$

However, by (2.35a),

$$\sup_{s\in J} v(s) \equiv \sup_{s\in J} \mathcal{E}(V(s,\Phi(s))) \leq c_2 \sup_{s\in J} \mathcal{E}(|\Phi(s)|^p) \leq c_2 \mathcal{E}(\sup_{s\in J} |\Phi(s)|^p) \tag{2.38a}$$

and

$$\mathcal{E}(|X(t)|^p) \;\leq\; \frac{1}{c_1} v(t), \tag{2.38b}$$

for $c_1 \neq 0$ so that from (2.37) and (2.38) we obtain the desired result, which by Definition 2.1 implies the exponential stability of the null solution. □

Lemma 2.4 *Suppose that $V(t,x)$ satisfies condition (2.35a), in the statement of Theorem 2.3, and it has continuous derivatives $V_t(t,x)$, $V_x(t,x)$, and $V_{xx}(t,x)$ for $t \geq t_0 - \tau$ and $x, y \in \mathbb{R}$. Suppose further that*

$$\begin{aligned} V_t(t,x) + V_x(t,x)\; F(t,x,y) + \frac{1}{2} V_{xx}(t,x) G^2(t,x,y) \\ \leq -\alpha V(t,x) + \beta V(t-\tau, y), \text{ for } 0 < \beta < \alpha, \end{aligned} \tag{2.39}$$

when $t \geq t_0$ and $x, y \in \mathbb{R}$. Then the inequality (2.35b) holds, and the conclusions of Theorem 2.3 apply.

Proof: By the integral form of the Itô formula (2.34) we obtain for $t \geq t_0$,

$$\begin{aligned} & V(t+\delta, X(t+\delta)) - V(t, X(t)) \\ & = \int_t^{t+\delta} V_t(s,X(s)) + V_x(s,X(s))\; F(s,X(s),X(s-\tau)) \\ & + \frac{1}{2}\int_t^{t+\delta} V_{xx}(s,X(s))\; G^2(s,X(s),X(s-\tau))\; ds \\ & + \int_t^{t+\delta} V_x(s,X(s))\; G(s,X(s),X(s-\tau))\; dW(s). \end{aligned}$$

Since $\mathcal{E}(\int_t^{t+\delta} V_x(s,X(s))\; G(s,X(s),X(s-\tau))\; dW(s)) = 0$, taking expectations to obtain $\mathcal{E}(V(t+\delta, X(t+\delta)) - V(t,X(t)))$ yields, for $t \geq t_0$,

$$\begin{aligned} & \mathcal{E}V(t+\delta, X(t+\delta)) - \mathcal{E}V(t, X(t)) \\ & = \mathcal{E}\Big(\int_t^{t+\delta} V_t(s,X(s)) + V_x(s,X(s))\; F(s,X(s),X(s-\tau)) \\ & + \frac{1}{2} V_{xx}(s,X(s))\; G^2(s,X(s),X(s-\tau))\; ds \Big) \end{aligned}$$

and thus

$$\begin{aligned} & \mathcal{E}(V(t+\delta, X(t+\delta)) - \mathcal{E}(V(t,X(t))) \\ & \leq \Big(\int_t^{t+\delta} \{ -\alpha\; \mathcal{E}(V(s,X(s))) + \beta\; \mathcal{E}(V(s-\tau, X(s-\tau)))\; ds \} \Big). \end{aligned}$$

Since the Dini derivative $D^+ v(t)$ is

$$D^+ \mathcal{E}(V(t,X(t))) := \limsup_{\delta \searrow 0} \frac{\mathcal{E}(V(t+\delta, X(t+\delta)) - \mathcal{E}(V(t,X(t)))}{\delta}$$

the preceding result leads directly to

$$D^+ \mathcal{E}(V(t,X(t))) \leq -\alpha\; \mathcal{E}(V(t,X(t))) + \beta\; \mathcal{E}(V(t-\tau, X(t-\tau))),$$

for $t \geq t_0$. Clearly, Theorem 2.3 now applies. □

An examination of the proof of Theorem 2.3 will reveal that (2.39) is stronger than required, since we only employ the bound

$$\mathcal{E}\big(V_t(t,X(t))+V_x(t,X(t))F(t,X(t),X(t-\tau))+\frac{1}{2}V_{xx}(t,X(t))G^2(t,X(t),X(t-\tau))\big)$$
$$\leq \mathcal{E}\big(-\alpha V(t,X(t))+\beta V(t-\tau,X(t-\tau))\big), \tag{2.40}$$

for $0<\beta<\alpha$, in the particular case that $X(t)$ satisfies the given SDDE for some $\Phi(\cdot)$.

Remarks:

- Mao [23, pp. 178 *et seq.*] considered multiple lags (see eqn. (4.20a) below) and established exponential stability in the p-th mean by a different proof, based on a Razumikhin-type analysis.
- Conditions (2.35) are required only for all sufficiently small x, y, uniformly for all sufficiently large t, to obtain *local* exponential stability in the p-th mean.
- In deterministic stability analysis for solutions of delay differential equations, one often utilizes a condition on the total derivative (suitably interpreted) of some *Lyapunov function* or *Lyapunov functional* along the solution trajectory. The approach followed here is an adaptation of the deterministic approach; one takes the Dini derivative of the expectation of $V(t,X(t))$ "along" $X(t)$.

The following result now follows as a corollary of Theorem 2.3.

Theorem 2.4 (A criterion for stability.) *Assume that there exists $\kappa>0$ such that*

$$xF(t,x,0) \;\leq\; -\kappa|x|^2 \quad \textit{for all} \quad (x,t)\in\mathbb{R}\times[0,\infty). \tag{2.41}$$

Assume also that there are nonnegative numbers α_0, α_1, β_0, β_1, such that

$$|F(t,x_1,0)-F(t,x_2,y)| \;\leq\; \alpha_0|x_1-x_2|+\alpha_1|y| \tag{2.42}$$

and

$$|G(t,x_1,y)|^2 \;\leq\; \beta_0|x_1|^2+\beta_1|y|^2 \tag{2.43}$$

for all $t\geq 0$ and $x_1,x_2,y\in\mathbb{R}$. If

$$p\geq 2 \textit{ and } \quad \kappa \;>\; \alpha_1+\frac{p-1}{2}(\beta_0+\beta_1) \tag{2.44}$$

then the null solution of eqn. (2.1) is globally exponentially stable in the p-th mean.

Proof: Our proof follows that in [23, p. 178–179] in most respects, but for an appeal (via Lemma 2.4) to Theorem 2.3 rather than an appeal to [23, Theorem 6.4, p. 177]. For completeness, we provide a detailed proof, borrowing from [23, p. 178–179].

We need to establish an inequality of the form (2.39). We shall require recourse to the inequalities

$$|x|^{p-1}|y| \;\leq\; \frac{p-1}{p}|x|^p+\frac{1}{p}|y|^p \tag{2.45a}$$
$$|x|^{p-2}|y|^2 \;\leq\; \frac{p-2}{p}|x|^p+\frac{2}{p}|y|^p \tag{2.45b}$$

which follow from the elementary inequality (see [10, p.37])

$$u^s v^{1-s} \;\leq su+(1-s)v, \quad \text{when } s\in(0,1) \text{ and } u,v\geq 0, \tag{2.45c}$$

on writing, respectively, either $u = |x|^p$, $s = \frac{p-1}{p}$ and $v = |y|^p$ (and observing that $|x|^{p-1}|y| = (|x|^p)^{\frac{p-1}{p}}(|y|^p)^{\frac{1}{p}}$), or $u = |x|^p$, $s = \frac{p-2}{p}$ and $v = |y|$. Let

$$V(t,x) = |x|^p. \tag{2.46}$$

We have (noting that $V(t,x)$ in (2.46) has been chosen to be independent of t)

$$\begin{aligned}
&V_t(t,x) + V_x(t,x)\ F(t,x,y) + \frac{1}{2}V_{xx}(t,x)G^2(t,x,y) \\
&= 0 + p|x|^{p-2}xF(t,x,0) + p|x|^{p-2}x\{F(t,x,y) - F(t,x,0)\} + \frac{p}{2}|x|^{p-2}|G(t,x,y)|^2 \\
&\qquad + \frac{p(p-2)}{2}|x|^{p-4}|xG(t,x,y)|^2 \\
&\le -\left(p\kappa - \frac{p(p-1)\beta_0}{2}\right)|x|^p + p\alpha_1|x|^{p-1}|y| + \frac{p(p-1)}{2}\beta_1|x|^{p-2}|y|^2
\end{aligned}$$

(for all $x, y \in \mathbb{R}$). We now appeal to (2.45a–b) and the conditions of our Theorem, and we obtain

$$\begin{aligned}
&V_t(t,x) + V_x(t,x)\ F(t,x,y) + \frac{1}{2}V_{xx}(t,x)G^2(t,x,y) \\
&\le -\left(p\kappa - \frac{p(p-1)}{2}\beta_0 - (p-1)\alpha_1 - \frac{(p-1)(p-2)}{2}\beta_1\right)|x|^p \\
&\qquad\qquad + \big(\alpha_1 + (p-1)\beta_1\big)|y|^p \qquad (2.47)\\
&\le -\alpha V(t,x) + \beta V(t-\tau, y) \qquad (2.48)
\end{aligned}$$

($V(t,x)$ is $|x|^p$, $V(t-\tau,y)$ is $|y|^p$) with

$$\alpha = p\kappa - \frac{p(p-1)}{2}\beta_0 - (p-1)\alpha_1 - \frac{(p-1)(p-2)}{2}\beta_1 \text{ and } \beta = \alpha_1 + (p-1)\beta_1.$$

By virtue of (2.44), $0 < \beta < \alpha$. The result (2.47) is obtained by Mao, who then appeals to [23, Theorem 6.4, p. 177] to establish Theorem 2.4; in our alternative proof of Theorem 2.4, the proof is immediately completed by an application of Lemma 2.4, using (2.48). □

2.8 A test equation

Here, we consider a linear stochastic delay differential equation on $t_0 \le t < \infty$ as a *test equation* for the discussion of stability, namely

$$\begin{aligned}
dX(t) &= \{-\alpha\ X(t)\ +\ \beta\ X(t-\tau)\}\ dt \qquad (2.49a)\\
&\qquad + \{\eta\ X(t)\ +\ \mu X(t-\tau)\}\ dW(t) \quad (t > t_0),\\
X(t) &= \Phi(t), \quad t \in [t_0 - \tau, t_0] \qquad (2.49b)
\end{aligned}$$

with $\alpha > 0$, $\mathcal{E}(\sup_{t\in J}|\Phi(t)|^2) < \infty$, where $W(t)$ is a standard Wiener process, and α, β, η and $\mu \in \mathbb{R}$. All the required conditions for us to apply Theorem 2.4 to the problem (2.49) are satisfied and we obtain the following result for mean-square stability ($p = 2$):

Corollary 2.1 *If*

$$\alpha > |\beta| + |\eta|^2 + |\mu|^2 \tag{2.50}$$

then the null solution of (2.49) is (globally) exponentially mean-square stable.

Proof: Set $F(t,x,y) := \{-\alpha x + \beta y\}$ and $G(t,x,y) = \{\eta x + \mu y\}$. By virtue of the fact that $2|\eta\mu xy| \le \eta^2 x^2 + \mu^2 y^2$, the conditions of Theorem 2.4 hold for the case $p = 2$ with $\beta_0 = \eta$, $\beta_1 = \mu$ and with $\kappa = \alpha$, $\alpha_0 = \alpha$, $\alpha_1 = \beta$. (Indeed, with the renaming of variables, for $p = 2$ conditions (2.50) and (2.44) are equivalent.)

Remarks:
- In the case that $\eta = \mu = 0$ the result in the Corollary reduces to a classical result corresponding to (2.18); see (2.19). For $\beta = \mu = 0$ we obtain an SODE for which the classical result for mean-square stability is $\alpha > \frac{1}{2}|\eta|^2$.
- Under condition (2.50) the null solution is also "exponentially almost surely stable" [23].

3 A Numerical Method

In this section we attempt to parallel, for the numerical solution, many of the features of our earlier discussion for the exact solution. Numerical methods were discussed (in particular, in terms of convergence) by Baker & Buckwar [2] and we recall the essential features of an Euler-type method[2]. Earlier results on numerical schemes were published by U. Küchler and E. Platen [16] and (in the late 1980's, in Romanian) by C. Tudor and M. Tudor [30, 31, 32]. See also Hu, Mohammed, and Fan [12, submitted for publication] for a rigorous and extensive discussion of schemes of Euler-Maruyama and Milstein-type for stochastic functional differential equations.

3.1 An Euler-type method

In order to define a numerical method for eqn. (2.49), for $\tau > 0$ we choose a step-size $h > 0$ of the form $h = \dfrac{\tau}{N}$ for some $N = N_{\tau,h} \in \mathbb{N}$. (If $\tau = 0$ we set $N = 0$ and take arbitrary $h > 0$. We leave the reader to make the necessary adjustments for this case.) Writing $t_n = t_0 + nh$ for $n \in \mathbb{Z}$, we define a mesh

$$\mathcal{T}_\infty := \{t_0, t_1, t_2, \ldots\}. \tag{3.1a}$$

The initial interval $[t_0 - \tau, t_0]$ is partitioned

$$t_0 - \tau \equiv t_{-N} < t_{-N+1} < \ldots < t_{-1} \equiv t_0 - h < t_0, \tag{3.1b}$$

and it is convenient to introduce the notation

$$\mathcal{J} \equiv \mathcal{J}_{\tau,h} := \{-N, 1-N, \ldots, -1, 0\} \tag{3.1c}$$

for the indices used in (3.1b). It is also convenient to write

$$\mathcal{T}^\infty := \{t_{-N}, t_{1-N}, \ldots, t_{-1}, t_0, t_1, t_2, \ldots\}. \tag{3.1d}$$

The Euler-Maruyama formulae for the problem

$$dX(t) = F(t, X(t), X(t-\tau))\, dt + G(t, X(t), X(t-\tau))\, dW(t) \;\; (t \ge t_0), \tag{3.2a}$$
$$\text{subject to } X(t) = \Phi(t), \;\; (t \in [t_0 - \tau, t_0]), \tag{3.2b}$$

read

$$\widetilde{X}_{n+1} = \widetilde{X}_n + h\, F(t_n, \widetilde{X}_n, \widetilde{X}_{n-N}) + G(t_n, \widetilde{X}_n, \widetilde{X}_{n-N})\sqrt{h}\,\xi_n \;\; (n \in \mathbb{N}), \tag{3.3a}$$
$$\text{subject to } \widetilde{X}_n = X(t_n) = \Phi(t_n) \quad (n \in \mathcal{J}_{\tau,h}), \tag{3.3b}$$

[2] The Euler-Maruyama formulae were initially introduced for stochastic differential equations with no time-lag but the extension to SDDEs is a natural one.

where ξ_n is a $\mathcal{N}(0,1)$ random variable, and ΔW_{n+1} has been replaced by $\sqrt{h}\,\xi_n$. One may indicate the dependence on the initial function by writing $\{\widetilde{X}_n\} \equiv \{\widetilde{X}_n(\Phi)\}$ for the solution of (3.3).

Definition 3.1 *The* error *of the approximation* $\{\widetilde{X}_n\}_{n\geq 0}$ *on the mesh-points* $\mathcal{T}_\infty \cap [t_0, T]$ *is the sequence of* $\mathcal{A}_{t_n}$*-measurable random variables* $\varepsilon_n := X(t_n) - \widetilde{X}_n$ *for* $n = 0, 1, \ldots,$ *with* $t_n \leq T$. *If*

$$\max_{t_n \in [t_0, T]} \left(\mathcal{E}|\varepsilon_n|^2\right)^{\frac{1}{2}} = \mathcal{O}(h^\rho) \quad \textit{(as } h \searrow 0 \textit{ with } \tau/h \in \mathbb{N}\textit{)}$$

then the approximation $\{\widetilde{X}_n\}$ *is* convergent in the mean-square sense, with order ρ to the solution $X(t)$, *on the mesh-points in* $[t_0, T]$.

The following convergence result was established in [2], and is restated here for completeness and ease of reference.

Theorem 3.1 *Under conditions C1 – C5 of Section 2.1, the Euler-Maruyama solution converges to the true solution with order of convergence* $\rho = \min(\gamma, 1/2)$ *in the mean-square sense, on the mesh-points.*

Remark: For the equations under consideration, with multiplicative noise, the above result suffices. However, we observe that Baker and Buckwar [2] also considered the consequences of the condition:

Condition C7: (a) The function G does not depend on X (the SDDE has only *additive noise*) and (b) the function $F(t, \phi, \psi)$ is *decomposable* as $f_1(\phi) + f_2(\psi)$,

in their discussion of (2.8). This Condition identified a subclass of problems (2.8), of the type

$$X(t) = X(t_0) + \int_{t_0}^{t} \{f_1(X(s)) + f_2(X(s-\tau))\}\, ds + g_0 \int_{t_0}^{t} dW(s),$$

for which the convergence result in [2] assumed a strengthened form. With the additional condition, the argument of [2, p. 328] could be refined and the order of convergence in the mean-square sense shown to be $\rho = \min(\gamma, 1)$. Whereas such special circumstances may result in improved rates of convergence, the general result $\rho = \min(\gamma, \frac{1}{2})$ of Theorem 3.1 cannot be bettered without making additional assumptions.

Example 3.1 *For eqn. (2.49) the Euler-Maruyama recurrence reduces to*

$$\widetilde{X}_{n+1} = (1 - \alpha h)\ \widetilde{X}_n + \beta h \widetilde{X}_{n-N} + (\eta\sqrt{h}\,\widetilde{X}_n + \mu\sqrt{h}\,\widetilde{X}_{n-N})\ \xi_n \tag{3.4a}$$

$$\widetilde{X}_n = X(t_n) = \Phi(t_n) \quad (n \in \{-N, 1-N, \ldots, 0\}), \tag{3.4b}$$

where $t_n = t_0 + nh$.

3.2 Numerical stability

We consider the stochastic *difference* equation with finite time-lag, of the form

$$\widetilde{X}_{n+1} - \widetilde{X}_n = hF(t_n, \widetilde{X}_n, \widetilde{X}_{n-N}) \ + \sqrt{h}G(t_n, \widetilde{X}_n, \widetilde{X}_{n-N})\ \xi_n,\ n \in \mathbb{N}, \tag{3.5a}$$

$$\widetilde{X}_n = \Phi(t_n), \quad n \in \mathcal{J}, \tag{3.5b}$$

where $t_n = t_0 + nh$. Since we assume (*cf.* Condition C6) that $F(t_n, 0, 0) = G(t_n, 0, 0) = 0$ for all $n \in \mathbb{N}$, (3.5) admits the null (zero) solution.

Definition 3.1 *For $p > 0$, the null solution of (3.5) is said to be*
(1) locally stable in the p-th mean *if, for each $\varepsilon > 0$, there exists a value $\delta > 0$ such that, whenever $\mathcal{E}(\sup_{n\in\mathcal{J}} |\Phi(t_n)|^p) < \delta$,*

$$\mathcal{E}(|x_n|^p) < \varepsilon \quad n \in \mathbb{N}; \tag{3.6a}$$

(2) locally asymptotically stable in the p-th mean *if it is stable in the p-th mean and if there exists a value $\delta > 0$ such that, whenever $\mathcal{E}(\sup_{n\in\mathcal{J}} |\Phi(t_n)|^p) < \delta$,*

$$\mathcal{E}(|x_n|^p) \to 0 \text{ as } n \to \infty; \tag{3.6b}$$

(3) locally exponentially p-stable *(with* exponent $\nu^\star$*) if it is stable in the p-th mean and if there exist a finite $C > 0$, a value $\nu^\star > 0$, and a value $\delta > 0$ such that, whenever $\mathcal{E}(\sup_{n\in\mathcal{J}} |\Phi(t_n)|^p) < \delta$,*

$$\mathcal{E}(|x_n|^p) \leq C \exp\{-\nu^\star(t_n - t_0)\} \text{ as } n \to \infty. \tag{3.6c}$$

If there is no restriction on the choice of δ, the stability is in each case global.

Remark:
- The above definition readily extends to a class of more general recurrence relations than that provided by the Euler-Maruyama scheme.

3.3 Insight from deterministic equations

As in the case of the SDDE, it is instructive to consider the deterministic case (2.17) for which we obtain the recurrence

$$x_{n+1} - x_n = hf(t_n, x_n, x_{n-N}) \quad (n \geq 0), \tag{3.7a}$$
$$x_n = \varphi(t_0 - nh) \quad (n \in \{-N, 1-N, \ldots, 0\}). \tag{3.7b}$$

Applied to the deterministic test equation (2.18), namely $x'(t) = -\alpha x(t) + \beta x(t-\tau)$, we obtain

$$x_{n+1} = x_n - \alpha h x_n + \beta h x_{n-N} \tag{3.8}$$

for $n = 0, 1, 2, \ldots$. The stability polynomial for the recurrence (3.8) is

$$\mathcal{R}_{N,\tau}(\zeta;\ \alpha, \beta) := \zeta^{N+1} - (1 - \alpha h)\zeta^N - \beta h \quad (Nh = \tau). \tag{3.9}$$

Remark:
- Denote by $\{\zeta_\ell\}_0^{N+1}$ the zeros of $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta)$ and by $\{\widehat{\zeta_\ell}\}$ the zeros of largest modulus. It is well-known that the null solution of the recurrence (3.8) is

  (*a*) globally stable if $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta)$ is *simple von Neumann*: that is, $|\zeta_\ell| \leq 1$ for all $\ell \in \{0, 1, \ldots, N+1\}$, and any $\widehat{\zeta_\ell}$ of modulus unity is simple;

  (*b*) globally asymptotically stable if $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta)$ is a *Schur polynomial*: that is, $|\zeta_\ell| < 1$ for all $\ell \in \{0, 1, \ldots, N+1\}$;

  (*c*) globally exponentially stable if, for some $\nu_\star > 0$, the corresponding polynomial $\mathcal{R}_{N,\tau}\big(\frac{\zeta}{\exp(-\nu_\star)};\ \alpha,\beta\big)$ in the variable ζ is *simple von Neumann*: that is, $|\zeta_\ell| \leq |\exp(-\nu_\star)|$ when $\mathcal{R}_{N,\tau}(\zeta_\ell;\ \alpha,\beta) = 0$, and any $\widehat{\zeta_\ell}$ of modulus $|\exp(-\nu_\star)|$ is simple.

Lemma 3.1 *If $0 < \beta < \alpha$ and $\alpha h < 1$, then (a) the polynomial $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta)$ has a single positive zero $\zeta_h^\star \in (1-\alpha h, 1)$ of the form $\zeta_h^\star = \exp(-\nu_h^\star h)$ where $0 < \nu_h^\star < \alpha$; furthermore, (b) for arbitrary $C > 0$ the sequence of positive and monotonic decreasing values $\{\widehat{v}_n\}_{-N}^\infty$ with*

$$\widehat{v}_\ell = C(\zeta_h^\star)^\ell \tag{3.10}$$

is a solution of the recurrence (3.8), that is $\widehat{v}_{n+1} - \widehat{v}_n = -\alpha h \widehat{v}_n + \beta h \widehat{v}_{n-N}$ $(n \geq 0)$ and, in consequence,

$$\widehat{v}_{n+1} - \widehat{v}_n = -\alpha h \widehat{v}_n + \beta h \sup_{\ell \in \{0,1,\cdots,N\}} \widehat{v}_{n-\ell} \qquad (n \geq 0). \tag{3.11}$$

Proof: Note that $\beta > 0$ and $(1-\alpha h) > 0$. Since $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta) = \zeta^{N+1} - (1-\alpha h)\zeta^N - \beta h$, we have $\mathcal{R}'_{N,\tau}(\zeta;\ \alpha,\beta) = \zeta^{N-1}\ \{(N+1)\zeta - N(1-\alpha h)\}$. We find $\mathcal{R}_{N,\tau}(0;\ \alpha,\beta) = -\beta h < 0$, $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta) < -\beta h$ for $\zeta \in (0, 1-\alpha h)$, and $\mathcal{R}_{N,\tau}((1-\alpha h);\ \alpha,\beta) = -\beta h < 0$. Now $\mathcal{R}_{N,\tau}(1;\ \alpha,\beta) = (\alpha-\beta)h > 0$ and $\mathcal{R}'_{N,\tau}(\zeta;\ \alpha,\beta) > 0$ for $\zeta > \frac{N}{1+N}(1-\alpha h)$ (hence for $\zeta > (1-\alpha h)$). Part (a) follows. Part (b) is established by substitution employing $\mathcal{R}_{N,\tau}(\zeta_h^\star;\ \alpha,\beta) = 0$ and observing the monotonicity of the sequence $\{\widehat{v}_n\}$. □

The estimate $\zeta_h^\star \in (1-\alpha h, 1)$ can be improved. We have $\mathcal{R}_{N,\tau}(1-(\alpha-\beta)h;\ \alpha,\beta) = \{(1-(\alpha-\beta)h)^N - 1\}\beta h < 0$ and hence

$$\zeta_h^\star \in (1-(\alpha-\beta)h, 1) \tag{3.12a}$$

and the lower bound on $\zeta_h^\star$ is $\exp\{-(\alpha-\beta)h\} + \mathcal{O}(h^2)$ as $h \searrow 0$ and $N \to \infty$ (that is, as $h \searrow 0$, α, β, τ being fixed with $Nh = \tau$), *cf.* Lemma 2.1. Indeed, $\zeta_h^\star$ approximates $\exp(-\nu^\star h)$ in the sense that

$$\zeta_h^\star = \exp(-\nu_h^\star h) \text{ with } \nu_h^\star \in (0, \alpha-\beta], \quad \text{when } h \in (0, \frac{1}{\alpha}), \tag{3.12b}$$

where $\nu_h^\star \to \nu^\star > 0$ and $\zeta_h^\star = \exp(-\nu^\star h) + \mathcal{O}(h^2)$, as $h \searrow 0$ and $N \to \infty$.

Remarks:
- The value $\zeta_h^\star$ can be termed the *principal root* of the equation $\mathcal{R}_{N,\tau}(\zeta;\ \alpha,\beta) = 0$.
- The following theorem provides an example of a discrete inequality of Halanay type, an analogue of Theorem 2.1. Other such results have been derived by Baker and Tang (see, for example, [4]).

3.4 A discrete inequality

Theorem 3.1 *Suppose, for some fixed $N \geq 0$, that $\{v_n\}_{-N}^\infty$ is a sequence of positive numbers that satisfies, where*

$$0 < \beta < \alpha \textit{ and } 0 < h < \tfrac{1}{\alpha}, \tag{3.13a}$$

the relation

$$v_{n+1} - v_n \leq -\alpha h v_n + \beta h \max_{\ell \in \{0,1,\ldots,N\}} v_{n-\ell} \textit{ for } n \in \mathbb{N}. \tag{3.13b}$$

Then $v_n \leq \left\{\max_{\ell \in \{-N,1-N,\ldots,0\}} v_\ell\right\} \exp\{-\nu_h^\star(t_n - t_0)\}$ where $\zeta_h^\star$, and $\nu_h^\star > 0$ are the values occurring in Lemma 3.1(a) and in (3.12).

Proof: With $\mathcal{J} := \{-N, 1-N, \ldots, 0\}$, take $C = \max_{\ell\in\mathcal{J}} v_\ell$, and define $\widehat{v}_\ell$ as in Lemma 3.1(b). Then $v_\ell \le \widehat{v}_\ell$ for $\ell \in \mathcal{J}$. Our theorem is established if we show that $v_\ell \le \widehat{v}_\ell$ for all $\ell \in \mathbb{N}$. Suppose, in contradiction, that there exists a least integer $m \ge 0$ such that $v_{m+1} - \widehat{v}_{m+1} > 0$. Then $\max_{\ell\in\mathcal{J}} v_{m+\ell} \le \max_{\ell\in\mathcal{J}} \widehat{v}_{m+\ell}$ and (3.11) and (3.13b) yield

$$\begin{aligned} v_{m+1} &\le (1-\alpha h)v_m + \beta h \max_{\ell\in\mathcal{J}} v_{m+\ell} \\ &\le (1-\alpha h)\widehat{v}_m + \beta h \max_{\ell\in\mathcal{J}} \widehat{v}_{m+\ell} = \widehat{v}_{m+1} \end{aligned} \tag{3.14}$$

so that $v_{m+1} \le \widehat{v}_{m+1}$, which contradicts our assumption. The result $v_n \le \{\max_{\ell\in\{-N,1-N,\ldots,0\}} v_\ell\}(\zeta_h^\star)^n$ (and hence the theorem) thus follows since we already have an explicit expression for $\widehat{v}_n$. □

3.5 A Lyapunov-type theorem

Theorem 3.2 *With $\tau > 0$, and $h = \tau/N > 0$ where $N \in \mathbb{N}$, suppose $\{\widetilde{X}_n\} \equiv \{\widetilde{X}_n(\Phi)\}$ is a solution of (3.3). Assume that there exists a positive-valued function $V(t,x)$ (for $t \in \mathcal{T}^\infty$ and $x \in \mathbb{R}$) such that (a) there exist positive constants c_1, c_2, and $p > 1$, for which*

$$c_1\ |x|^p \le V(t_n, x)\ \le\ c_2|x|^p \quad (t_n \in \mathcal{T}^\infty,\ \ x \in \mathbb{R}) \tag{3.15a}$$

and (b) such that, for

$$0 < \beta < \alpha, \text{ where } \alpha h < 1, \tag{3.15b}$$

$$\begin{aligned} &\mathcal{E}(V(t_{n+1}, \widetilde{X}_{n+1})) - \mathcal{E}(V(t_n, \widetilde{X}_n)) \\ &\qquad\le -\alpha h \mathcal{E}(V(t_n, \widetilde{X}_n)) + \beta h \mathcal{E}(V(t_{n-N}, \widetilde{X}_{n-N})), \end{aligned} \tag{3.15c}$$

for all $n \in \mathbb{N}$. Then

$$\mathcal{E}(|\widetilde{X}_n|^p) \le \frac{c_2}{c_1}\{\mathcal{E}(\max_{\ell\in\mathcal{J}} |\Phi(t_\ell)|^p)\} \exp\{-\nu_h^\star(t_n - t_0)\}, \tag{3.16}$$

for all $n \in \mathbb{N}$, where $\nu_h^\star$ is given by (3.12b) in terms of α and β. Hence, the null solution of (3.3) is exponentially stable.

Proof: If we define $v_n = \mathcal{E}(V(t_n, \widetilde{X}_n)$ then condition (3.15c) yields

$$v_{n+1} - v_n\ \le\ -\alpha v_n + \beta \max_{\ell\in\mathcal{J}} v_{n+\ell} \tag{3.17}$$

(recall that $\mathcal{J} := \{-N, 1-N, \ldots, -1, 0\}$), and we can use Theorem 3.1 to obtain the exponentially decreasing bound on v_n:

$$v_n \le \{\max_{\ell\in\mathcal{J}} v_\ell)\}\ \exp\{-\nu^\star(t_n - t_0)\} \quad \text{for } t_0 \le t < \infty. \tag{3.18}$$

However, by (3.15a),

$$\max_{\ell\in\mathcal{J}} v(t_\ell) \equiv \max_{\ell\in\mathcal{J}} \sup_{\varsigma\in(0,1]} \mathcal{E}(V(t_{\ell+\varsigma}, \Phi(t_\ell))) \le c_2 \max_{\ell\in\mathcal{J}} \mathcal{E}(|\Phi(t_\ell)|^p) \le c_2 \mathcal{E}(\max_{\ell\in\mathcal{J}} |\Phi(t_\ell)|^p). \tag{3.19a}$$

Again by (3.15a),

$$\mathcal{E}(|\widetilde{X}(t_n)|^p)\ \le\ \frac{1}{c_1} v_n, \tag{3.19b}$$

for $c_1 \ne 0$, since $v_n = \mathcal{E}(V(t_n, \widetilde{X}(t_n))$. Thus, from (3.18) and (3.19), we obtain the desired result (3.16). By Definition 3.1, this implies the exponential stability of the null solution. □

3.6 A worked example

Consider the example, appearing in (3.4),

$$\widetilde{X}_{n+1} = (1-\alpha h)\ \widetilde{X}_n + \beta h \widetilde{X}_{n-N} + (\eta\sqrt{h}\,\widetilde{X}_n + \mu\sqrt{h}\,\widetilde{X}_{n-N})\ \xi_n. \tag{3.20}$$

The deterministic recurrence of which this is a generalization reads

$$\widetilde{x}_{n+1} = (1-\alpha h)\ \widetilde{x}_n + \beta h \widetilde{x}_{n-N}. \tag{3.21}$$

In the stochastic case, the analysis depends upon our choice of p and we will here take $p = 2$ (the mean-square case). We can write the recurrence (3.20) in the form

$$\widetilde{X}_{n+1} - \widetilde{X}_n = \Delta_n; \quad \Delta_n := (a' + a''\xi_n)\widetilde{X}_n + (b' + b''\xi_n)\widetilde{X}_{n-N}, \tag{3.22a}$$

where for ease of notation we set

$$a' = -\alpha h,\ \ a'' = \eta\sqrt{h}; b' = \beta h,\ \ b'' = \mu\sqrt{h}. \tag{3.22b}$$

Now $\widetilde{X}_{n+1}^2 - \widetilde{X}_n^2 = \Delta_n(\widetilde{X}_{n+1} + \widetilde{X}_n) = \Delta_n(2\widetilde{X}_n + \Delta_n)$ whence $\widetilde{X}_{n+1}^2 - \widetilde{X}_n^2 = \{(a' + a''\xi_n)\widetilde{X}_n + (b' + b''\xi_n)\widetilde{X}_{n-N}\} \times \{(2 + a' + a''\xi_n)\widetilde{X}_n + (b' + b''\xi_n)\widetilde{X}_{n-N}\}$. Introducing appropriate real numbers $A_i = A_i(\alpha, \beta, \eta, \mu, h)$, $B_i = B_i(\alpha, \beta, \eta, \mu, h)$, $C_i = C_i(\alpha, \beta, \eta, \mu, h)$, this is of the form

$$\begin{aligned}\widetilde{X}_{n+1}^2 - \widetilde{X}_n^2 = \{A_0 + A_1\xi_n + A_2\xi_n^2\}\widetilde{X}_n^2 + \ \{B_0 + B_1\xi_n + B_2\xi_n^2\}\widetilde{X}_n\widetilde{X}_{n-N} \\ + \{C_0 + C_1\xi_n + C_2\xi_n^2\}\widetilde{X}_{n-N}^2.\end{aligned}$$

Hence, taking account of the fact that ξ_n is a standard normal random variable[3] and ξ_n and $\widetilde{X}_r$ are independent for $r \le n$, we have, if $-N \le r, s$ $(r, s \in \mathbb{N})$, and if

$$r, s \le n,$$

$$\mathcal{E}(\xi_n \widetilde{X}_r\widetilde{X}_s) = \mathcal{E}(\xi_n)\mathcal{E}(\widetilde{X}_r\widetilde{X}_s) = 0, \text{ and } \mathcal{E}(\xi_n^2 \widetilde{X}_r\widetilde{X}_s) = \mathcal{E}(\xi_n^2)\mathcal{E}(\widetilde{X}_r\widetilde{X}_s) = \mathcal{E}(\widetilde{X}_r\widetilde{X}_s).$$

We therefore obtain

$$\begin{aligned}\mathcal{E}(\widetilde{X}_{n+1}^2) - \mathcal{E}(\widetilde{X}_n^2) = \{A_0 + A_2\}\ \mathcal{E}(\widetilde{X}_n^2) + \ \{B_0 + B_2\}\mathcal{E}(\widetilde{X}_n\widetilde{X}_{n-N}) \\ + \{C_0 + C_2\}\mathcal{E}(\widetilde{X}_{n-N}^2).\end{aligned} \tag{3.23}$$

To this point, the analysis is exact; but we now introduce bounds (and therefore have some choice). For example, we have $\mathcal{E}(\widetilde{X}_n\widetilde{X}_{n-N}) \le \frac{1}{2}\{\mathcal{E}(\widetilde{X}_n^2) + \mathcal{E}(\widetilde{X}_{n-N}^2)\}$ and hence

$$\begin{aligned}&\mathcal{E}(\widetilde{X}_{n+1}^2) - \mathcal{E}(\widetilde{X}_n^2) \le \\ &\quad \{A_0 + A_2 + \frac{1}{2}\{B_0 + B_2\}\}\ \mathcal{E}(\widetilde{X}_n^2) + \ \{C_0 + C_2 + \frac{1}{2}\{B_0 + B_2\}\}\mathcal{E}(\widetilde{X}_{n-N}^2).\end{aligned} \tag{3.24}$$

Evaluating $A_i = A_i(\alpha, \beta, \eta, \mu, h)$, $B_i = B_i(\alpha, \beta, \eta, \mu, h)$, and $C_i = C_i(\alpha, \beta, \eta, \mu, h)$, for $i = 0$ and $i = 2$ we obtain (on setting $v_n = \mathcal{E}(\widetilde{X}_n^2)$, and bounding v_{n-N} by $\max_{r\in\{0,1,\cdots,N\}} v_{n-r}$) a result

$$v_{n+1} - v_n \le -\alpha_h v_n + \beta_h \max_{\ell\in\{0,1,\cdots,N\}} v_{n-\ell}, \tag{3.25}$$

with

[3] That is, an $N(0,1)$ random variable, with zero mean and unit variance.

$$\alpha_h = \{A_0 + A_2 + \frac{1}{2}\{B_0 + B_2\} = \alpha h + (\alpha h - \beta h)(1 - \alpha h) - (\eta^2 + \eta\mu)h \quad (3.26a)$$

$$\beta_h = \{C_0 + C_2 + \frac{1}{2}\{B_0 + B_2\}\} = \beta h\{1 - (\alpha - \beta)h\} + (\mu^2 + \eta\mu)h, \quad (3.26b)$$

to which Theorem 3.1 may be applied if we have

$$0 < \beta_h < \alpha_h < 1. \quad (3.26c)$$

Now if $0 < \beta < \alpha$ and $0 < \alpha h < 1$ we have $0 < \alpha_h < 1$ and $0 < \beta_h < \alpha_h$ provided, respectively $0 < \alpha h + (\alpha h - \beta h)(1 - \alpha h) - (\eta^2 + \eta\mu)h < 1$ and $0 < \beta h\{1-(\alpha-\beta)h\} + (\mu^2 + \eta\mu)h < \alpha h + (\alpha h - \beta h)(1-\alpha h) - (\eta^2 + \eta\mu)h$. Rearranging, we require

$$\frac{(\eta^2 + \eta\mu) - \alpha}{1 - \alpha h} h < (\alpha - \beta)h \ < 1 + \frac{(\eta^2 + \eta\mu)h}{1 - \alpha h} \quad (3.27a)$$

$$\frac{(\mu^2 + \eta\mu)}{1 - \beta h} h < (\alpha - \beta)h \ < 1 + \frac{(\mu^2 + \eta\mu)}{\beta} \quad (3.27b)$$

Our conditions already assume that $0 < \beta h < \alpha h < 1$ and (3.27) provide additional bounds (i) on h in terms of the noise (which are stricter for smaller $\alpha - \beta$) and (ii) on the noise (which are stricter for smaller $\alpha - \beta$ and for βh closer to 1). If we introduce a scaling factor and write $\eta = \varepsilon\eta_0$, $\mu = \varepsilon\mu_0$, in order to regard ε as a measure of the size of the noise, we see that the above inequalities indicate a dependency on ε^2.

4 Some extensions

Much remains to be explored and we give here some indication of possible extensions that have attracted our attention for future study. We note again that results for SODEs can be obtained as a special case of the results for SDDEs.

4.1 Use of the continuous Halanay-type inequality

The preceding stability results for the discretized scheme (in section 3.2 *et seq.*) relied upon a discrete Halanay-type inequality. Our objective in this subsection is to indicate that if we define a process $\widetilde{X}(t)$ for all $t \geq t_0$ (not merely for $t \in \mathcal{T}^\infty$) in terms of the sequence $\{\widetilde{X}_\ell\}$, there may be an opportunity to employ this to derive a relation for $\mathcal{E}(V(t, \widetilde{X}(t)))$ to which the continuous Halanay-type inequality can be applied. We have

$$\widetilde{X}_{n+1} = \widetilde{X}_n + h\ F(t_n, \widetilde{X}_n, \widetilde{X}_{n-N}) + G(t_n, \widetilde{X}_n, \widetilde{X}_{n-N})\sqrt{h}\,\xi_n, \ (n \in \mathbb{N}), \quad (4.1a)$$

$$\widetilde{X}_n = X(t_n) = \Phi(t_n), \quad (n \in \mathcal{J}_{\tau,h}\ ), \quad (4.1b)$$

and a necessary step is to introduce a *continuous extension* $\widetilde{X}(t)$ of the sequence $\{\widetilde{X}_\ell\}$, here achieved by defining

$$\widetilde{X}(t_{n+\varsigma}) := \widetilde{X}_n + \varsigma h\ F(t_n, \widetilde{X}_n, \widetilde{X}_{n-N}) + \sqrt{\varsigma h} G(t_n, \widetilde{X}_n, \widetilde{X}_{n-N})\,\xi_n \quad (4.2a)$$
$$(\text{for } n \in \mathbb{N}, \varsigma \in [0, 1]),$$

$$\widetilde{X}(t) = X(t) = \Phi(t) \quad (t \in J\ ). \quad (4.2b)$$

Given a solution $\{\widetilde{X}_n\}$ of the discrete recurrence (4.1a), define (recalling that $\tau = Nh$)

$$\widetilde{F}_n := F(t_n, \widetilde{X}_n, \widetilde{X}_{n-N}), \quad \widetilde{G}_n := G(t_n, \widetilde{X}_n, \widetilde{X}_{n-N}). \quad (4.3)$$

(The shorthand notation (4.3) is used in the following.) Now (4.2a) implies that

$$d\widetilde{X}(t_n+\varsigma h) = \widetilde{F}_n\,dt + \widetilde{G}_n\,dW(t) \text{ (for } n\in\mathbb{N} \text{ and } \varsigma\in[0,1)). \tag{4.4}$$

Theorem 4.1 *Assume that there exists a positive, continuous function $V(t,x)$ (with continuous derivatives $V_t(t,x), V_x(t,x)$, and $V_{xx}(t,x)$), for $t\geq t_0-\tau$ and $x\in\mathbb{R}$, and that there exist positive constants c_1, c_2, and $p>1$, such that*

$$c_1\,|x|^p \;\leq\; V(t,x) \;\leq\; c_2\,|x|^p, \tag{4.5}$$

when $t\geq t_0-\tau$ and $x\in\mathbb{R}$. Suppose further that, for $0<\widetilde{\beta}(h)<\widetilde{\alpha}(h)$,

$$\mathcal{E}\left(\; V_t(s,\widetilde{X}(s)) + V_x(s,\widetilde{X}(s))\;\widetilde{F}_n + \frac{1}{2}V_{xx}(s,\widetilde{X}(s))\;\widetilde{G}_n^2\;\right)$$
$$\leq -\widetilde{\alpha}(h)\;\mathcal{E}V(t,\widetilde{X}(t)) + \widetilde{\beta}(h)\;\mathcal{E}V(t-\tau,\widetilde{X}(t-\tau)) \tag{4.6}$$

(for $n\geq 0$) whenever $t\in[t_n,t_{n+1})$. Then

$$\mathcal{E}(|\widetilde{X}(t;t_0,\Phi)|^p) \leq \frac{c_2}{c_1}\mathcal{E}(\sup_{s\in[t_0-\tau,t_0]}|\Phi(s)|^p)\;\exp(-\widetilde{\nu}^\star(h)(t-t_0)), \tag{4.7}$$

with $\widetilde{\nu}^\star(h)$ given in terms of $\widetilde{\alpha}(h)$, $\widetilde{\beta}(h)$ by (2.21)

$$-\widetilde{\nu}^\star(h) = \widetilde{\nu}_\star(h) - \alpha \text{ where } \widetilde{\nu}_\star(h)\exp\{\widetilde{\nu}_\star(h)\tau\} = \widetilde{\beta}(h)\exp\{\widetilde{\alpha}(h)\tau\} \tag{4.8}$$

(so that $0<\widetilde{\nu}_\star(h)\in(\widetilde{\beta}(h),\widetilde{\alpha}(h))$), and the null solution of eqn. (4.2) is therefore globally exponentially stable in the p-th mean.

The close affinity between the condition (4.6) and (2.40) will be apparent.

Proof: By the integral form of the Itô formula (2.34) we obtain for $t\geq t_0$, and when

$$\delta\geq 0 \text{ and } t, t+\delta\in[t_n,t_{n+1}], \tag{4.9a}$$

$$\begin{aligned} &V(t+\delta,\widetilde{X}(t+\delta)) - V(t,\widetilde{X}(t)) \\ &= \int_t^{t+\delta} V_t(s,\widetilde{X}(s)) + V_x(s,\widetilde{X}(s))\;\widetilde{F}_n + \frac{1}{2}\int_t^{t+\delta} V_{xx}(s,\widetilde{X}(s))\;\widetilde{G}_n^2\;ds \\ &\qquad + \int_t^{t+\delta} V_x(s,\widetilde{X}(s))\;\widetilde{G}_n\;dW(s). \end{aligned} \tag{4.9b}$$

Since $\mathcal{E}(\int_t^{t+\delta} V_x(s,\widetilde{X}(s))\;\widetilde{G}_n\;dW(s)) = 0$, taking expectations yields, for $t\geq t_0$, with (for some $n\in\mathbb{N}$) $t,t+\delta\in[t_n,t_{n+1}]$ and $\delta\geq 0$,

$$\begin{aligned} &\mathcal{E}V(t+\delta,\widetilde{X}(t+\delta)) - \mathcal{E}V(t,\widetilde{X}(t)) \\ &= \mathcal{E}(\int_t^{t+\delta} V_t(s,\widetilde{X}(s)) + V_x(s,\widetilde{X}(s))\;\widetilde{F}_n + \frac{1}{2}V_{xx}(s,\widetilde{X}(s))\;\widetilde{G}_n^2\;ds) \end{aligned}$$

for the given n, and thus (by our assumption (4.6))

$$\begin{aligned} &\mathcal{E}(V(t+\delta,\widetilde{X}(t+\delta)) - \mathcal{E}(V(t,\widetilde{X}(t))) \\ &\leq \Big(\int_t^{t+\delta}\{-\widetilde{\alpha}(h)\;\mathcal{E}(V(s,\widetilde{X}(s))) + \widetilde{\beta}(h)\;\mathcal{E}(V(s-\tau,\widetilde{X}(s-\tau)))\;ds\}\Big). \end{aligned}$$

Since the Dini derivative $D^+ v(t)$ is

$$D^+\mathcal{E}(V(t,\widetilde{X}(t))) := \limsup_{\delta\searrow 0} \frac{\mathcal{E}(V(t+\delta,\widetilde{X}(t+\delta)) - \mathcal{E}(V(t,\widetilde{X}(t)))}{\delta}$$

the preceding result leads directly to

$$D^+\mathcal{E}(V(t,\widetilde{X}(t))) \le -\widetilde{\alpha}(h)\ \mathcal{E}(V(t,\widetilde{X}(t))) + \widetilde{\beta}(h)\ \mathcal{E}(V(t-\tau,\widetilde{X}(t-\tau))),$$

for $t \ge t_0$. Clearly, Theorem 2.3 now applies and if we invoke (4.5) we obtain (4.7) following arguments similar to those yielding (2.36). □

The verification of (4.6) is in any application a technical exercise but it will be transparent that since $\widetilde{X}(t)$ depends upon h so does verification of (4.6). Indeed, the condition (2.39) (which was employed in Lemma 2.4) reads

$$\begin{aligned} V_t(t,x) + V_x(t,x)\ F(t,x,y) + \frac{1}{2}V_{xx}(t,x)G^2(t,x,y) \qquad\qquad (4.10)\\ \le -\alpha V(t,x) + \beta V(t-\tau,y),\ \text{ for } 0<\beta<\alpha, \end{aligned}$$

when $t \ge t_0$ and $x, y \in \mathbb{R}$, and we can readily deduce from (2.39) the result

$$\mathcal{E}\big(V_t(t,\widetilde{X}(t)) + V_x(t,\widetilde{X}(t))\ F(t,\widetilde{X}(t),\widetilde{X}(t-\tau)) + \frac{1}{2}V_{xx}(t,\widetilde{X}(t))G^2(t,\widetilde{X}(t),\widetilde{X}(t-\tau))\big)$$

$$\le -\alpha\mathcal{E}V(t,\widetilde{X}(t)) + \beta\mathcal{E}V(t-\tau,\widetilde{X}(t-\tau)). \qquad (4.11)$$

In comparison, to apply the Theorem 4.1 we need to verify condition (4.6), namely that for $t \in [t_n, t_{n+1})$

$$\mathcal{E}\big(V_t(t,\widetilde{X}(t)) + V_x(t,\widetilde{X}(t))\ F(t_n,\widetilde{X}(t_n),\widetilde{X}(t_n-\tau)) + \frac{1}{2}V_{xx}(t,\widetilde{X}(t))G^2(t_n,\widetilde{X}(t_n),\widetilde{X}(t_n-\tau))\big)$$

$$\le -\widetilde{\alpha}(h)\ \mathcal{E}V(t,\widetilde{X}(t)) + \widetilde{\beta}(h)\ \mathcal{E}V(t-\tau,\widetilde{X}(t-\tau)) \qquad (4.12)$$

4.2 Lag-dependent results

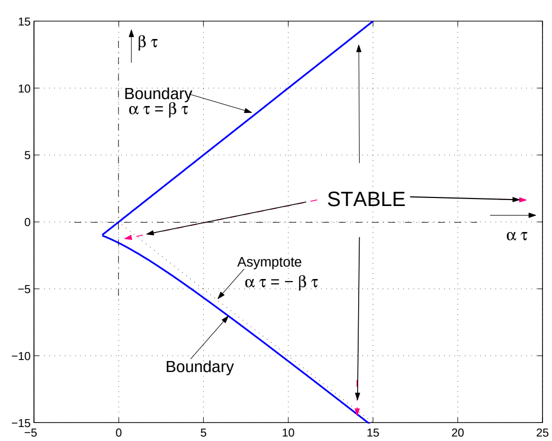

Figure 4.1: Stability region for the true solutions of $x'(t) = -\alpha x(t) + \beta x(t-\tau)$

For the deterministic equation $x'(t) = -\alpha x(t) + \beta x(t-\tau)$ (with, as usual, $\tau > 0$) we know that the null solution is stable whenever $|\beta| \leq \alpha$. The solution is asymptotically stable (and indeed[4] exponentially stable) when $|\beta| < \alpha$, irrespective of $\tau > 0$. However, the stated conditions are sufficient but not necessary for stability (*cf.* [1, 8] and their references). For the parameters $(\alpha_\tau, \beta_\tau) = (\alpha\tau, \beta\tau)$, the region of parameter space which exactly corresponds to stability is that containing the region $|\beta_\tau| \leq \alpha_\tau$ which is bounded by the line $\alpha_\tau = -\beta_\tau$ and the curve

$$\alpha_\tau = -s\ \cot(s), \quad \beta_\tau = -s/\sin(s), \quad s \in (0, \pi), \tag{4.13}$$

and the region of stability therefore includes (see Figure 4.2) the line segment $\alpha_\tau = 0$, $-\frac{1}{2}\pi \leq \beta_\tau \leq 0$, *i.e.*.

$$\alpha = 0, \qquad \beta \in (-\frac{\pi}{2\,\tau}, 0).$$

Thus, even for a simple linear equation, the complete stability analysis requires that we take account of the lag τ as well as the coefficients α, β. This is illustrative of results that are said to be *lag-dependent* (in contrast to the lag-independent result that $|\beta| < \alpha$ ensures asymptotic stability); we expect to be able to find lag-dependent results for SDDEs as much as for their deterministic counterparts. We draw attention to the lag-dependent result of Mao [23, p.180], as a complementary result to that provided in Theorem 2.4.

4.3 The restriction $\alpha h < 1$

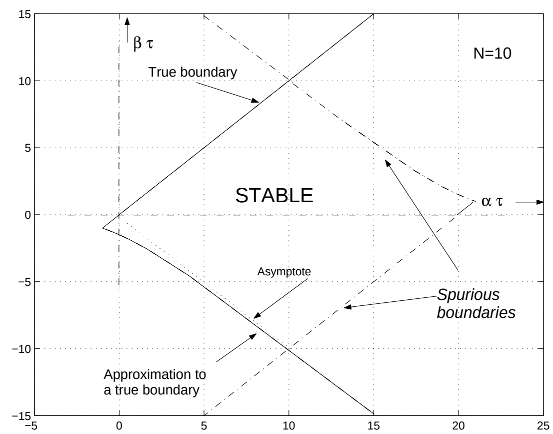

Figure 4.2: Stability region for the numerical solutions of $x'(t) = -\alpha x(t) + \beta x(t-\tau)$

In our discussion of the Euler-Maruyama method with condition $0 < \beta < \alpha$ we asked that $\alpha h < 1$. The reason for this restriction can be seen if one examines the stability region for Euler's method applied to the deterministic equation $x'(t) = -\alpha x(t) + \beta x(t-\tau)$ (with $0 < \tau = Nh$ and $N \in \mathbb{N}$). The stability polynomial for the recurrence $x_{n+1} = (1-\alpha h)x_n + \beta h x_{n-N}$ has the form $\zeta^N\{\zeta - (1 - \frac{\alpha\tau}{N})\} - \frac{\beta\tau}{N}$. The boundary locus technique for obtaining, given N, the parameters $\alpha\tau$ and $\beta\tau$ that correspond to stability involves plotting the loci corresponding to $|\zeta| = 1$; equivalently, finding $\alpha\tau$ and $\beta\tau$ such that (setting $\zeta = \exp(\mathrm{i}\,\theta)$)

$$\{\exp(\mathrm{i}\,\theta) - (1 - \frac{\alpha\tau}{\mathrm{N}})\} = \frac{\beta\tau}{\mathrm{N}}\exp(-\mathrm{i}\,\mathrm{N}\theta) \tag{4.14a}$$

[4] The *rate constant* of the exponential stability then depends upon α, β *and* τ.

or, (setting $\zeta = -\exp(\mathrm{i}\,\vartheta)$, where $\vartheta = \theta + \pi$),

$$\{\exp(\mathrm{i}\,\vartheta) + (1 - \frac{\alpha\tau}{\mathrm{N}})\} = (-1)^{\mathrm{N}}\frac{\beta\tau}{\mathrm{N}}\exp(-\mathrm{i}\,\mathrm{N}\vartheta). \tag{4.14b}$$

We have given the two equivalent formulations because they make it transparent[5] that, by symmetry, every section of the boundary in the half plane with $\alpha\tau/N < 1$ has a corresponding section in the half plane with $\alpha\tau/N > 1$ (its location $\beta\tau > 0$ or $\beta\tau < 0$ depending on the parity of N). The loci are obtained by equating real and imaginary parts; for example, with $h = \frac{\tau}{N}$,

$$1 - \alpha h = \cos(\theta) + \cot(N\theta)\sin(\theta), \quad \beta h = -\frac{\sin(\theta)}{\sin(N\theta)} \tag{4.15a}$$

from (4.14a), when $\sin(N\theta) \neq 0$, and equally (from (4.14b)),

$$\alpha h - 1 = \cos(\vartheta) + \cot(N\vartheta)\sin(\vartheta), \quad \beta h = (-1)^{N+1}\frac{\sin\vartheta}{\sin(N\vartheta)} \tag{4.15b}$$

when $\sin(N\vartheta) \neq 0$. The region $\beta\tau < \alpha\tau$ is restricted (i) for $(-1)^N\beta < 0$ by the straight line $\alpha\tau = \beta\tau + 2N$ that passes through the point $(2\frac{\tau}{h}, 0)$ and which is *spurious*[6] in the sense that it is due to the discretization (it is the symmetrical counterpart corresponding to $\zeta = -1$ of the locus $\alpha\tau = \beta\tau$ that corresponds to $\zeta = 1$) (ii) for $(-1)^N\beta > 0$, by another spurious boundary[7] (see Figure 4.3) that also restricts the region. What limits the results available for Euler's method is that the region $|\beta| < \alpha$, which is a subset of the true stability region, is curtailed by the spurious boundaries; with Euler's method, stability occurs for $\alpha = \beta$ only if $\alpha\tau < N$, that is, $\alpha h < 1$. Of course, the restriction on h in the Euler method is less important for large N (for small h). The results carry over as a stability limitation[8] for the Euler-Maruyama technique but, of course, the low rate of convergence in the stochastic case requires the stepsize to be taken small in order to obtain accurate results.

4.4 Stepsizes incommensurable with the lag τ

The analysis of the Euler-Maruyama method was based upon the assumption that the stepsize was selected so that $\tau = Nh$ for some $N \in \mathbb{N}$. If this restriction is dropped, the numerical method must be re-defined, and it is then natural to introduce for the problem

$$dX(t) = F(t, X(t), X(t-\tau))dt + G(t, X(t), X(t-\tau))dW(t), \; (t \geq t_0),$$
$$\text{subject to } X(t) = \Phi(t), \; (t \in [t_0 - \tau, t_0]),$$

the *densely-defined* approximation $\widetilde{X}(t)$ satisfying

$$\widetilde{X}(t) = \widetilde{X}(t_n) + (t - t_n)\; F(t_n, \widetilde{X}(t_n), \widetilde{X}(t_n - \tau)) + G(t_n, \widetilde{X}(t_n), \widetilde{X}(t_n - \tau))\{W(t) - W(t_n)\}, \tag{4.17}$$

for $t_n \leq t \leq t_{n+1}$ $(n \geq 0)$ with $t_n = t_0 + nh$, subject to $\widetilde{X}(t) = \Phi(t)$, for $t \in J$. We now write $\xi_n\sqrt{t - t_n}$ in place of $\{W(t) - W(t_n)\}$.

[5] The situation is illustrated for $N = 10$ by our Figure 4.3.

[6] This 'spurious' line might be thought of as approximating a line at infinity as $h \searrow 0$.

[7] This is the symmetrical counterpart of the approximation to the true 'lower' boundary having, see Figure 4.2, the line $\alpha\tau = -\beta\tau$ as asymptote.

[8] It should be possible to obtain stability results for $\alpha h < 2$, rather than $\alpha h < 1$, if one amends the discussion and places additional restrictions on β.

4.5 Generalizations of the basic equation

The extension of the Euler method indicated above also allows the definition of an Euler method for equations with variable lag:

$$dX(t) = F(t, X(t), X(t-\tau(t)))dt + G(t, X(t), X(t-\tau(t)))dW(t) \ (t \geq t_0),$$
$$\text{subject to } X(t) = \Phi(t), \ (t \in [t_0 - \sup_{t \geq t_0} \tau(t), t_0]),$$

where $\tau(t) > 0$. Then $\widetilde{X}(t)$ satisfies

$$\widetilde{X}(t) = \widetilde{X}(t_n) + (t - t_n)\ F(t_n, \widetilde{X}(t_n), \widetilde{X}(t_n - \tau(t_n))) + G(t_n, \widetilde{X}(t_n), \widetilde{X}(t_n - \tau(t_n)))\{W(t) - W(t_n)\}, \tag{4.19}$$

for $t_n \leq t \leq t_{n+1}$ $(n \geq 0)$. The generalized (stochastic) pantograph equation [3] provides an example of variable lag.

Other variations on the equation with one fixed lag present themselves, as suitable for further study, and Baker, Buckwar & Ford are currently examining such problems. Examples are:

- **Multiple lag equations**

$$dX(t) = \mathcal{F}(t, X(t), X(t-\tau_1), \ldots, X(t-\tau_M))\, dt + \mathcal{G}(t, X(t), X(t-\tau_1), \ldots, X(t-\tau_M))\, dW(t) \quad (t \geq t_0), \tag{4.20a}$$

  subject to the initial condition (2.1b) and with $\tau := \max_\ell \tau_\ell$;

- **Multiple dW_ℓ**

$$dX(t) = F(t, X(t), X(t-\tau))\, dt + \sum_\ell G_\ell(t, X(t), X(t-\tau))\, dW_\ell(t) \quad (t \geq t_0), \tag{4.20b}$$

  subject to the initial condition (2.1b) — see [16];

- **Variable lag equations with more than one lag**

$$dX(t) = F(t, X(t), X)(t - \tau_1(t))\, dt + G(t, X(t), X(t-\tau_2(t)))\, dW(t) \quad (\tau_{1,2}(t) \geq 0). \tag{4.20c}$$

  Generalizing the previous two examples we have

$$dX(t) = \sum_{\ell=0}^{M} F_\ell(t, X(t), X(t-\tau_\ell(t)))\ dt + \sum_{\ell=0}^{M} G_\ell(t, X(t), X(t-\tau_\ell(t)))\ dW_\ell(t) \quad (t \geq t_0), \tag{4.20d}$$

  where $X(t) = \Phi(t)$ for $t \in J := [t_0 - \tau_\star, t_0]$, $\tau_\ell(t) \geq 0$ for $\ell \in \{0, 1, \ldots, M\}$ and where $\tau_\star = \max_\ell \sup_{t \geq t_0} \tau_\ell(t)$ (where $\tau_\star$ is assumed to be finite);

- **Pure delay equations**

$$dX(t) = \mathfrak{F}(t, X(t-\tau))\, dt + \mathfrak{G}(t, X(t-\tau))\, dW(t) \quad (t \geq t_0), \tag{4.21}$$

  subject to the initial condition (2.1b). We observe (assuming the existence of the derivative $\mathfrak{F}_x$ and invoking continuity of $X(\cdot)$) that

$$dX(t) = \Big\{\mathfrak{F}(t, X(t)) + \{X(t-\tau)) - X(t)\}\mathfrak{F}_x(t, X(t - \tau_\star(t, X(t))))\Big\}dt + \mathfrak{G}(t, X(t-\tau))\, dW(t),$$

with $\tau_\star(t, X(t)) \in [0, \tau]$, (for $t \geq t_0$). The latter equation has an unknown state-dependent lag but since $\tau_\star(t, X(t)) \in [0, \tau]$, some progress may be made with the theory.

4.6 Approximating equations

The theorems presented here relate to the effect of perturbations in the initial data. The effect of perturbations in the equation itself (persistent perturbations) remains to be explored.

In deterministic problems, it is commonplace to employ linear test equations

$$x'(t) = -\alpha x(t) + \beta x(t - \tau) \tag{4.22}$$

as a focus in the discussion of the stability of the null solution of a nonlinear equation

$$x'(t) = f(t, x(t), x(t - \tau)) \quad (t \geq t_0), \tag{4.23}$$

$$x(t) = \varphi(t) \quad (t \in [t_0 - \tau, t_0]). \tag{4.24}$$

A justification for this can be found when (for example)

$$\left\| f(t, u(t), u(t - \tau)) + \alpha u(t) - \beta u(t - \tau) \right\| \tag{4.25}$$

is, in an appropriate sense, uniformly small for all $\sup_{s \in [0,1]} |u(t - s\tau)|$ that are sufficiently small and all sufficiently large t (α, β not simultaneously vanishing). The linear theory has to be extended if

$$f(t, u(t), u(t - \tau)) \sim -\alpha\{u(t)\}^p + \beta\{u(t - \tau)\}^q, \quad p, q > 1,$$

for small $\sup_{s \in [0,1]} |u(t - s\tau)|$. Similar results for SDDEs are currently under investigation by the authors, and it is hoped will be published elsewhere by Baker & Buckwar, in due course. Note that Kolmanovskii and Nosov [17] comment on a type of "first approximation" for SDDEs in which the noise is neglected to yield a deterministic equation as the approximating equation. There is more to study here.

4.7 Lyapunov functionals versus Lyapunov functions

The use of Lyapunov functionals in the stability analysis of DDEs gave rise to a major theoretical advance. Returning to deterministic problems to gain ready insight, we set, for $t \geq t_0$,

$$z(t) := |x(t)|^2 + \alpha \int_{t-\tau}^{t} |x(s)|^2 ds$$

for $\alpha > 0$ when $x \in C[t_0 - \tau, \infty)$. Now z is an example of a Lyapunov *functional* (there is no underlying function defined on $\mathbb{R}$, similar to (2.22a) or (2.24)); previously we have dealt with Lyapunov *functions*. Here we have

$$|x(t)|^2 \leq z(t)$$

(and, indeed, $z(t) \leq \{1 + \alpha\} \sup_{s \in [0,\tau]} |x(t - s)|^2$) and, if $x(t)$ is a solution of (2.18), $z'_+(t) = 2x(t)x'_+(t) + \alpha\{|x(t)|^2 - |x(t - \tau)|^2$ or

$$z'_+(t) \leq \; -\alpha|x(t)|^2 + 2\beta x(t)x(t - \tau) - \alpha|x(t - \tau)|^2$$

and, based on the discriminant $4\{\beta^2 - \alpha^2\}$ of the quadratic form $\alpha X^2 - 2\beta XY + \alpha Y^2$, the right-hand side of this inequality is negative if $|\beta| < \alpha$. Since it follows that $z(t)$ is then non-increasing, so is $|x(t)|^2$ and we deduce stability of the null solution of (2.18), if $|\beta| < \alpha$. It is possible to use an appropriate functional for non-linear equations such as

$$x'_+(t) = -\alpha\{x(t)\}^3 + \beta\{x(t-\tau)\}^3$$

(in the latter case the null solution is stable if $|\beta| < \alpha$). Lyapunov functionals can also be defined [20] for stochastic problems, and their use and numerical simulation provides avenues for further investigation.

4.8 Further reading

A discussion of (global) exponential stability and numerical exponential stability in the case of SODEs is available in, for example [13] and also [14]. The papers on SODEs suggest further lines of enquiry in the case of SDDEs. The work [14] came to our attention in the closing stages of publishing our Report; these recent papers have additional citations to earlier work.

5 Conclusions

We have indicated a number of results on the convergence and stability of SDDEs and of the Euler-Maruyama method for SDDEs using a fixed step h. Perhaps it would be wise to emphasize that convergence results and stability results are, in their respective manners, *asymptotic results*. It is obvious that convergence theorems relate to what happens to numerical approximations as $h \searrow 0$; stability results relate to boundedness for $t \geq t_0$ and what happens as $t \to \infty$. Such results should not be misinterpreted as relating to a given $h > 0$ or to transient behaviour of induced perturbations.

Although the equations and their discrete versions are stochastic, we have relied upon deterministic results for expectations in order to establish the stability results. Both the order of convergence and the stability results for the Euler-Maruyama formula suggest the need to choose a relatively small step. The limitations of the Euler-Maruyama formula may prove to be significant in practice when other methods prove to be either more efficient, more accurate, more stable, or more controllable.

6 Acknowledgements

We express our thanks to Neville Ford (Chester) and Judy Ford (Liverpool) for comments on a draft version.

7 A code for implementing the Euler-Maruyama method

We include here a code for implementing the Euler-Maruyama method for SDDEs with multiplicative noise, to enable the reader to explore by experiment the effect of the choice of stepsize h. The code was written in the language `C` by Evelyn Buckwar.

```
/* This program permits investigation of */
/* the stochastic delay differential equation (SDDE) */
/* dX(t) = f(X(t), X(t-tau))dt + (sigma1+ sigma2 X(t)+ sigma3 X(t-tau))dW(t)*/
/* for t in [0,T], with  X(t) = 1 + t   for t in  [-tau,0], */
/* where W(t) is a Wiener process. */
/* Equidistant approximation of X(t) by the explicit Euler scheme  */
```

```
/* i) with the time step size delta_xt to provide an `exact solution' */
/* ii) with the time step size delta_y to give an `approximate solution' */
/* It uses the Polar Marsaglia method to generate Gaussian random numbers */
/* (Kloeden and Platen, Numerical solution of stochastic differential */
/* equations, Springer 1992, 1995 ISBN 3-540-54062-8) */

#include <math.h>
#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#define frand() ((double) rand() / RAND_MAX)

const int NumberSteps=2048;
typedef double vector[NumberSteps+1];
FILE *ResultsOut;

char fileout[]= "ResultsOut.dat";
time_t times;
const double t0 = 0.0;  /* left end point  */

double t_end =  1.0;     /* right end point  */
double tau= 1.0;     /* lag */
double alpha=-1.0;    /* parameter in the drift f(t,x,y)*/
double beta=-0.5;    /* parameter in the drift f(t,x,y) (lag)   */
double sigma1=0.0;   /* parameter in the diffusion sigma(t,x,y) */
double sigma2=0.2;   /* parameter in the diffusion sigma(t,x,y)  */
double sigma3=0.2;   /* parameter in the diffusion sigma(t,x,y) */
double deltadiv = 8.0; /* divider for the time step size  */

int taudelta= ((int)(NumberSteps * tau))/((int)(t_end - t0));
                            /* number of steps used for the delay */
int i,j,k,g;     /* counters                                        */
int Ratio;       /* ratio of the time step sizes  */
double tn;       /* subinterval point                              */
double zt;       /* the delay term               */
double delta_yt; /* time step size of the 'approximate solution' */
double delta_xt; /* time step size to the 'exact solution'  */
double sqrtdelta_xt; /* square root of the time step size delta_xt  */
double GRand1, GRand2;  /* gaussian random numbers                   */
double dwtn;      /* wiener process increment  w(t_(n+1)) - w(t_n)*/
double eps;       /* mean square error */
vector dwt;         /* values of the wiener process increments      */
vector xt;          /* values of the 'exact solution' */
vector yt;         /* values of the 'approximate solution'  */

void InputData( void );
void simulGRand(double *GRand1, double *GRand2);

/* drift function */
double f(double tn, double xi, double yi)
{
 return ( alpha*xi + beta * yi);
}/* f */
/* diffusion function */
double sigmaf(double tn, double xi, double yi)
{
 return (sigma1+ sigma2 * xi+ sigma3 * yi);
}/* sigma */
```

```
/* initial function  */
double Psi(double tt)
{
 return (1.0 +tt);
}/* Psi */

/* measures the mean square error  */
double sqrerr(double xt, double yt)
{
 double  result;
 result= fabs(xt-yt)*fabs(xt-yt);
 return( result );
}/* sqrerr */

/* main program : */
void main()
{
 InputData();
 ResultsOut = fopen(fileout,"w");
 srand((unsigned) time(&times));
 delta_yt = (t_end-t0)/deltadiv; /* time step size, for approximate solution */
 Ratio=(int)(NumberSteps/deltadiv); /* ratio delta_yt: delta_xt               */

 delta_xt=(t_end-t0)/NumberSteps;   /* time step size for the exact solution */
 sqrtdelta_xt=sqrt(delta_xt);


 tn=t0;                   /* initialize the time step */
 xt[0]=Psi(t0);           /* value of the exact solution at zero  */

 /* generation of the 'explicit solution'  */
 for ( i=1; i <= NumberSteps; i++ )
  {
   tn+=delta_xt;  /* time     */
   if( i%2 ) simulGRand(&GRand1,&GRand2); /* with polar marsaglia method */
   else GRand1=GRand2;
   dwt[i-1]=GRand1*sqrtdelta_xt; /* wiener process increment w(t_(n+1))-w(t_n)*/
   /* Evaluation of the delay term */
   if ( (tn - delta_xt - tau) <= 0.0 )
        zt = Psi(tn - delta_xt - tau);
   else zt = xt[i-1- taudelta];
   xt[i] = xt[i-1] + f(tn-delta_xt,xt[i-1],zt) * delta_xt
            + sigmaf(tn-delta_xt,xt[i-1], zt)*dwt[i-1];
  }  /* end for - loop  */

/* generation of the 'approximate solution' with explicit Euler-Maruyama */
 i=0;
 tn=t0;     /* initial time   */
 zt = 0.0;
 yt[0]=Psi(t0); /* initial value of the explicit euler approximation */
 while ( tn < t_end )
  {
   i++;
   tn+=delta_yt;

   dwtn=0.0; /* sums up the wiener process increments for the current time step */
   for (j=1; j <= Ratio; j++) dwtn += dwt[(i-1)*Ratio+j-1];
```

```
    /* Evaluation of the delay term */
    if ( (tn-delta_yt - tau) <= 0.0 )
          zt = Psi(tn-delta_yt - tau);
    else zt = yt[(i-1)*Ratio - taudelta];
     /* explicit euler scheme : */
    yt[i*Ratio]= yt[(i-1)*Ratio]
            + f(tn-delta_yt, yt[(i-1)*Ratio], zt )* delta_yt
            + sigmaf(tn-delta_yt, yt[(i-1)*Ratio],zt)*dwtn;

   /* interpolation for the other values for illustration purposes: */
    if ( Ratio>1 )
     for ( j=1; j <= Ratio-1; j++ )
       {
    yt[(i-1)*Ratio+j]=yt[(i-1)*Ratio]+j*(yt[i*Ratio]-yt[(i-1)*Ratio])/Ratio;
      }
   }/* end of while- loop */
 eps=sqrerr(xt[NumberSteps],yt[NumberSteps]);
 /* printout : */
  tn=t0;
  for (j=0; j <= NumberSteps; j++)
      { fprintf(ResultsOut,"%.7G %.7G %.7G\n",tn,xt[j],yt[j]);
          tn+=delta_xt;
       }

      printf("finished!, eps = %.7G\n\n", eps);
fclose(ResultsOut);
}/* end of main () */
void InputData( void )
{
char istr[20];
printf("\n\nThis program approximates the strong solution of the stochastic\n");
printf("delay differential equation (Ito) on the interval [0,T]\n");
printf("dX(t)=f(X(t),X(t-tau))dt + (sigma1+sigma2 X(t)+sigma3 X(t-tau))dW(t)\n");
printf("with f(X(t),X(t-tau)) = a * X(t) ++ X(t-tau)  \n");
printf("with the initial function Psi(t) = 1 + t  on [-tau,0] \n");
printf("using the Euler method and stepsize h = T/N. \n");
printf("It prints the results to the file %s in the form\n",fileout);
printf("t_n X(t_n) Y_n  for n = 0 to %d. \n\n",NumberSteps);
printf("Press <Enter> to accept the choices, otherwise write yours.\n");
printf("T = %.2f    ", t_end );
gets(istr);
if ( istr[0] ) t_end = atof ( istr );
printf("tau = %.2f ", tau );
gets(istr);
if ( istr[0] ) tau = atof ( istr );
printf("N = %.2f   ", deltadiv );
 gets(istr);
if ( istr[0] ) deltadiv = atof ( istr );
printf("a = %.2f   ", alpha );
gets(istr);
if ( istr[0] ) alpha = atof ( istr );
printf("b = %.2f   ", beta );
gets(istr);
if ( istr[0] ) beta = atof ( istr );
printf("sigma1 = %.2f   ", sigma1 );
gets(istr);
if ( istr[0] ) sigma1 = atof ( istr );
printf("sigma2 = %.2f   ", sigma2 );
```

```
gets(istr);
if ( istr[0] ) sigma2 = atof ( istr );
printf("sigma3 = %.2f   ", sigma3 );
gets(istr);
if ( istr[0] ) sigma3 = atof ( istr );
}
/* generates two independent N(0,1) random numbers */
/* by the polar-marsaglia method                    */
void simulGRand(double *GRand1, double *GRand2)
{
  double V1, V2, W, root;
  do
     { V1 = 2.0 * frand()-1.0;
        V2 = 2.0 * frand()-1.0;
        W  = V1 * V1 + V2 * V2;
     }
  while (!((W<=1.0) && (W>0.0))); /* exclude ln(0) */

  root = sqrt(-2.0 * log(W)/W);
  *GRand1 = V1*root;
  *GRand2 = V2*root;
}/* simulGRand */
```

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