

COMPENSATOR-BASED SIMULATION OF CORRELATED DEFAULTS

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Abstract

The market for derivatives with payoffs contingent on the credit quality of a number of reference entities has grown considerably over recent years. The risk analysis and valuation of such multi-name structures often relies on simulating the performance of the underlying credits. In this paper we discuss the simulation of correlated unpredictable default arrival times. Our algorithm is based on the compensator of default. We construct this compensator explicitly in a multi-firm structural model with correlated defaults and imperfect asset and default threshold observation. It is shown how the model parameters can be estimated from readily available equity and single-name credit derivatives market data.

Key words: correlated defaults; simulation; default compensator.

JEL Classification: G12; G13

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1 Introduction

Credit risk refers to the risk of incurring losses due unexpected changes in a counterparty's credit quality. Credit derivatives allow to isolate and trade that risk by providing a payoff upon a credit event arrival with respect to a reference entity, such as rating downgrade, failure to pay, or bankruptcy. Multi-name or basket credit derivatives have payoffs contingent on the credit quality of a number of reference entities. They provide a means to reduce or gain credit risk exposure related to a basket of debt securities. Popular products include first-to-default swaps, in which the contingent payment by the protection selling party is triggered by the first-to-occur default event. More complex multi-name instruments include Collateralized Debt Obligations, which are structured fixed income transactions. They involve prioritized tranches whose cash flows are linked to the performance of a pool of debt instruments.

The risk analysis and valuation of complex multi-name structures often relies on simulating the performance of the reference entities. In credit risk management applications, where one is concerned with the aggregated credit risk associated with some portfolio of credit-risky securities, simulation of event arrivals is often essential to estimate the distribution of aggregated losses due to credit event arrivals.

One existing approach to the simulation of correlated and unpredictable default times is based on the *default intensity*. The intensity process directly prescribes the stochastic structure of default; it can be interpreted as the conditional arrival rate of default. To model default dependence, one can introduce correlation between the intensities [Duffie & Singleton (1998)], impose the default dependence structure directly [Schönbucher & Schubert (2001)], or introduce joint shock events which lead to simultaneous defaults of several firms [Giesecke (2002)]. While individual intensity process parameters might be calibrated from market data, the calibration of the correlation structure is difficult. This is in part due to the ad-hoc nature of this approach, in which the default event is not causally modeled.

We propose an alternative *structural* approach to the simulation of correlated and *unpredictable* default times. Taking as given the asset dynamics, a default event is explicitly defined in terms of a firm's assets and liabilities. This facilitates the calibration of the model, in particular with respect to the correlation structure.

Our simulation methodology involves a recursive algorithm which is based on the *compensator* of default. This compensator counteracts the jump in the default indicator such that the difference between compensator and indica-

tor becomes a martingale. In that sense the compensator can be viewed as the fair fixed rate premium for a default insurance payment of one unit of account upon the default incidence. While the compensator always uniquely exists, the existence of an intensity as the density of the compensator is not always granted. We construct the compensator of the k th default explicitly in a multi-firm structural model with correlated defaults. Taking as given a firm's asset dynamics, we define a default event as the first time the assets fall to some lower threshold relative to liabilities. In practice, however, it is typically difficult to directly observe a firm's assets and default threshold. We take this into account and suppose that assets and default thresholds are not publicly known, cf. Duffie & Lando (2001) and Giesecke (2001*b*). In this situation one is always uncertain about the nearness of the assets to the threshold. Therefore defaults are surprise events; they are completely unpredictable. This is realistic and implies credit spread term structure properties that match empirical observations. In particular, bond prices are subject to surprise jumps upon default and credit spreads are bounded away from zero even if maturities are going to zero. In the usual structural approach with a continuous asset process and perfect information, bond prices converge continuously to their default-contingent values and spreads go to zero with maturity going to zero. This behavior is however empirically not plausible.

Following Giesecke (2001*a*), dependence between default events is introduced through correlation of firms' assets and correlated default thresholds. The former can be thought of as arising from firms' common dependence of general (macro-) economic factors, while the latter corresponds to direct firm inter-linkages, such as parent-subsidiary relationships or substantial mutual capital holdings. Either 'correlation mechanism' potentially induces the full range of possible default dependence, from perfect negative to perfect positive default correlation.

The parameters of our multi-firm model (asset volatilities, asset correlations, and default thresholds or their distribution) can be estimated from readily available equity and single-name credit derivatives market data.

The remainder of this paper is organized as follows. In Section 2, we introduce our basic algorithm for the simulation of an individual default time. The underlying structural model and parameter estimation is discussed in some detail. In Section 3, we extend to the practically relevant case of first-to-default simulation. The modeling of default correlation is emphasized. A general algorithm for simulating successive correlated event times is provided in Section 4. Section 5 concludes.

2 Single-Entity Default Simulation

In this section we discuss an algorithm for default arrival time simulation. In Sections 3 and 4, we will extend this basic algorithm to multi event time simulation.

2.1 Basic Algorithm

We fix a probability space (Ω, \mathcal{G}, P) equipped with a right-continuous and complete filtration $(\mathcal{G}_t)_{t \geq 0}$ describing the information flow over time. In risk analysis applications, P is the physical probability; for derivatives valuation purposes P is taken to be some risk-neutral probability.

The default time of some given firm is a random variable τ taking values in $(0, \infty]$. We assume that $P[\tau > t] > 0$ for all $t \geq 0$. The default indicator process is denoted $N = (N_t)_{t \geq 0}$, where $N_t = 1_{\{t \geq \tau\}}$. We assume that

$$\mathcal{G}_t = \sigma(N_s : s \leq t). \quad (1)$$

In accordance with empirical observations, we suppose furthermore that the default event occurs completely unexpectedly; by surprise, so to speak. That means τ is a totally inaccessible stopping time.

Our goal is to simulate an inaccessible $(0, \infty]$ -valued stopping time which is equal in conditional distribution to τ . This conditional distribution can be characterized through the *default compensator*, i.e. the unique, increasing, and predictable process $K = (K_t)_{t \geq 0}$ with $K_0 = 0$ and such that the difference process $N - K$ is a martingale. The default compensator can be calculated in terms of the survival function $L(t) = P[\tau > t]$ as follows:

$$K_t = - \int_0^{t \wedge \tau} \frac{dL(s)}{L(s-)}, \quad (2)$$

cf. Dellacherie (1970). If τ is inaccessible, then L is continuous and the continuous compensator is given by

$$K_t = - \ln L(t \wedge \tau). \quad (3)$$

In this case, on the set $\{t < \tau\}$ the default probability satisfies

$$P[\tau \leq T | \mathcal{G}_t] = 1 - \frac{L(T)}{L(t)} = 1 - e^{K_t - K_T}, \quad t \leq T. \quad (4)$$

Now let some continuous increasing function A with $A(0) = 0$ be given. By means of the following algorithm we can construct an inaccessible stopping

time τ having default compensator $A^\tau = A(\cdot \wedge \tau)$. Interpreting A^τ as the compensator of some given event time, according to (4) the simulated τ is equal in distribution to that event time.

Basic Algorithm.

- (1) Simulate a standard uniform random variable U .
- (2) Set $\tau = \inf\{t \geq 0 : e^{-A(t)} \leq U\}$.

From Step (2) we obtain immediately

$$L(t) = P[U < e^{-A(t)}] = e^{-A(t)}, \tag{5}$$

so that L is continuous. In view of (3), it follows that the simulated stopping time τ has indeed compensator $A^\tau = K$ where $A_t^\tau = A(t \wedge \tau) = -\ln L(t \wedge \tau)$.

2.2 Constructing the Survival Function

Having established our basic single entity default simulation algorithm, we need a default model which can be easily calibrated to observable market variables and for which the default compensator can be tractably calculated. We will insist on a model such that the survival function L is continuous, so that the default event is unpredictable and the default compensator is given by (3). We then simply put $A(t) := -\ln L(t)$ in Step (2) of our basic algorithm.

Our default model belongs to the structural class. That is, we take as given the dynamics of the firm’s asset value V and assume that the firm defaults when the assets fall below some threshold D for the first time. We assume V to be continuous and without loss of generality we normalize $V_0 = 0$. The running minimum asset process is denoted $M = (M_t)_{t \geq 0}$ with

$$M_t = \min\{V_s \mid 0 \leq s \leq t\}.$$

In the classical structural approach investors are assumed to have complete information on assets and threshold. In this case the default occurs never unexpectedly, because investors can observe the nearness of the assets to the default threshold, and therefore, they are warned in advance when a default is imminent. In practice, however, it is typically difficult if not impossible to directly observe the assets of an issuer and the threshold asset level at which the firm will be liquidated, cf. Duffie & Lando (2001) and Giesecke (2001*b*).

Taking this into account, in (1) we assume that default events are publicly observable but the firm's assets V and its default threshold D are unknown. Note that in this case one is always uncertain about the nearness of the assets to the default threshold, so that a default occurs completely unexpectedly. This is in accordance with empirical observations.

Let D have density g on $(-\infty, 0)$ and suppose throughout that D is independent from assets V . We then have for the survival probability

$$L(t) = 1 - P[M_t \leq D] = 1 - \int_{-\infty}^0 H(t, x)g(x)dx, \quad (6)$$

where $H(t, \cdot)$ is the distribution function of M_t . Clearly, L is continuous as soon as $H(t, x)$ is continuous in t for fixed x . Now suppose the default threshold D is a priori known, $D \in \mathcal{G}_0$. We then have simply

$$L(t) = P[D < M_t] = 1 - H(t, D). \quad (7)$$

Obviously the efficiency of default arrival time simulation depends on the structure of the functions $H(t, \cdot)$ and g . While the former is determined by the assumed asset dynamics, the latter can be specified 'more freely'. If these functions are in closed-form, then simulation using our basic algorithm is very efficient, especially when D is known.

2.3 Estimating Parameters

To implement our basic algorithm, we need to specify the asset dynamics, estimate the parameters of the assumed process, and estimate the threshold value D or its distribution G , respectively.

In line with the majority of structural approaches, let us specify a Brownian motion with drift μ and volatility σ for assets V . Then the distribution function $H(t, x)$ of M_t is continuous in (t, x) and for $x \leq 0$ and $t > 0$ given by

$$H(t, x) = 1 - \Phi\left(\frac{\mu t - x}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\mu x}{\sigma^2}\right)\Phi\left(\frac{x + \mu t}{\sigma\sqrt{t}}\right), \quad (8)$$

where Φ is the standard normal distribution function. Of course, if we work under some risk-neutral probability P , the drift equals the riskless short rate. Estimates for the volatility σ can be obtained from standard credit risk management software packages such as KMV's Credit Monitor. Their default modeling approach and ours are consistent; both are based on similar structural arguments. For an exposition of the estimation methodology we refer to Crosbie (1997) or Crouhy, Galai & Mark (2000). In case the name to be simulated

is not monitored, one can easily infer estimates from monitored competitors or take average industry values.

Alternatively, the asset volatility σ can be calibrated from market data, for example (liquid) credit default swap prices or corporate bond prices. Our default model is then consistent with observed prices. Once a (risk-neutral) survival probability $L(T)$ for the fixed horizon T is implied out from a price quote, σ can be found (numerically) via (6) or (7).

Balance-sheet data can be the basis for estimating the default threshold value D . Based on extensive empirical observations, KMV advocates to set D equal to the face value of short term (up to one year) liabilities plus some fraction of the face value of longer term liabilities (see Crosbie (1997)). Alternatively, given some market-implied survival probability, D can be calibrated via (7), cf. Hull & White (2000), for example.

In case reliable data for the estimation of D is not available, one has to specify some possibly parametric distribution G for D (the parameter can again be calibrated from market data). A pragmatic choice of G can be based on tractability considerations. For example, one can set $G(x) = g(x) = e^x$ for $x \leq 0$, which would correspond to a uniformly distributed threshold if we start off with a geometric Brownian motion for assets. In order to exploit the closed-form solution for L in (7) when D is known, one can also easily simulate D by drawing an additional independent $W \sim U[0, 1]$ and setting $D = G^{-1}(W)$ for continuous and strictly increasing G .

In Figure 1, we plot the term structure of survival probabilities L based on (6) for varying asset volatility. We set $\mu = 6\%$ (the riskless rate) and $g(x) = e^x$. Clearly, the higher the business risk of the firm, the higher are default probabilities for a given time horizon.

3 Simulating the First-to-Default

We now extend our basic one-name setup to n correlated names, where we focus on first-to-occur default event simulation, which is instructive and practically relevant. A general algorithm for simulating successive defaults is provided in Section 4.

3.1 Basic Algorithm

The default time of firm $i \in \{1, 2, \dots, n\}$ is denoted τ_i with indicator process N^i . We assume that the probability of two defaults occurring simultaneously

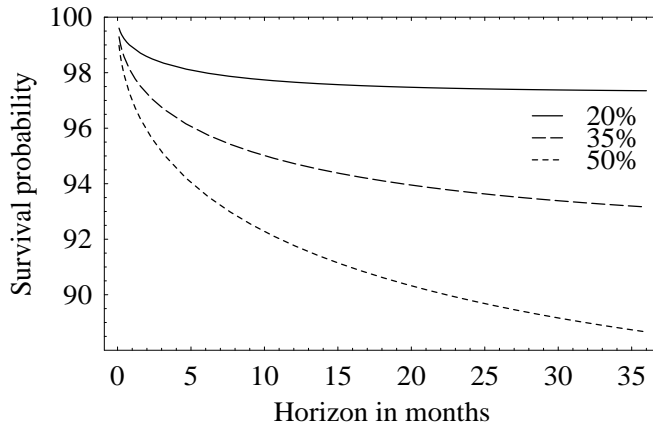


Figure 1: Term structure of survival probabilities, varying asset volatility.

is zero. The public information flow is defined by

$$\mathcal{G}_t = \sigma(N_s^i : s \leq t, i = 1, \dots, n). \quad (9)$$

The ordered sequence of default times is denoted by (T_i) and we set $N_t = \sum_i 1_{\{t \geq T_i\}}$. In analogy to Section 2, the distribution of the *first-to-default* time T_1 can be characterized through the first-to-default compensator. Letting $L_1(t) = P[T_1 > t]$ be continuous, the process K^1 given by

$$K_t^1 = -\ln L_1(t \wedge T_1) \quad (10)$$

is the compensator of T_1 , cf. Chou & Meyer (1975). On the set $\{t < T_1\}$, the conditional first-to-default probability is then given by

$$P[T_1 \leq T | \mathcal{G}_t] = 1 - \frac{L_1(T)}{L_1(t)} = 1 - e^{K_t^1 - K_T^1}, \quad t \leq T.$$

We wish to simulate an inaccessible stopping time which is equal in distribution to T_1 . Our algorithm proceeds in two steps: Given some continuous increasing function A_1 with $A_1(0) = 0$, we first construct an unpredictable stopping time T_1 having compensator $A^{T_1} = A_1(\cdot \wedge T_1)$. The second step consists of simulating the identity of the first-to-defaulter given T_1 .

First-to-Occur Event Algorithm.

- (1) Simulate a standard uniform random variable U .

- (2) Set $T_1 = \inf\{t \geq 0 : e^{-A_1(t)} \leq U\}$.
- (3) Simulate the identity of the first defaulter, i.e. simulate an $\{1, \dots, n\}$ -valued random variable I having given conditional distribution

$$P[I = i | \sigma(T_1)] = \frac{q_1(i, T_1)}{q_1(1, T_1) + \dots + q_1(n, T_1)},$$

where the quantities $q_1(i, t)$ are given by

$$\begin{aligned} q_1(i, t)dt &= P[T_1 = \tau_i, T_1 \in (t, t + dt)] \\ &= P[\tau_i \in (t, t + dt), \tau_j > t (j \neq i)]. \end{aligned} \quad (11)$$

3.2 Constructing the First-to-Default Survival Function

For first-to-default simulation we need a multi-firm default model for which the first-to-default compensator can be efficiently calculated. In analogy to Section 2.2, we insist on a model such that the survival function L is continuous, so that the defaults are inaccessible and we can put $A_1(t) := -\ln L_1(t)$ in our algorithm.

Given the dynamics of its asset value V^i , we assume that firm i defaults when assets fall to the threshold D_i for the first time. We assume V^i to be continuous and without loss of generality we normalize $V_0^i = 0$. Throughout, we suppose that the threshold vector $D = (D_1, \dots, D_n)$ is independent of assets $V = (V^1, \dots, V^n)$. We denote by G the joint distribution function of D on $(-\infty, 0]^n$, and we assume that G is continuous. Letting M^i be the running minimum asset process of firm i , we have for the survival function

$$\begin{aligned} L_1(t) &= P[\min_i(\tau_i) > t] \\ &= P[M_t^1 > D_1, \dots, M_t^n > D_n] \\ &= \int_{(-\infty, 0]^n} G(x) h(t, x) dx, \end{aligned} \quad (12)$$

which is continuous as soon as the density $h(t, x)$ of $M_t = (M_t^1, \dots, M_t^n)$ is continuous in t for fixed $x \in (-\infty, 0]^n$. In view of (11), for the first-to-default identity probability we get

$$q_1(i, t) = \int_{(-\infty, 0]^n} G_{z_i}(x) h(t, x) dx, \quad (13)$$

where $G_{z_i}(z_1, \dots, z_n)$ denotes the partial derivative of G with respect to its i th argument. Now suppose that the thresholds D_i are known a priori: $D \in \mathcal{G}_0$. Denoting by $\bar{H}(t, \cdot)$ the joint survival function of M_t , from (12) we obtain

$$L_1(t) = \bar{H}(t, D), \quad (14)$$

which is continuous as soon as $\bar{H}(t, x)$ is continuous in t for fixed $x \in (-\infty, 0)^n$. Likewise, we have for the first-to-default identity probability

$$q_1(i, t) = -\bar{H}_{z_i}(t, D). \quad (15)$$

The efficiency of first-to-default simulation depends critically on the structure of the density h and the survival function \bar{H} under the assumed asset dynamics, to which we turn next.

3.3 Default Correlation and Estimation

In order to implement our n -name structural model, we need to specify/estimate individual firm parameters (asset dynamics, default threshold). This proceeds as discussed in Section 2.3. Additionally, we have to specify the correlation between individual firms.

Our structural model allows for two natural and intuitive ways of incorporating correlation between individual firms. The first is via the correlated evolution of firms' assets and the second is via dependence of firms' default thresholds, cf. Giesecke (2001a). These 'mechanisms' can be imposed simultaneously or alternatively, depending on what correlation characteristics is needed, which data as a basis for estimation is available, or simply on computational tractability.

Asset value correlation corresponds to an indirect interrelation between firms arising from the dependence of firms on common macro-economic factors. Assuming that assets V follow an n -dimensional Brownian motion, this is formally represented by the correlation matrix $\Sigma = (\rho_{ij})_{n \times n}$, where ρ_{ij} denotes the (constant) linear correlation coefficient between V^i and V^j . There are well-established methods for estimating Σ from equity market data. For an exposition, see Kealhofer (1998) and Crouhy et al. (2000). These methods are implemented in standard software packages (KMV, CreditMetrics). Given the joint asset dynamics, for the calculation of L_1 and q_1 we need the running minimum asset density $h(t, \cdot)$ or the survival function $\bar{H}(t, \cdot)$. Iyengar (1985) provides these functions in case $n = 2$; see also Zhou (2001). He, Keirstead & Rebholz (1998) calculate the 2-dimensional distribution function $H(t, \cdot, \cdot)$ of M_t in terms of modified Bessel functions. Then $\bar{H}(t, x, y) =$

$1 - H(t, x, 0) - H(t, 0, y) + H(t, x, y)$. Unfortunately, results for the general multi-dimensional case $n \geq 3$ have not yet been worked out, to our knowledge.

Default threshold dependence corresponds to direct inter-firm linkages such as parent-subsidiary relationships or mutual substantial capital holdings. We can separate the threshold dependence structure from their marginal behavior by means of their copula C , satisfying

$$G(x_1, \dots, x_n) = C(G_1(x_1), \dots, G_n(x_n)), \quad x_i \leq 0, \quad (16)$$

where G_i is the distribution function of the default threshold D_i of firm i . The copula function C captures the complete dependence between the random thresholds D_1, \dots, D_n . While G_i is determined by idiosyncratic factors, C represents threshold determinants which are due to the direct links between firms. Suppose we have fixed the G_i (see Moraux (2001) for a related problem). It then remains to specify C , which is much more difficult than estimating asset correlations Σ . If sufficient historical default data is available, one can directly estimate the empirical copula \hat{C} of D , cf. Nelsen (1999) for a definition. Durrleman, Nikeghbali & Roncalli (2001) review approximation methods for the empirical copula. In lack of appropriate default data, one can specify some parametric copula family C_θ . Many parametric families with different properties are available, see Lindskog (2000) for some. A choice can be led by desired mathematical properties (for tractability, say), or by the desired statistical properties. The parameter vector θ might then be estimated from risky bond prices or credit swap spreads. Durrleman et al. (2001) discuss some suitable estimation methods. Jouanin, Rapuch, Riboulet & Roncalli (2001) show how to calibrate θ from Moody's Diversity Score.

3.4 Threshold Copula

In order to examine the effects of the threshold copula on the distribution of the first-to-default time T_1 , we consider a homogeneous first-to-default basket with $n = 5$ names. Assume that issuers' assets follow a standard 5-dimensional Brownian motion. This means that assets are independent and we can focus on the effects of threshold dependence.¹ The density of M_t is then given by

¹For an analysis of the joint default behavior with respect to asset correlation under perfect information we refer to Frey & McNeil (2001).

$h(t, \cdot) = h_1(t, \cdot) \cdots h_5(t, \cdot)$, where

$$h_i(t, x) = \frac{1}{\sigma_i \sqrt{t}} \phi \left(\frac{-x + \mu_i t}{\sigma_i \sqrt{t}} \right) + \exp \left(\frac{2\mu_i x}{\sigma_i^2} \right) \left[\frac{2\mu_i}{\sigma_i^2} \Phi \left(\frac{x + \mu_i t}{\sigma_i \sqrt{t}} \right) + \frac{1}{\sigma_i \sqrt{t}} \phi \left(\frac{x + \mu_i t}{\sigma_i \sqrt{t}} \right) \right], \quad (17)$$

is the density of M_t^i , which is straightforwardly derived from (8) by differentiation. ϕ is the standard normal density function.

The threshold dependence structure is modeled by the parametric Clayton copula family. As is not uncommon for first-to-default baskets, we suppose that the correlation structure in the basket is symmetric. We can therefore choose the one-parameter version of the Clayton family, which is given by

$$C_\theta^C(u_1, \dots, u_5) = (u_1^{-\theta} + \dots + u_5^{-\theta} - 4)^{-\frac{1}{\theta}}, \quad u_i \in [0, 1], \quad \theta > 0. \quad (18)$$

The parameter θ controls the degree of threshold dependence: $\theta \rightarrow \infty$ reflects perfect positive dependence, and $\theta \rightarrow 0$ corresponds to independence. The degree of monotonic threshold dependence can be expressed in terms of Kendall's pairwise rank correlation $\rho^K \in [-1, 1]$. We have $\rho^K = -1$ iff the thresholds are perfectly negatively related, $\rho^K = 1$ iff they are perfectly positively related, and $\rho^K = 0$ in case of independence. In contrast to linear correlation, ρ^K is defined on copula level. For the Clayton family we have $\rho^K = \theta/(\theta + 2)$ (with $\theta > 0$ ρ^K is positive as well and (18) expresses positive dependence). Assuming that the default threshold of firm i has distribution function $G^i(x) = e^{-x}$, from (16) the joint threshold distribution is

$$G^C(x_1, \dots, x_5) = (e^{-\theta x_1} + \dots + e^{-\theta x_5} - 4)^{-\frac{1}{\theta}}, \quad x_i \leq 0. \quad (19)$$

In Figure 2, we plot the term structure of risk-neutral first-to-default survival probabilities L_1 for varying degrees of rank threshold correlation ρ^K (we set $\mu = 6\%$, the riskless rate). The asset volatility σ_i is set to 20% for all names (this is the average volatility of a conservative low-risk S&P 500 firm). In a high quality basket individual default probabilities are low, and with a positive firm association the probability of some default in the basket is small. The better the basket quality and the higher the firm correlation, the lower are first-to-default probabilities. In other words, for a given horizon L_1 is increasing in the degree of firm correlation, cf. Figure 3, which shows $L_1(1)$ as a function of rank threshold correlation ρ^K for varying asset volatility $\sigma = \sigma_i$. This observation is also consistent with the price behavior of a first-to-default contract paying off upon the the first default in the basket: with increasing

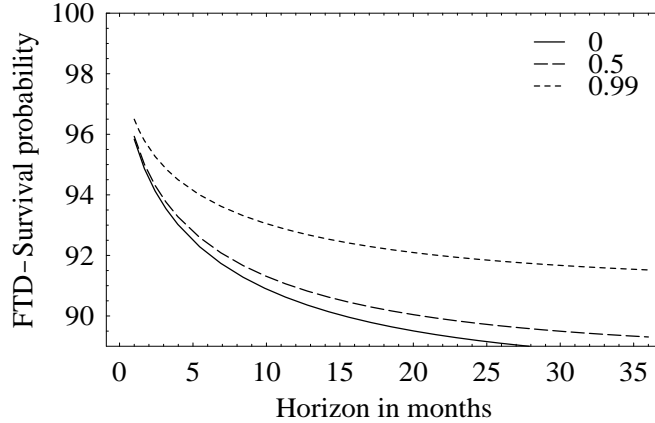


Figure 2: Term structure of first-to-default survival probabilities, varying rank threshold correlation.

firm dependence, the survival probability of T_1 increases, the payoff probability decreases, and the contract price decreases. In line with intuition, the higher the correlation in the basket, the less value has the contract's insurance capability for the holder. If the first default occurs, the defaults of the other names is likely as well, but the contract aims at covering the first default only. As for asset volatility, the lower the quality of the basket, the lower L_1 , and the higher the premium for the first-to-default insurance contract.

So far we have examined the sensitivity of the survival probability with respect to threshold correlation and asset volatility for the Clayton family. The survival probability is however also sensitive to the choice of the family itself. To study this, let us introduce the Gumbel family with parameter $\theta \geq 1$:

$$C_\theta^G(u_1, \dots, u_5) = \exp\left(-\left[(-\ln u_1)^\theta + \dots + (-\ln u_5)^\theta\right]^{\frac{1}{\theta}}\right), \quad u_i \in [0, 1]. \quad (20)$$

The value $\theta = 1$ corresponds to independence, while $\theta \rightarrow \infty$ reflects perfect positive dependence. For the Gumbel family the pairwise rank correlation is $\rho^K = 1 - 1/\theta$. With $G^i(x) = e^x$, we have for the joint threshold distribution

$$G^G(x_1, \dots, x_5) = \exp\left(-\left[(-x_1)^\theta + \dots + (-x_5)^\theta\right]^{\frac{1}{\theta}}\right), \quad x_i \leq 0. \quad (21)$$

The choice of the copula family has indeed significant effects on the resulting arrival probabilities. Figure 4 displays the (risk-neutral) 1-year first-to-default survival probability $L_1(1)$ as a function of rank threshold correlation ρ^K for both Clayton and Gumbel threshold copulas. The asset volatility σ_i is set to 20% for all names. The differences in the survival probability for the two

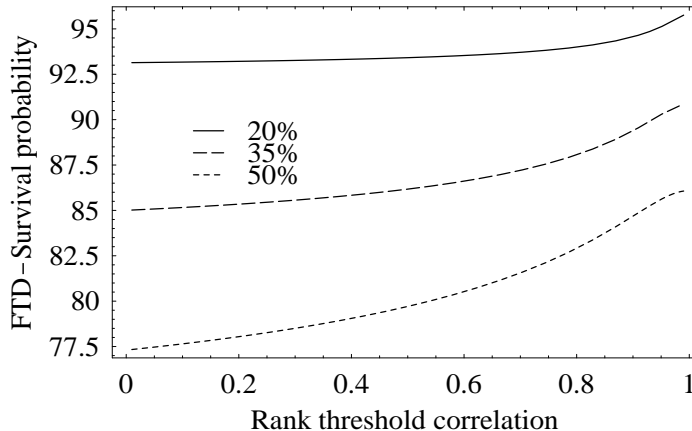


Figure 3: First-to-default survival probability as a function of rank threshold correlation, varying asset volatility.

families are due to their tail dependence properties. The Gumbel copula exhibits upper tail dependence, which refers to the pronounced generation of high threshold values in all marginals simultaneously (for a formal definition we refer to Nelsen (1999)). All else being equal, this implies in turn an increased likelihood of joint defaults, which leads to higher survival probabilities L_1 of T_1 (see Section 3.5 below for the structure of joint default probabilities and their relation to the threshold copula). This can be easily seen in case $n = 2$:

$$L_1(t) = 1 - P[\tau_1 \leq t] - P[\tau_2 \leq t] + P[\tau_1 \leq t, \tau_2 \leq t].$$

The Clayton copula exhibits lower tail dependence, which leads to opposite effects. Consequently, for a given horizon the first-to-default survival probability with the Gumbel threshold copula is at least as high as with the Clayton copula.

If we have evidence for such particular default characteristics in our basket, we can model them by choosing tail-dependent threshold copula families. In view of the uncertainty surrounding the choice of a copula family, if such evidence is not available it seems reasonable to confine on families which display asymptotic independence in both tails. A simple closed-form family that satisfies this property is the Frank family, which is in the one-parameter version for $\theta > 0$ defined by

$$C_\theta^F(u_1, \dots, u_5) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta u_1} - 1) \dots (e^{-\theta u_5} - 1)}{(e^{-\theta} - 1)^4} \right), \quad u_i \in [0, 1].$$

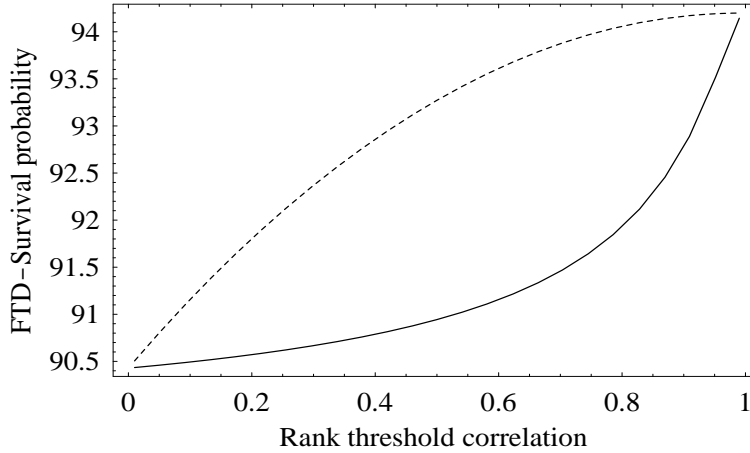


Figure 4: First-to-default survival probability as a function of rank threshold correlation for different copula families (solid line: Clayton, dashed line: Gumbel).

For more families suitable for modeling the threshold copula, and methods to construct copula families, we refer to Nelsen (1999).

3.5 Threshold Dependence vs. Asset Correlation

In general it seems much more difficult to estimate the threshold dependence structure C than the asset correlation matrix Σ in the canonical Brownian motion case. On the other hand, using default threshold dependence instead of asset dependence in order to induce default correlation leads to more tractability in the calculation of the survival function L_1 and the identity density q_1 . This is due to the fact that for $n \geq 3$ explicit results on the joint density $h(t, \cdot)$ of the running minimum asset vector M_t are hardly available. In this section we discuss a way to resolve this trade-off.

Let us consider the pairwise joint default probability for horizon $T > 0$:

$$F_{ij}(T) = P[\tau_i \leq T, \tau_j \leq T] = P[M_T^i \leq D_i, M_T^j \leq D_j] \quad (22)$$

In case the default thresholds are independent,

$$F_{ij}(T) = \int_{-\infty}^0 \int_{-\infty}^0 H_{\rho_{ij}}(T, x, y) g_i(x) g_j(y) dx dy, \quad (23)$$

where g_k is the density of D_k and where we have used the independence of the D_k . We have made the dependence of the joint law of (M^i, M^j) on the asset

correlation ρ_{ij} explicit by the notation $H_{\rho_{ij}}$. Since for planar Brownian motion $H_{\rho_{ij}}$ is known [Iyengar (1985) and He et al. (1998)], F_{ij} can be calculated by numerical integration for general densities g_k .

If assets are independent, joint default probabilities are given by

$$F_{ij}(T) = \int_{-\infty}^0 \int_{-\infty}^0 H_i(T, x) H_j(T, y) g(x, y) dx dy, \quad (24)$$

where H_k is the law of M^k given by (8) and g is the joint density of (D_i, D_j) . Fixing some parametric copula family C_θ for the threshold vector D and assuming that C_θ admits a density c_θ , we have for the joint threshold density

$$g(x, y) = c_{\theta_{ij}}(G_i(x), G_j(y)) g_i(x) g_j(y), \quad (25)$$

where θ_{ij} is the copula parameter controlling the dependence between the random thresholds D_i and D_j . Setting T equal to the simulation horizon, we are thus able to calculate θ_{ij} from ρ_{ij} , and vice versa.

The trade-off mentioned above can now be resolved as follows. We first estimate an asset correlation matrix Σ . This is well-known and implemented in standard industry software. Fixing some threshold family C_θ , the parameter vector θ can be calculated element by element as shown above. Given C_θ together with the G_i , we can now compute L_1 and q_1 via (12) and (13) under the hypothesis that thresholds are correlated according to C_θ while assets are independent. This is generally tractable because $h(t, x, y) = h_1(t, x) \cdots h_n(t, y)$, where the density $h_i(t, \cdot)$ of M_t^i is given in (17) in closed-form. This auxiliary procedure of 'converting' asset correlation to threshold correlation for purposes of computing L_1 and q_1 is purely technical. It can be justified since (pairwise) joint default probabilities for the simulation horizon, including default correlation of course, remain unchanged.

With the above described procedure and the hypothesis that thresholds are correlated with C_θ while assets are independent, we can also exploit the closed-form expressions for L_1 and q_1 arising from (14) and (15) for a given D . For the survival function we then have

$$L_1(t) = (1 - H_1(t, D_1)) \cdots (1 - H_n(t, D_n)), \quad (26)$$

where $H_i(t, \cdot)$ is the distribution function of M_t^i , given in closed-form in (8). In analogy to the one-firm case, in order to exploit (26) we simply simulate the vector D using the copula C_θ and the marginals G_i of D : given a realization (W_1, \dots, W_n) from C_θ , the vector $D = (G_1^{-1}(W_1), \dots, G_n^{-1}(W_n))$ has joint distribution G . For the generation of realizations from a copula several efficient

algorithms are available, see Embrechts, McNeil & Straumann (2001) and Devroye (1986). Lindskog (2000) provides specialized algorithms for families C_θ belonging to the class of Archimedean copulas (the families we introduced in Section 3.4 belong to this class, for example).

4 Simulating Successive Default Arrivals

4.1 General Algorithm

In this section we extend our first-to-occur algorithm to general multi-event time simulation. The setting is as in Section 3: the ordered sequence of default times is denoted by (T_i) with $T_0 := 0$ and we set $N_t = \sum_i 1_{\{t \geq T_i\}}$. Let us consider the compensator of N , which characterizes the distribution of the T_i . For the k th arrival time T_k , we define the survival function L_k by

$$L_k(t) = P[T_k > t \mid \sigma(Z_{k-1})], \quad k \geq 1, \quad (27)$$

where $Z_k = (T_i, I_i)_{i \leq k}$ and $I_k \in \{1, \dots, n\}$ is the identity of the k th defaulter. Note that $L_k(t) = 1$ for all $t \leq T_{k-1}$. From the results in Chou & Meyer (1975), if L_k is continuous, then the process K given by

$$K_t = \begin{cases} -\ln L_1(T_1) - \dots - \ln L_k(t) & : T_{k-1} \leq t < T_k \\ -\ln L_1(T_1) - \dots - \ln L_n(T_n) & : T_n \leq t \end{cases} \quad (28)$$

is the compensator of the process N . This implies that the compensator K^k of the k th default is given by

$$K_t^k = -\ln L_k(t \wedge T_k). \quad (29)$$

Note that $K_t^k = 0$ for all $t \leq T_{k-1}$.

Simulation of $m \leq n$ successive correlated arrival times T_1, \dots, T_m is possible by iterating the first-to-occur algorithm. Given some continuous increasing function A_1 with $A_1(t) = 0$, one starts by constructing an unpredictable first-to-default stopping time T_1 having compensator $A^{T_1} = A_1(\cdot \wedge T_1)$. Afterwards the identity of the first-to-default is simulated. Given some continuous increasing \mathcal{G}_{T_1} -measurable function A_2 with $A_2(t) = 0$ for $0 \leq t \leq T_1$, one next simulates the second-to-default time with identity, and so on. We let R_k denote the set of surviving entities after the k th event arrival.

Multi-Event Algorithm.

- (1) Initialize $R_0 = \{1, \dots, n\}$ and $k = 1$.

- (2) Simulate a standard uniform random variable U_k , which is independent of the U_1, \dots, U_{k-1} .
- (3) Set $T_k = \inf\{t \geq 0 : e^{-A_k(t)} \leq U_k\}$.
- (4) Simulate the identity of the k th defaulter, i.e. simulate an R_{k-1} -valued random variable I_k having given conditional distribution

$$P[I_k = i \mid \sigma(T_k, Z_{k-1})] = \frac{q_k(i, T_k)}{\sum_{i \in R_{k-1}} q_k(i, T_k)},$$

where we define the density

$$q_k(i, t)dt = P[\tau_i \in (t, t + dt), \tau_j > t (j \in R_{k-1} - \{i\}) \mid \sigma(Z_{k-1})]. \quad (30)$$

- (5) Set $R_k = R_{k-1} - I_k$.
- (6) If $k = m$ then stop, else set $k = k + 1$ and go back to Step (2).

From Steps (2) and (3) it follows immediately that

$$L_k(t) = P[U_k < e^{-A_k(t)} \mid \sigma(Z_{k-1})] = e^{-A_k(t)}, \quad (31)$$

which is continuous and equal to one for $t \leq T_{k-1}$. In view of (29), the process $A^{T_k} = A_k(\cdot \wedge T_k)$ is the compensator of T_k , as desired.

4.2 Constructing the Survival Function

The structural multi-firm default model introduced in Section 3.2 remains unchanged. If L_k is continuous in this model, then defaults are inaccessible and we can put $A_k(t) := -\ln L_k(t)$ in our multi-event algorithm. Without loss of generality we assume that the first k default times have (simulated) identities $I_i = i$ for $1 \leq i \leq k$. Then, from (27),

$$L_k(t) = P[\tau_k > t, \dots, \tau_n > t \mid \sigma(\tau_1, \dots, \tau_{k-1})].$$

Let h_{t_1, \dots, t_n} denote the density of the vector $(M_{t_1}^1, \dots, M_{t_n}^n)$ and g the density of D . Using the structural definition of default, with an application of Bayes' rule we obtain

$$L_k(t) = \frac{\int_{(-\infty, 0]^n} \int_{x_k}^0 \cdots \int_{x_n}^0 h_{\tau_1, \dots, \tau_{k-1}, t, \dots, t}(x_1, \dots, x_{k-1}, y_k, \dots, y_n) g(x) dy dx}{\int_{(-\infty, 0]^n} \int_{x_k}^0 \cdots \int_{x_n}^0 h_{\tau_1, \dots, \tau_{k-1}, \dots, \tau_{k-1}}(x_1, \dots, x_{k-1}, y_k, \dots, y_n) g(x) dy dx}$$

for $t \geq \tau_{k-1}$ and $L_k(t) = 1$ for $t < \tau_{k-1}$. L_k is continuous if $h_{t_1, \dots, t_n}(x)$ is jointly continuous in (t_1, \dots, t_n) for fixed x . Assuming, without loss of generality, that $i = k \in R_{k-1}$, we have from (30) for the identity density

$$g_k(k, t)dt = P[\tau_k \in (t, t + dt), \tau_{k+1} > t, \dots, \tau_n > t | \sigma(\tau_1, \dots, \tau_{k-1})].$$

Using the structural definition of default and Bayes' rule, for $t \geq \tau_{k-1}$ we get

$$g_k(k, t) = \frac{\int_{(-\infty, 0]^n} \int_{x_{k+1}}^0 \cdots \int_{x_n}^0 h_{\tau_1, \dots, \tau_{k-1}, t, \dots, t}(x_1, \dots, x_k, y_{k+1}, \dots, y_n) g(x) dy dx}{\int_{(-\infty, 0]^n} \int_{x_k}^0 \cdots \int_{x_n}^0 h_{\tau_1, \dots, \tau_{k-1}, \dots, \tau_{k-1}}(x_1, \dots, x_{k-1}, y_k, \dots, y_n) g(x) dy dx}.$$

For planar Brownian motion, $h_{t,s}$ is calculated in Iyengar (1985); the general multi-dimensional case $n \geq 3$ has not yet been worked out, to our knowledge. If the threshold distribution is difficult to estimate, we therefore have to rely on the procedure for 'converting' estimated asset correlation into threshold correlation (Section 3.5) when we want to calculate L_k and q_k . We can then work under the hypothesis of correlated thresholds and independent assets. In this case we have simply

$$\begin{aligned} & \int_{x_k}^0 \cdots \int_{x_n}^0 h_{t_1, \dots, t_n}(x_1, \dots, x_{k-1}, y_k, \dots, y_n) dy_k \cdots dy_n \\ &= h_1(t_1, x_1) \cdots h_{k-1}(t_{k-1}, x_{k-1}) (1 - H_k(t_k, x_k)) \cdots (1 - H_n(t_n, x_n)), \end{aligned} \quad (32)$$

where $h_i(t, \cdot)$ is the density of M_t^i and $H_i(t, \cdot)$ is the associated distribution function. In the Brownian motion case these quantities are given in closed-form in (17) and (8), respectively. The efficiency of successive default time simulation hinges then only on the ability to perform an n -dimensional numerical integration.

As described in Section 3.5, once the joint threshold distribution has been fixed, we can simulate a vector D from the copula C_θ and the marginals G_i . Given D , we have for the survival function

$$L_k(t) = \frac{\int_{D_k}^0 \cdots \int_{D_n}^0 h_{\tau_1, \dots, \tau_{k-1}, t, \dots, t}(D_1, \dots, D_{k-1}, y_k, \dots, y_n) dy}{\int_{D_k}^0 \cdots \int_{D_n}^0 h_{\tau_1, \dots, \tau_{k-1}, \dots, \tau_{k-1}}(D_1, \dots, D_{k-1}, y_k, \dots, y_n) dy}$$

and the identity density (again with $i = k \in R_{k-1}$)

$$q_k(k, t) = \frac{\int_{D_{k+1}}^0 \cdots \int_{D_n}^0 h_{\tau_1, \dots, \tau_{k-1}, t, \dots, t}(D_1, \dots, D_k, y_{k+1}, \dots, y_n) dy}{\int_{D_k}^0 \cdots \int_{D_n}^0 h_{\tau_1, \dots, \tau_{k-1}, \dots, \tau_{k-1}}(D_1, \dots, D_{k-1}, y_k, \dots, y_n) dy}.$$

Successive default arrival time simulation using our general algorithm is very efficient if these functions are in closed-form. This is the case if we work under the hypothesis that assets are independent, i.e. if we induce default correlation exclusively through default threshold dependence (we then exploit (32)).

5 Conclusion

In this paper we describe an efficient algorithm for the simulation of successive correlated and unpredictable default arrival times. The methodology is based on the compensator of default. We construct this compensator explicitly in a structural multi-firm model, where we suppose that a firm's assets and default threshold cannot be observed directly. This is realistic and leads to defaults being unpredictable; the model-implied credit spread properties are empirically plausible. The structural interpretation of default greatly facilitates the calibration of the model, in particular with respect to the correlation structure. In particular, there are two ways of inducing default correlation: through correlation in firms' assets and default thresholds. The former can be thought of as arising from firms' common dependence on general economic factors. The latter corresponds to direct inter firm linkages, such as parent-subsidiary relationships.

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