Asymptotic Minimax Risk in the Uniform Norm for the White Noise Model on the Sphere *

Jussi Klemelä
Rolf Nevanlinna Institute
University of Helsinki

Abstract

Estimation of the signal function defined on the unit sphere of the Euclidean space is considered. Gaussian continuous time white noise model is supposed. Uniform norm is chosen as a loss function and exact asymptotic minimax risk is derived extending the result of Korostelev (1993). The exact asymptotic minimax risk is given also for the L_2-loss applying the result of Pinsker (1982).

Mathematics Subject Classifications: 62G07, 62C20, 62H11
Key Words and Phrases: asymptotic minimax risk, spherical data, uniform norm, white noise model.

1 Introduction

Let $S_d = \{x \in \mathbf{R}^{d+1} : \|x\| = 1\}$. Consider the observation of the form

$$dX(x) = f(x)dx + \epsilon dW(x), \;\; x \in S_d$$

where $\epsilon > 0$ is the noise size and $dW(x)$ is the Gaussian white noise in S_d, that is for measurable sets $B, B_1, B_2 \subset S_d$,

*This research was supported by Sonderforschungsbereich 373 “Quantifikation und Simulation Ökonomischer Prozesse”.

(i) $\mathcal{L}\left(\int_B dW(x)\right) = N(0, \mu(B))$ where μ is the Lebesgue measure of S_d,

(ii) if $B_1 \cap B_2 = \emptyset$, then $\int_{B_1} dW(x)$ and $\int_{B_2} dW(x)$ are independent.

The Lebesgue measure of the sphere can be defined with the help of the Lebesgue measure of $\mathbf{R}^{d+1}$, see Rudin (1986, page 175). It is a Borel measure, invariant with respect to rotations and $\mu(S_d) = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$.

Let $\Lambda(\beta, L)$ be the class of Hölder functions

$$\Lambda(\beta, L) = \left\{ f : M \to \mathbf{R} \;\middle|\; |f(x) - f(y)| \leq L\delta(x,y)^\beta,\ x, y \in S_d \right\}$$

where $0 < \beta < 1$ and δ is the Riemannian metric of S_d, that is, $\delta(x,y) = \arccos(x'y)$. We define the discrepancy of an arbitrary estimator $\hat{f}_\epsilon(x)$ and the true signal function $f(x)$ by the sup norm

$$\|\hat{f}_\epsilon - f\|_\infty = \sup_{x \in S_d} |\hat{f}_\epsilon(x) - f(x)|.$$

Let $w(u)$, $u \geq 0$, be a loss function, i. e. a continuous bounded monotone function of u, $w(0) = 0$. Introduce the minimax risk

$$r_\epsilon = r_\epsilon(w, \beta, L) = \inf_{\hat{f}_\epsilon} \sup_{f \in \Lambda(\beta, L)} E_{\epsilon, f} w\left(\psi_\epsilon^{-1} \left\| \hat{f}_\epsilon - f \right\|_\infty\right)$$

where $\psi_\epsilon = (\epsilon^2 \log \epsilon^{-2})^{\beta/(2\beta+d)}$.

Theorem 1 *Let* $A_{\beta,d} = \lim_{\kappa \to \infty} \|\kappa^{d/2} K_{\kappa,\eta}\|_2^{-2\beta/(2\beta+d)}$ *where* $K_{\kappa,\eta} : S_d \to \mathbf{R}$,

$$K_{\kappa,\eta}(x) = \left(1 - (\kappa\delta(x,\eta))^\beta\right)_+ \tag{1}$$

with $\kappa > 0$ *and* $\eta \in S_d$. *Let*

$$C_0 = L^{d/(2\beta+d)} A_{\beta,d} \left(\frac{2d}{2\beta + d}\right)^{\beta/(2\beta+d)}.$$

Then

$$\lim_{\epsilon \to 0} r_\epsilon = w(C_0).$$

The constant $A_{\beta,d}$ can be calculated explicitily,

$$\begin{aligned} \|\kappa^{d/2}K_{\kappa,\eta}\|_2^2 &= \kappa^d \mu_{d-1}(S_{d-1}) \int_0^{\pi} \left(1-(\kappa\theta)^\beta\right)_+^2 \sin^{d-1}\theta \, d\theta \\ &= \kappa^d \mu_{d-1}(S_{d-1}) \int_0^{\kappa\pi} \left(1-t^\beta\right)_+^2 \sin^{d-1}(\kappa^{-1}t)\kappa^{-1} dt \\ &\sim \mu_{d-1}(S_{d-1}) \int_0^1 \left(1-t^\beta\right)^2 t^{d-1} dt \qquad (2) \\ &= \mu_{d-1}(S_{d-1}) \frac{2\beta^2}{(2\beta+d)(\beta+d)d} \end{aligned}$$

where $\mu_{d-1}(S_{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$, $d \geq 2$ and we use the convention $\mu_0(S_0) = 2$. Thus,

$$\lim_{\epsilon\to 0} r_\epsilon = w\left(L^{d/(2\beta+d)}\left(\frac{(\beta+d)d^2}{\mu_{d-1}(S_{d-1})\beta^2}\right)^{\beta/(2\beta+d)}\right).$$

Korostelev (1993) proved corresponding result for estimating regression function defined on the unit interval from n samples $X_i = f(i/n) + \epsilon\xi_i$, $\xi_i \overset{i.i.d}{\sim} N(0,1)$. Donoho (1994) presented results for the continuous time white noise model on the real line, and gave also evaluation of the asymptotic minimax risk for estimating derivatives of the signal function.

Previously for instance Hall, Watson, and Cabrera (1987) have studied kernel density estimation on the sphere and Hendriks (1990) has studied Fourier series density estimates on Riemannian manifolds. However, exact minimax constants have not been previously given in the case of spherical data.

The proof of Theorem 1 is given in sections 2 and 3. The upper bound is proved in Section 2 using ideas presented in Korostelev (1993) and the lower bound is proved in Section 3 using ideas presented in Donoho (1994). In Section 4 exact asymptotic minimax risk is given when estimating signal function belonging to the spherical Sobolev ball and the L_2-loss is used. This extends the result in Pinsker (1982).

2 The Upper Bound

Let us define the estimator

$$\hat{f}(x) = \sum_{i=1}^{N} \hat{a}_{Ni} I_{A_{Ni}}(x)$$

where

$$\hat{a}_{Ni} = c(\kappa) \int_{S_d} K_{\kappa,\eta(N,i)}(x) X(dx),$$

$K_{\kappa,\eta}$ was defined in (1), $c(\kappa)^{-1} = \int_{S_d} K_{\kappa,\eta} d\mu$,

$$\kappa = \left(\frac{C_0 \psi_\epsilon}{L} \right)^{-1/\beta},$$

$$N = \lceil \chi^{-1} (\epsilon^2 \log \epsilon^{-2})^{-d/(2\beta+d)} \rceil = \lceil \chi^{-1} \psi_\epsilon^{-d/\beta} \rceil$$

where $\lceil b \rceil$ is the smallest integer $\geq b$,

$$\chi = \frac{1}{\mu(S_d)} \left(\frac{\gamma C_0 d}{L(2\beta + d)} \right)^{d/\beta}$$

where $\gamma > 0$ is arbitrary, $\eta(N,i) \in S_d$, $i = 1, \ldots, N$, are asymptotically equidistant on S_d, and A_{Ni}, $i = 1, \ldots, N$, is a partition of S_d such that A_{Ni} is the set of those $x \in S_d$ for which $\eta_{N,i}$ is the closest member of the set $\{\eta_{N,1}, \ldots, \eta_{N,N}\}$ (ties can be broken arbitrarily). Thus

$$A_{Ni} \subset \{x \in S_d : i = \text{argmin}_{i=1,\ldots,N} \delta(\eta(N,i), x)\}.$$

Let us start with the variation of the estimator.

Lemma 2

$$\lim_{\epsilon \to 0} P\left(\psi_\epsilon^{-1} \|\hat{f} - E\hat{f}\|_\infty > (1+\gamma) C_0 \frac{2\beta}{2\beta + d} \right) = 0.$$

Proof. Let us denote $Z_\epsilon(x) = \hat{f}(x) - E\hat{f}(x)$. Then

$$\begin{aligned} D_\epsilon^2 &\stackrel{def}{=} Var\left(\psi_\epsilon^{-1} Z_\epsilon(\eta(N,1))\right) = \psi_\epsilon^{-2} Var(\hat{a}_{N1}) \\ &= \psi_\epsilon^{-2} \epsilon^2 c(\kappa)^2 \int_{S_d} K^2_{\kappa,\eta(N,1)}(x) d\mu(x) \\ &\sim \psi_\epsilon^{-2} \epsilon^2 \kappa^d \mu_{d-1}(S_{d-1})^{-1} \frac{2d(\beta+d)}{2\beta+d} \end{aligned}$$

as is seen from Equation (2) and from

$$c(\kappa)^{-1} = \int_{S_d} \left(1 - (\kappa \delta(x,\eta))^\beta\right)_+ d\mu(x)$$

$$\begin{aligned}
&= \mu_{d-1}(S_{d-1}) \int_0^{\pi} \left(1-(\kappa\theta)^{\beta}\right)_+ \sin^{d-1}\theta \, d\theta \\
&= \mu_{d-1}(S_{d-1}) \int_0^{\kappa\pi} \left(1-t^{\beta}\right)_+ \sin^{d-1}(\kappa^{-1}t)\kappa^{-1} dt \\
&\sim \kappa^{-d}\mu_{d-1}(S_{d-1}) \int_0^{1} \left(1-t^{\beta}\right) t^{d-1} dt \qquad (3) \\
&= \kappa^{-d}\mu_{d-1}(S_{d-1}) \frac{\beta}{(\beta+d)d}
\end{aligned}$$

Let us denote $y = (1+\gamma)C_0 2\beta/(2\beta+d)$. Then it can be checked that

$$\frac{y^2}{D_\epsilon^2} = 2d(1+\gamma)^2 \log(\epsilon^{-2/(2\beta+d)})$$

and for sufficiently small ϵ,

$$\begin{aligned}
P\left(\|\psi_\epsilon^{-1} Z_\epsilon\|_\infty > y\right) &\leq P\left(\max_{i=1,\ldots,N} \psi_\epsilon^{-1} |Z_\epsilon(\eta(N,i))| > y\right) \\
&\leq NP\left(D_\epsilon^{-1}\psi_\epsilon^{-1}|Z_\epsilon(\eta(N,i))| > \frac{y}{D_\epsilon}\right) \\
&\leq N\exp\left\{-\frac{1}{2}\frac{y^2}{D_\epsilon^2}\right\} \\
&\leq N\exp\left\{-\frac{d(1+\gamma)^2\log\epsilon^{-2}}{2\beta+d}\right\} \\
&\leq \chi^{-1}(\log\epsilon^{-2})^{-d/(2\beta+d)}(\epsilon^2)^{d((1+\gamma)^2-1)/(2\beta+d)},
\end{aligned}$$

which approaches to 0 as $\epsilon \to 0$. □

Next, let us consider the bias of the estimator.

Lemma 3

$$\limsup_{\epsilon\to 0} \sup_{f\in\Lambda(\beta,L)} \psi_\epsilon^{-1}\|f - E\hat{f}\|_\infty \leq (1+\gamma)C_0\frac{d}{2\beta+d}.$$

Proof. We have that

$$\begin{aligned}
&\|E\hat{f} - f\|_\infty \\
&\leq \max_{i=1,\ldots,N} |E\hat{f}(\eta(N,i)) - f(\eta(N,i))| + L\max_i \min_{j\neq i} \delta(\eta(N,i),\eta(N,j))^{\beta}
\end{aligned}$$

Now

$$\limsup_{\epsilon\to 0} \max_i \min_{j\neq i} \delta(\eta(N,i),\eta(N,j)) \leq \limsup_{\epsilon\to 0} \left(\frac{\mu(S_d)}{N}\right)^{1/d}$$

and thus

$$\begin{aligned}
&\limsup_{\epsilon\to 0} \psi_\epsilon^{-1} \max_i \min_{j\neq i} \delta(\eta(N,i),\eta(N,j))^\beta \\
\leq\ & \limsup_{\epsilon\to 0} \psi_\epsilon^{-1} \left(\frac{\mu(S_d)}{N}\right)^{\beta/d} \leq (\mu(S_d)\chi)^{\beta/d} \leq \frac{\gamma C_0 d}{L(2\beta+d)}.
\end{aligned}$$

On the other hand, for $i = 1,\ldots,N$,

$$\begin{aligned}
&|E\hat{f}(\eta(N,i)) - f(\eta(N,i))| \\
=\ & |E\hat{a}_{N,i} - f(\eta(N,i))| \\
=\ & \left| c(\kappa)\int_{S_d} K_{\kappa,\eta(N,i)}(x) f(x) d\mu(x) - f(\eta(N,i))\right| \\
\leq\ & c(\kappa)\int_{S_d} K_{\kappa,\eta(N,i)}(x)\, |f(x) - f(\eta(N,i))|\, d\mu(x) \\
\leq\ & c(\kappa)L\int_{S_d} (1-(\kappa\delta(\eta(N,i),x))^\beta)_+ \delta(\eta(N,i),x)^\beta d\mu(x) \\
=\ & c(\kappa)L\mu_{d-1}(S_{d-1})\int_0^\pi \left(1-(\kappa\theta)^\beta\right)_+ \theta^\beta \sin^{d-1}\theta\, d\theta \\
=\ & c(\kappa)L\mu_{d-1}(S_{d-1})\int_0^{\kappa\pi} \left(1-t^\beta\right)_+ (\kappa^{-1}t)^\beta \sin^{d-1}(\kappa^{-1}t)\kappa^{-1} dt \\
\sim\ & c(\kappa)L\kappa^{-\beta-d}\mu_{d-1}(S_{d-1})\int_0^1 \left(1-t^\beta\right) t^{\beta+d-1} dt \\
=\ & c(\kappa)L\kappa^{-\beta-d}\mu_{d-1}(S_{d-1})\frac{\beta}{(2\beta+d)(\beta+d)} \\
\sim\ & \kappa^{-\beta}L\frac{d}{2\beta+d} = \psi_\epsilon C_0\frac{d}{2\beta+d}
\end{aligned}$$

where we used Equation (3). □

The upper bound follows from lemma 2 and lemma 3 by an similar argument as was given in Korostelev (1993).

3 The Lower Bound

We will give the proof for the lower bound. Let $J_{\kappa,\eta} : S_d \to \mathbf{R}$,

$$J_{\kappa,\eta}(x) = L\kappa^{-\beta}K_{\kappa,\eta}(x) = L\kappa^{-\beta}\left(1-(\kappa\delta(x,\eta))^\beta\right)_+$$

where $\kappa > 0$, $\eta \in S_d$. Let $N = N(\kappa)$ be the greatest integer such that there exists such $\eta(N,i) \in S_d$, $i = 1, \ldots, N$, that the functions $J_{\kappa,\eta(N,i)}$ have disjoint supports. Now we have that

$$\liminf_{\kappa\to\infty} \kappa^{-d} N(\kappa) \geq Const. \tag{4}$$

Let

$$\mathcal{C}(\kappa, \{\eta(N,i)\}) = \left\{ \sum_{i=1}^{N} \theta_i J_{\kappa,\eta(N,i)} : |\theta_i| \leq 1,\ i = 1, \ldots, N \right\}.$$

Now $\mathcal{C}(\kappa, \{\eta(N,i)\}) \subset \Sigma(\beta, L)$. The complete class of estimators for estimating $f \in \mathcal{C}(\kappa, \{\eta(N,i)\})$ consists of all procedures of the form

$$\hat{f}_\epsilon = \sum_{i=1}^{N} \hat{\theta}_i J_{\kappa,\eta(N,i)} \tag{5}$$

where $\hat{\theta}_i = \delta_i(y_1, \ldots, y_N)$, $i = 1, \ldots, N$, for measurable functions δ_i and

$$y_i = \int_{S_d} \xi_i(x) X(dx),\ i = 1, \ldots, N$$

with $\xi_i = J_{\kappa,\eta(N,i)}/\|J_{\kappa,\eta(N,i)}\|_2^2$.

When $\hat{f}_\epsilon$ is of the form (5) and $f \in \mathcal{C}(\kappa, \{\eta(N,i)\})$ then

$$\begin{aligned} \left\| \hat{f}_\epsilon - f \right\|_\infty &\geq \max_{i=1,\ldots,N} \left| \hat{f}_\epsilon(\eta(N,i)) - f(\eta(N,i)) \right| \\ &= \left| J_{\kappa,\eta(N,1)}(\eta(N,1)) \right| \left\| \hat{\theta} - \theta \right\|_{l_N^\infty} \\ &= L\kappa^{-\beta} \left\| \hat{\theta} - \theta \right\|_{l_N^\infty}. \end{aligned}$$

Thus

$$\begin{aligned} r_\epsilon &\geq \inf_{\hat{f}_\epsilon} \sup_{f \in \mathcal{C}(\kappa,\{\eta(N,i)\})} E_{\epsilon,f} w\left(\psi_\epsilon^{-1} \left\| \hat{f}_\epsilon - f \right\|_\infty \right) \\ &\geq \inf_{\hat{\theta}} \sup_{|\theta_i| \leq 1} E_{G(\theta,\sigma_N^2 I_N)} w\left(\psi_\epsilon^{-1} L\kappa^{-\beta} \left\| \hat{\theta} - \theta \right\|_{l_N^\infty} \right) \end{aligned}$$

where $G(\theta, \sigma_n^2 I_N)$ is the normal distribution with the expectation vector θ and the covariance matrix $\sigma_N^2 I_N$, where I_N is the $N \times N$ identity matrix and $\sigma_N^2 = Var(y_1) = \epsilon^2/\|J_{\kappa,\eta(N,1)}\|_2^2$.

By Korostelev's lemma (Korostelev 1993, Donoho 1994), if

$$\sigma_N^{-1} \leq \sqrt{2-\gamma}\sqrt{\log N}, \tag{6}$$

for some $0 < \gamma < 2$, then

$$\inf_{\hat\theta} \sup_{|\theta_i|\leq 1} E_{G(\theta,\sigma_N^2 I_N)} w\left(\left\|\hat\theta - \theta\right\|_{l_N^\infty}\right) \longrightarrow w(1), \text{ when } N \longrightarrow \infty.$$

Let $0 < \gamma < 2$,

$$C_0' = L^{d/(2\beta+d)} A_{\beta,d} \left(\frac{(2-\gamma)d}{2\beta+d}\right)^{\beta/(2\beta+d)},$$

and

$$\kappa = \left(\frac{C_0'\psi_\epsilon}{L}\right)^{-1/\beta} = \left(L^{d/\beta} A_{\beta,d}^{(2\beta+d)/\beta} \frac{(2-\gamma)d}{2\beta+d} \epsilon^2 \log \epsilon^{-2}\right)^{-1/(2\beta+d)}.$$

Now

$$\begin{aligned}
\sigma_N^{-1} &= \epsilon^{-1} \left\|J_{\kappa,\eta(N,1)}\right\|_2 \\
&= \epsilon^{-1} \left\|L\kappa^{-\beta} K_{\kappa,\eta(N,1)}\right\|_2 \\
&= \epsilon^{-1} L\kappa^{-\beta-d/2} \left\|\kappa^{d/2} K_{\kappa,\eta(N,1)}\right\|_2 \\
&\sim \epsilon^{-1} L\kappa^{-\beta-d/2} A_{\beta,d}^{-(2\beta+d)/(2\beta)} \\
&= \left(\frac{(2-\gamma)d}{2\beta+d} \log \epsilon^{-2}\right)^{1/2} \\
&\leq (2-\gamma)^{1/2} \left(\log\left(\epsilon^2 \log \epsilon^{-2}\right)^{-d/(2\beta+d)}\right)^{1/2} \\
&= (2-\gamma)^{1/2} \sqrt{\log(Const. \times \kappa^d)}.
\end{aligned}$$

Thus (6) is satisfied ultimately as $\epsilon \to 0$ by Equation (4).

The lower bound follows by Korostelev's lemma because

$$\psi_\epsilon^{-1} L\kappa^{-\beta} \sim C_0' \sim L^{d/(2\beta+d)} A_{\beta,d} \left(\frac{(2-\gamma)d}{2\beta+d}\right)^{\beta/(2\beta+d)}$$

and $0 < \gamma < 2$ was chosen arbitrarily. □

4 Estimation with the L_2-loss

Let us also consider estimation with the L_2-loss assuming that the signal function belongs to the Sobolev ball. We will give exact asymptotic minimax constant applying the result of Pinsker (1982) to the spherical case.

Let $H_s(S_d)$, $s > 0$, be the Sobolev space of S_d. Then $f \in L_2(S_d, \mu)$ is in $H_s(S_d)$ if and only if

$$\|f\|_s^2 \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \sigma_k^s \sum_{f_j \in E_k} < f, f_j >^2_{L_2(S_d,\mu)} < \infty, \tag{7}$$

where $(\sigma_k)_{k=0}^{\infty}$ is the set of different eigenvalues of the Laplace-Beltrami operator Δ in increasing order, $(E_k)_{k=0}^{\infty}$ is the set of eigenspaces of Δ (E_k corresponds to σ_k for each k), and $(f_j)_{j=0}^{\infty}$ is an orthonormal basis of $L_2(S_d, \mu)$ consisting of real eigenfunctions of Δ (Giné, 1975).

Let $\Sigma(s, M) = \{f \in L_2(S_d, \mu) : \|f\|_s^2 \leq M\}$, where $s > 0$ and $\|\cdot\|_s$ is defined in (7). Define

$$R_\epsilon(\Sigma(s, M)) = \inf_{\hat{f}} \sup_{f \in \Sigma(s,M)} E_{\epsilon,f} \left\| \hat{f} - f \right\|^2_{L_2(S_d,\mu)}.$$

Now we can state the following theorem.

Theorem 4

$$R_\epsilon(\Sigma(s, M)) \sim \epsilon^{4s/(2s+d)} M^{d/(2s+d)} P_{s,d}, \quad \epsilon \longrightarrow 0,$$

where

$$P_{s,d} = \left(\frac{2s}{(s+d)(d-1)!} \right)^{2s/(2s+d)} \frac{(2s+d)^{d/(2s+d)}}{d}.$$

The proof of the theorem follows from Pinsker (1980, Theorem 1). In fact, consider the observations

$$y_{k,i} = \theta_{k,i} + \epsilon \xi_{k,i}, \; k = 1, 2, \ldots, \; i = 1, \ldots, \tau_k,$$

where $\tau_k \geq 1$ are integers, $\xi_{k,i}$ are i. i. d. $N(0, 1)$, $\epsilon > 0$ is the noise size, and the sequence $\theta = (\theta_{k,i})$ is in l_2. Consider the parameter space

$$\Theta = \left\{ \theta \in l_2 : \sum_{k=1}^{\infty} a_k \sum_{i=1}^{\tau_k} \theta_{k,i}^2 \leq M \right\}$$

where $a_k \to \infty$ as $k \to \infty$. Define

$$R_\epsilon^*(\Theta) = \inf_{\hat{\theta}} \sup_{\theta \in \Theta} E_{\epsilon,\theta} \left\| \hat{\theta} - \theta \right\|^2_{l_2}.$$

From Pinsker (1980, Theorem 1) it follows that

$$R^*_\epsilon(\Theta) \sim \omega^2 \quad (\epsilon \longrightarrow 0)$$

where

$$\omega^2 = \epsilon^2 \sum_{\mu_\epsilon a_k < 1} \tau_k \left(1 - (\mu_\epsilon a_k)^{1/2}\right) \tag{8}$$

and μ_ϵ is the unique solution of

$$\epsilon^2 \sum_{\mu_\epsilon a_k < 1} \tau_k a_k \left((\mu_\epsilon a_k)^{-1/2} - 1\right) = M. \tag{9}$$

If one chooses $a_k = \sigma_k^s$ and $\tau_k = \dim E_k^{(d)}$, where σ_k are the eigenvalues and $E_k^{(d)}$ are the eigenspaces of the Laplace-Beltrami operator, respectively, then it holds that

$$R_\epsilon(\Sigma(s, M)) = R^*_\epsilon(\Theta).$$

For the circle S_1 it is known that when $k \geq 1$, $\sigma_k = k^2$, $\dim E_k = 2$, and orthonormal basis of E_k is $\{\pi^{-1/2}\cos k\theta, \pi^{-1/2}\sin k\theta\}$. For the sphere S_2 it is known that $\sigma_k = k(k+1)$ and $\dim E_k = 2k+1$, when $k \geq 1$. For the d-dimensional sphere S_d it is known that $\sigma_k = k(k+d-1)$ and

$$\begin{aligned} \dim E_k^{(d)} &= \text{card}\left\{(l_2, \ldots, l_d) \in \mathcal{Z}_+^{d-1} : k \geq l_2 \geq l_3 \geq \cdots \geq l_{d-1} \geq |l_d|\right\} \\ &= \frac{(2k+d-1)(k+d-2)!}{k!(d-1)!} \end{aligned}$$

(Takeuchi, 1994, pages 216, 223). It can be seen that

$$\dim E_k^{(d)} = \sum_{i=0}^{k} \dim E_i^{(d-1)} = \sum_{i=0}^{d-1} \alpha_i k^i,$$

where α_i are integers and $\alpha_{d-1} = 2/(d-1)!$.

Thus Theorem 4 follows by applying (8) if we choose $a_k = [k(k+d-1)]^s$ and $\tau_k = \sum_{i=0}^{d-1} \alpha_i k^i$, where α_i are integers and $\alpha_{d-1} = 2/(d-1)!$.

References

[1] Giné, E. M. (1975). Invariant tests for uniformity on compact Riemannian manifolds based on Sobolev norms. *Ann. Statist.* **3** 1243–1266.

[2] Donoho, D. L. (1994). Asymptotic minimax risk for sup-norm loss: Solution via optimal recovery. *Probab. Theory Relat. Fields* **99** 145–170

[3] Hall, P., Watson, G. S., and Cabrera, J. (1987). Kernel density estimation with spherical data. *Biometrika* **74** 751–62.

[4] Hendriks, H. (1990). Nonparametric estimation of a probability density on a Riemannian manifold using Fourier expansions. *Ann. Statist.* **18** 832–849.

[5] Korostelev, A. P. (1993). Exact asymptotically minimax estimator for nonparametric regression in uniform norm. *Theory Probab. Appl.* **38** 775–782.

[6] Pinsker, M. S. (1980). Optimal filtering of square integrable signals in Gaussian white noise. *Problems Inform. Transmission* **16** 52–68.

[7] Takeuchi, M. (1994). *Modern Spherical Functions.* Translations of Mathematical Monographs Vol. 135. American Mathematical Society, Providence, Rhode Island.