LARGE SAMPLE THEORY OF THE ESTIMATION OF THE ERROR DISTRIBUTION FOR A SEMIPARAMETRIC MODEL *

Hua Liang[a] Wolfgang Härdle[b]
Institut für Statistik und Ökonometrie
Humboldt-Universität zu Berlin
D-10178 Berlin, Germany [a,b]
Institute of Systems Science, Beijing 100080, China [a]

Abstract

The paper studies large sample theory of estimators of the error distribution for the semiparametric model $Y = X^\tau\beta + g(T) + \varepsilon$. Under appropriate conditions, we prove that the estimators converge in probability, almost surely converge and uniformly almost surely converge. Asymptotic normality and the rates of convergence of the estimators are also investigated. Finally we establish the law of the iterated logarithm for the estimators.

Key Words and Phrases: *Weak, strong consistency; uniformly strong consistency; rates of convergence; asymptotic normality; law of the iterated logarithm; semiparametric model.*

1 INTRODUCTION

Consider the model given by

$$Y_i = X_i^\tau\beta + g(T_i) + \varepsilon_i, i = 1, \ldots, \tag{1}$$

where $X_i = (x_{i1}, \ldots, x_{ip})^\tau (p \geq 1)$ and $T_i(T_i \in [0, 1])$ are known fixed design points, $\beta = (\beta_1, \ldots, \beta_p)^\tau$ is an unknown parameter vector and g is an unknown function, and $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. random variables with a common unknown density function $f(u)$, and mean 0 and finite variance σ^2. The model was introduced by Engle, et al. (1986) to study the effect of weather or electricity demand. More recent work dealt with the estimation of β at a parametric rate. Chen

*The research for both authors was supported by Sonderforschungsbereich 373 "Quantifikation und Simulation Ökonomischer Prozesse". The first author was also supported by Alexander von Humboldt Foundation .

(1988), Chen and Shiau (1991), Heckman (1986, 1988), Robinson (1988), Schick (1996) and Speckman (1988) constructed $\sqrt{n}-$consistent estimates of β under the nonsingularity of the matrix $E[\{X_1 - E(X_1|T_1)\}\{X_1 - E(X_1|T_1)\}^\tau]$ which guarantees the identifiability of the parameter β, under various conditions on $E(X_1|T_1)$ and $g(\bullet)$ when (X_i, T_i) are random design points. Cuzick (1992a) constructed efficient estimates of β when the error density is known and has finite Fisher information. The same problem was solved by Cuzick (1992b) and Schick (1993) when the error distribution is unknown.

In this paper, the estimators of $f(u)$, $\widehat{f}_n(u)$, are obtained by using nonparametric regression to approximate $g(t)$. Under appropriate conditions, we prove that $\widehat{f}_n(u)$ converges in probability, almost surely converges and uniformly almost surely converges. Then we consider asymptotic normality and the convergence rates of $\widehat{f}_n(u)$. Finally we establish the law of the iterated logarithm for $\widehat{f}_n(u)$.

The paper is organized as follows. In the following we give the assumptions on the X_i and T_i. Section 2 lists some lemmas. Section 3 proves that $\widehat{f}_n(u)$ converges in probability, almost surely converges and uniformly almost surely converges. Section 4 gives the convergence rates of $\widehat{f}_n(u)$. Section 5 obtains asymptotic normality and the law of the iterated logarithm. For the convenience and simplicity, we shall employ $C(0 < C < \infty)$ to denote some constant not depending on n but may assume different values at each appearance.

Assume $\{X_i = (x_{i1}, \ldots, x_{ip})^\tau, T_i, Y_i, i = 1, \ldots, n.\}$ satisfy model (1). Let $W_{ni}(t) = W_{ni}(t; T_1, \ldots, T_n)$ be probability weight functions depending only on the design points $T_1, \ldots, T_n$. Denote $\widetilde{\mathbf{X}}^\tau = (\widetilde{X}_1, \ldots, \widetilde{X}_n)$, $\widetilde{X}_i = X_i - \sum_{j=1}^n W_{nj}(T_i)X_j$, and $\widetilde{\mathbf{Y}} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_n)^\tau$, $\widetilde{Y}_i = Y_i - \sum_{j=1}^n W_{nj}(T_i)Y_j$.

If β were known, we could take $g_n(t) = \sum_{j=1}^n W_{nj}(t)(Y_j - X_j^\tau\beta)$ as the estimator of $g(t)$. Generally we can take $W_{nj}(t)$ as Nadaraya-Watson kernel. Surveys of nonparametric methods can be found in Härdle (1990), which gives extensive discussions of various statistical estimation. Based on the modified model $\{Y_i = X_i^\tau\beta + g_n(T_i) + \varepsilon_i, \quad i = 1, \ldots, n\}$ with $g_n(t)$, we get the estimator $\widehat{\beta}_n$ of β

$$\widehat{\beta}_n = (\widetilde{\mathbf{X}}^\tau\widetilde{\mathbf{X}})^{-1}\widetilde{\mathbf{X}}^\tau\widetilde{\mathbf{Y}}$$

and "true" estimator $\widehat{g}_n(t)$ of $g(t)$

$$\widehat{g}_n(t) = \sum_{i=1}^n W_{nj}(t)(Y_i - X_i^\tau\widehat{\beta}_n)^2$$

Set $\widehat{\varepsilon}_i = Y_i - X_i^\tau \widehat{\beta}_n - \widehat{g}_n(T_i)$ for $i = 1, \ldots, n$. Define the estimators of $f(u)$ as follows,

$$\widehat{f}_n(u) = \frac{1}{2na_n} \sum_{i=1}^{n} I_{(u - a_n \le \widehat{\varepsilon}_i \le u + a_n)}, \quad u \in R^1 \tag{2}$$

where $a_n (> 0)$ is a bandwidth, and I_A denotes the indicator function of the set A.

In the following we list the sufficient conditions for our main result.

Condition 1. *There exist functions $h_j(\cdot)$ defined on $[0, 1]$ such that*

$$x_{ij} = h_j(T_i) + u_{ij} \quad 1 \le i \le n, \quad 1 \le j \le p$$

where u_{ij} is a sequence of real numbers such that

$$B = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} u_i u_i^\tau$$

is a positive matrix, and

$$\lim_{n \to \infty} n^{-1/2} \log^{-1} n \left| \sum_{i=1}^{n} u_{ij} \right| < \infty \quad \textit{holds for all } 1 \le j \le p,$$

for $u_i = (u_{i1}, \ldots, u_{ip})^\tau$.

Condition 2. *$g(\cdot)$ and $h_j(\cdot)$ are Lipschitz continuous of order 1.*

Condition 3. *Weight functions $W_{ni}(\cdot)$ satisfy:*

$$\begin{aligned}
&(i) && \max_{1 \le i \le n} \sum_{j=1}^{n} W_{ni}(T_j) = O(1), \\
&(ii) && \sup_t \max_{1 \le i \le n} W_{ni}(t) = O(h_n), \quad h_n = n^{-2/3}, \\
&(iii && \sup_t \sum_{j=1}^{n} W_{nj}(t) I_{(|T_j - t| > c_n)} = O(c_n), \quad c_n = n^{-1/2}, \\
&(iv) && \max_{1 \le i \le n} |W_{ni}(t) - W_{ni}(s)| \le M_0 |s - t| \quad \textit{for } s, t \in [0, 1] \\
& && \qquad\qquad\qquad \textit{and some positive number } M_0 > 0.
\end{aligned}$$

Remark. Here we give two weight functions that satisfy Condition 3 to demonstrate the reasonability of Condition 3.

$$W_{ni}^{(1)}(t) = \frac{1}{h_n} \int_{s_{i-1}}^{s_i} K\Big(\frac{t - s}{h_n}\Big) ds, \text{ or } \quad W_{ni}^{(2)}(t) = K\Big(\frac{t - T_i}{h_n}\Big) \Big/ \sum_{j=1}^{n} K\Big(\frac{t - T_j}{h_n}\Big),$$

where $s_i = \frac{1}{2}(T_i + T_{i-1})$, $i = 1, \ldots, n-1$, $s_0 = 0$, $s_n = 1$. $K(\cdot)$ is a Parzen-Rosenblatt kernel function, h_n is a bandwidth parameter.

2 SOME LEMMAS

In this section, we list some lemmas proved which are used in the following for proving the main results. First we give an exponential inequality for bounded independent random variables.

Lemma 1 (Bernstein inequality). *Let $Z_1, \ldots, Z_n$ be independent r.v's satisfying $P\{|Z_i| \leq m\} = 1$, each i, where $m < \infty$. Then, for $\eta > 0$,*

$$P\Big\{\Big|\sum_{i=1}^{n} Z_i\Big| \geq n\eta\Big\} \leq 2\exp\Big\{-n^2\eta^2/[2\sum var(Z_i) + \frac{2}{3}mn\eta]\Big\}$$

for all $n = 1, \ldots$.

Lemma 2 (Gao, et al. (1995)). *Suppose Conditions 1-3 hold. If $E|\varepsilon_1|^3 < \infty$ and $\max_i \sum_{j=1}^{p} u_{ij}^2 \leq C_0 < \infty$. Then*

$$\sup_t |\widehat{g}_n(t) - g(t)| = O_p(n^{-1/3}\log n),$$

and

$$\limsup_{n\to\infty} \Big(\frac{n}{\log\log n}\Big)^{1/2} \|\widehat{\beta}_n - \beta\| < \infty \quad a.s.$$

Here and below we denote Euclidean norm by $\|\cdot\|$.

Lemma 3 (See Devroye, et al. (1980)) *Let μ_n and μ be 1-dimensional empirical distribution and theoretical distribution, respectively, $a > 0$ and Ia be an interval with length a. Then for any $\zeta > 0$, $0 < b \leq 1/4$ and $n \geq \max\{1/b, 8b/\zeta^2\}$,*

$$\begin{aligned} P\Big(\sup\{|\mu_n(Ia) - \mu(Ia)| : 0 < \mu(Ia) \leq b\} \geq \zeta\Big) &\leq 16n^2\exp\{-n\zeta^2/(64b+4\zeta)\} \\ &\quad +8n\exp\{-nb/10\}. \end{aligned}$$

3 CONSISTENCY

In this section we shall prove that $\widehat{f}_n(u)$ converges in probability, almost surely converges and uniformly almost surely converges. Below we always denote

$$f_n(u) = \frac{1}{2na_n}\sum_{i=1}^{n} I_{(u-a_{\leq}\varepsilon_i \leq u+a_n)},$$

for fixed $u \in C(f)$, where $C(f)$ in the set of continuous points of f.

Theorem 3.1. *There exists a $M > 0$ such that $\|X_i\| \leq M$ for $i = 1 \sim n$. Under the assumptions of lemma 2. If*

$$0 < a_n \to 0, \quad n^{1/3}a_n\log^{-1} n \to \infty.$$

Then $\widehat{f}_n(u) \to f(u)$ *in probability as* $n \to \infty$.

Proof. Simply calculation shows that the mean of $f_n(u)$ converges to $f(u)$, and its variance does to 0. This implies that $f_n(u) \to f(u)$ in probability as $n \to \infty$.

Now, we prove $\widehat{f}_n(u) - f_n(u) \to 0$ in probability.

If $\varepsilon_i < u - a_n$, then $\widehat{\varepsilon}_i \in (u - a_n, u + a_n)$ implies that $u - a_n + X_i^\tau(\widehat{\beta}_n - \beta) + (\widehat{g}_n(T_i) - g(T_i)) < \varepsilon_i < u - a_n$. If $\varepsilon_i > u + a_n$, then $\widehat{\varepsilon}_i \in (u - a_n, u + a_n)$ implies that $u + a_n < \varepsilon_i < u + a_n + X_i^\tau(\widehat{\beta}_n - \beta) + (\widehat{g}_n(T_i) - g(T_i))$. Write

$$C_{ni} = X_i^\tau(\widehat{\beta}_n - \beta) + (\widehat{g}_n(T_i) - g(T_i)) \quad \text{for } i = 1, \ldots, n.$$

It follows from lemma 2 that, for any $\zeta > 0$, there exists a $\eta_0 > 0$ such that

$$P\{n^{1/3} \log^{-1} n \sup_i |C_{ni}| > \eta_0\} \le \zeta$$

The above arguments yield that

$$\begin{aligned} |\widehat{f}_n(u) - f_n(u)| &\le \frac{1}{2na_n} I_{(u \pm a_n - |C_{ni}| \le \varepsilon_i \le u \pm a_n)} + \frac{1}{2na_n} I_{(u \pm a_n \le \varepsilon_i \le u \pm a_n + |C_{ni}|)} \\ &\stackrel{\text{def}}{=} I_{1n} + I_{2n}, \end{aligned}$$

where

$$I_{(u \pm a_n - |C_{ni}| \le \varepsilon_i \le u \pm a_n)} = I_{(u + a_n - |C_{ni}| \le \varepsilon_i \le u + a_n) \cup (u - a_n - |C_{ni}| \le \varepsilon_i \le u - a_n)}.$$

We complete the proof of the theorem by dealing with I_{1n} and I_{2n}. For any $\zeta' > 0$ and large enough n,

$$\begin{aligned} P\{I_{1n} > \zeta'\} &\le \zeta + P\{I_{1n} > \zeta', \sup_i |C_{ni}| \le \eta_0\} \\ &\le \zeta + P\Big(\sum_{i=1}^n I_{(u \pm a_n - C\eta_0 n^{-1/3} \log n \le \varepsilon_i \le u \pm a_n)} \ge 2na_n\zeta'\Big) \end{aligned}$$

According to the continuity of f on u, using Chebyshev's inequality we know the second term above is less than

$$\frac{1}{2a_n\zeta'} P\Big(u \pm a_n - C\eta_0 n^{-1/3} \log n \le \varepsilon_i \le u \pm a_n\Big) = C\eta_0 \log n \Big/ (2\zeta' n^{1/3} a_n)(f(u) + o(1)).$$

It follows from $a_n n^{1/3} \log^{-1} n \to \infty$ that

$$\limsup_{n \to \infty} P\{I_{1n} > \zeta'\} \le \zeta.$$

Since ζ is arbitrary, we obtain $I_{1n} \to 0$ in probability as $n \to \infty$. We can similarly prove that I_{2n} tends to zero in probability as $n \to \infty$. Thus, we complete the proof of Theorem 3.1.

Theorem 3.2 *Under the assumptions of Theorem 3.1. If*

$$0 < a_n \to 0, \quad n^{1/3} a_n \log^{-2} n \to \infty. \tag{3}$$

Then $\widehat{f}_n(u) \to f(u)$ *for* $u \in C(f)$ *a.s. as* $n \to \infty$.

Proof. Set $f_n^E(u) = Ef_n(u)$, for $u \in C(f)$. Using the continuity of f on u and $a_n \to 0$ we can show that

$$f_n^E(u) \to f(u) \quad \text{as } n \to \infty \tag{4}$$

Now let us consider $f_n(u) - f_n^E(u)$.

$$\begin{aligned} f_n(u) - f_n^E(u) &= \frac{1}{2na_n} \sum_{i=1}^{n} \Big\{ I_{(u-a_n \le \varepsilon_i \le u+a_n)} - EI_{(u-a_n \le \varepsilon_i \le u+a_n)} \Big\} \\ &\overset{\text{def}}{=} \frac{1}{2na_n} \sum_{i=1}^{n} U_{ni}. \end{aligned}$$

Then $U_{n1}, \ldots, U_{nn}$ are independent with $EU_{ni} = 0$, and $|U_{ni}| \le 1$, moreover

$$\operatorname{var}(U_{ni}) \le P(u - a_n \le \varepsilon_i \le u + a_n) = 2a_n f(u)(1 + o(1)) \le 4a_n f(u),$$

for large enough n. It follows from lemma 1 that, for any $\zeta > 0$,

$$\begin{aligned} P\{|f_n(u) - f_n^E(u)| \ge \zeta\} &= P\{|\sum_{i=1}^{n} U_{ni}| \ge 2na_n\zeta\} \\ &\le 2\exp\{-4n^2 a_n^2 \zeta^2 \Big/ [8na_n f(u) + 4/3na_n\zeta]\} \\ &= 2\exp\{-3na_n\zeta^2 \Big/ [6f(u) + \zeta]\}. \end{aligned} \tag{5}$$

Condition (3) and Borel-Cantelli lemma imply

$$f_n(u) - f_n^E(u) \to 0 \quad a.s. \tag{6}$$

In the following, we shall prove

$$\widehat{f}_n(u) - f_n(u) \to 0 \quad a.s. \tag{7}$$

According to lemma 2, we have with probability one that

$$\begin{aligned} |\widehat{f}_n(u) - f_n(u)| &\le \frac{1}{2na_n} I_{(u \pm a_n - Cn^{-1/3}\log n \le \varepsilon_i \le u \pm a_n)} + \frac{1}{2na_n} I_{(u \pm a_n \le \varepsilon_i \le u \pm a_n + Cn^{-1/3}\log n)} \\ &\overset{\text{def}}{=} J_{1n} + J_{2n}. \end{aligned} \tag{8}$$

Denote

$$f_{n1}(u) = \frac{1}{2a_n} P(u \pm a_n - Cn^{-1/3}\log n \le \varepsilon_i \le u \pm a_n). \tag{9}$$

Then $f_{n1}(u) \le Cf(u)(n^{1/3}a_n)^{-1}\log n$, for large enough n. By the condition (3), we obtain

$$f_{n1}(u) \to 0, \quad \text{as } n \to \infty. \tag{10}$$

Now let us deal with $J_{n1} - f_{n1}(u)$. Set

$$Q_{ni} = I_{(u\pm a_n - Cn^{-1/3}\log n \le \varepsilon_i \le u \pm a_n)} - P(u \pm a_n - Cn^{-1/3}\log n \le \varepsilon_i \le u \pm a_n),$$

for $i = 1, \ldots, n$. Then $Q_{n1}, \ldots, Q_{nn}$ are independent, and $|Q_{ni}| \le 1$, $EQ_{ni} = 0$, and

$$\mathrm{Var}(Q_{ni}) \le 2Cn^{-1/3}(\log n)f(u)$$

By lemma 1, we have

$$\begin{aligned} P\{|J_{n1} - f_{n1}(u)| > \zeta\} &= P\{|\sum_{i=1}^{n} Q_{ni}| > \zeta\} \\ &\le 2\exp\{-Cna_n\zeta^2 \Big/ (n^{-1/3}a_n^{-1}f(u)\log^{-1} n + \zeta)\} \\ &\le 2\exp\{-Cna_n\zeta\}. \end{aligned} \tag{11}$$

Employing Borel-Cantelli lemma we conclude that

$$J_{n1} - f_{n1}(u) \to 0 \quad a.s.$$

Combining (10) with the above conclusion, we obtain $J_{n1} \to 0$ a.s. Similar argument yields $J_{n2} \to 0$ a.s. Moreover, (8) implies (7). From (4), (6) and (7), we complete the proof of Theorem 3.2.

Theorem 3.3. *Under the assumptions of Theorem 3.2. If f is uniformly continuous on R^1 and*

$$0 < a_n \to 0, \quad n^{1/3}a_n \log^{-2} n \to \infty. \tag{12}$$

Then $\sup_u |\widehat{f}_n(u) - f(u)| \to 0$ *a.s.*

Proof. We still use the notations in the proof of Theorem 3.2 to denote the empirical distribution of $\varepsilon_1, \ldots, \varepsilon_n$ by μ_n and the distribution of ε_1 by μ. Since f is uniformly continuous, thus $\sup_u f(u) = f_0 < \infty$. It is easy to show

$$\sup_u |f(u) - f_n^E(u)| \to 0 \quad \text{as } n \to \infty \tag{13}$$

Write

$$f_n(u) - f_n^E(u) = \frac{1}{2a_n}\{\mu_n([u-a_n, u+a_n]) - \mu([u-a_n, u+a_n])\}$$

and denote $b_n^* = 2f_0 a_n$, $\zeta_n = 2a_n\zeta$ for any $\zeta > 0$. Then for large enough n, $0 < b_n^* < 1/4$ and $\sup_u \mu([u-a_n, u+a_n]) \le b_n^*$ for all n. From lemma 3, we have, for large enough n,

$$\begin{aligned} P\{\sup_u |f_n(u) - f_n^E(u)| \ge \zeta\} &= P\{\sup_u |\mu_n([u-a_n, u+a_n]) - \mu([u-a_n, u+a_n])| \ge 2a_n\zeta\} \\ &\le 16n^2 \exp\{-na_n^2\zeta^2 \big/ (32f_0 a_n + 2a_n\zeta)\} + 8n\exp\{-na_n^2 f_0/5\}. \end{aligned}$$

From (12) and Borel-Cantelli lemma, it follows that

$$\sup_u |f_n(u) - f_n^E(u)| \to 0 \quad \text{a.s.} \tag{14}$$

Combining (14) with (13), we obtain

$$\sup_u |f_n(u) - f(u)| \to 0 \quad \text{a.s.} \tag{15}$$

In the following we shall prove that

$$\sup_u |\widehat{f}_n(u) - f_n(u)| \to 0 \quad \text{a.s.} \tag{16}$$

It is obvious that $\sup_u |f_{n1}(u)| \to 0$, as $n \to \infty$. Set $d_n = f_0 n^{-1/3}\log n$. For large enough n, we have $0 < d_n < 1/4$ and

$$\sup_u \mu\{(u \pm a_n - Cn^{-1/3}\log n, u \pm a_n)\} \le Cd_n \text{ for all } n.$$

It follows from lemma 3 that

$$\begin{aligned} P(\sup_u |J_{n1} - f_{n1}(u)| > \zeta) &\le P[|\mu_n\{(u \pm a_n - Cn^{-1/3}\log n, u \pm a_n)\} \\ &\quad -\mu\{(u \pm a_n - Cn^{-1/3}\log n, u \pm a_n)\}| \ge 2a_n\zeta] \\ &\le 16n^2 \exp\Big(-\frac{4na_n^2\zeta^2}{64f_0 n^{-1/3}\log n + 8a_n\zeta}\Big) + 8n\exp(-n^{2/3}\log n/10). \end{aligned}$$

By (12) and the above arguments, it follows that $\sup_u |J_{n1} - f_{n1}(u)| \to 0$ a.s., and hence $\sup_u |J_{n1}| \to 0$ a.s. We have $\sup_u |J_{n2}| \to 0$ similarly. In the proof of Theorem 3.2, it can be shown that, with probability one and for large enough n,

$$\sup_u |\widehat{f}_n(u) - f_n(u)| \le \sup_u |J_{n1}| + \sup_u |J_{n2}|.$$

This implies (16), and so does the conclusion of Theorem 3.3.

4 CONVERGENCE RATE

Theorem 4.1. *Under the assumptions of Theorem 3.2. If f is locally Lipschitz continuous of order 1 on u. Then for $a_n = n^{-1/6}\log^{1/2} n$,*

$$\widehat{f}_n(u) - f(u) = O(n^{-1/6}\log^{1/2} n). \quad \text{a.s} \tag{17}$$

Proof. The proof is completely analogous to Theorem 3.2. By the assumption of Theorem 4.1, there exist $c_0 > 0$ and $\delta_1 = \delta_1(u) > 0$ such that $u' \in (u-\delta_1, u+\delta_1)$ implying $|f(u')-f(u)| \le c_0|u'-u|$. Hence for large enough n,

$$\begin{aligned} |f_n^E(u) - f(u)| &\le \frac{1}{2a_n}\int_{u-a_n}^{u+a_n} |f(u) - f(u')|du' \\ &\le c_0 a_n/2 = O(n^{-1/6}\log^{1/2} n). \end{aligned} \tag{18}$$

Since f is bounded on $(u-\delta_1, u+\delta_1)$, we have, for large enough n, that

$$\begin{aligned} f_{n1}(u) &= \frac{1}{2a_n}P(u \pm a_n - Cn^{-1/3}\log n \le \varepsilon_i \le u \pm a_n) \\ &\le Cn^{-1/3}a_n^{-1}\log n \sup_{u'\in(u-\delta_1,u+\delta_1)} f(u') \\ &= O(n^{-1/6}\log^{1/2} n). \end{aligned}$$

Replacing ζ by $\zeta_n = \zeta n^{-1/6}\log^{1/2} n$ in (5), then for large enough n,

$$P(|f_n(u) - f_n^E(u)| \ge 2\zeta n^{-1/6}\log^{1/2} n) \le 2\exp\{-3n^{1/2}\log^{3/2} n\zeta \big/ (6f_0 + \zeta)\},$$

here $f_0 = \sup_{u'\in(u-\delta_1,u+\delta_1)} f(u')$. Instead of (15), we have

$$f_n(u) - f_n^E(u) = O(n^{-1/6}\log^{1/2} n). \quad \text{a.s.} \tag{19}$$

The similar argument as (11) yields

$$P\{|J_{n1} - f_{n1}(u)| > \zeta n^{-1/6}\log^{1/2} n\} \quad \le \quad 2\exp(-Cn^{2/3}\log^{1/2} n).$$

Hence, $J_{n1} - f_{n1}(u) = O(n^{-1/6}\log^{1/2} n)$ a.s. (18) and (19) imply that we have proved

$$f_n(u) - f(u) = O(n^{-1/6}\log^{1/2} n). \quad \text{a.s.}$$

Using the arguments below (8) in the proof of Theorem 3.2, the proof is completed.

5 ASYMPTOTIC NORMALITY AND LAW OF THE ITERATED LOGARITHM

Theorem 5.1. *Under the assumptions of Theorem 3.2. If f is locally Lipschitz continuous of order 1 on u and*

$$0 < na_n^3 \to 0, \quad n^{5/12} a_n \log^{-1} n \to \infty.$$

Then

$$\sqrt{2na_n/f(u)}\{\widehat{f}_n(u) - f(u)\} \to N(0,1) \quad \textit{in distribution as } n \to \infty.$$

Theorem 5.2. *Under the assumptions of Theorem 3.2. If f is locally Lipschitz continuous of order 1 on u and*

$$\lim_{n\to\infty}(na_n^3/\log\log n) = 0, \quad \lim_{n\to\infty}(n^{1/2}a_n \log\log n \log^{-2} n) = \infty.$$

Then

$$\limsup_{n\to\infty} \pm\Big\{\frac{na_n}{f(u)\log\log n}\Big\}^{1/2}\{\widehat{f}_n(u) - f(u)\} = 1, \quad a.s.$$

The proofs of the above two theorems can be completed by slightly modifying the proofs of theorems 2 and 3 of Chai and Li(1993), we omitted the details.

REFERENCES

Bickel, P.J. (1982). On Adaptive Estimation. *Annals of Statistics*, **10** 647-671.

Chai, G. X. and Li, Z.Y. (1993). Asymptotic Theory for Estimation of Error Distributions in Linear Model. *Science in China, Ser. A* **4** 408-419.

Chen, H. (1988). Convergence Rates for Parametric Components in a Partly Linear Model. *Annals of Statistics*, **16** 136-146.

Chen, H. and Shiau, J.G. (1991). A Two-Stage Spline Smoothing Method for Partially Linear Models. *Journal of Statistical Planning & Inference*, **25** 187-201.

Cuzick, J. (1992a). Semiparametric Additive Regression. *Journal of the Royal Statistical Society, Series B*, **54** 831-843.

Cuzick, J. (1992b). Efficient Estimates in Semiparametric Additive Regression Models with Unknown Error Distribution. *Annals of Statistics*, **20** 1129-1136.

Devroye, L. P. and Wagneer, T.J. (1980). The Strong Uniform Consistency of Kernel Estimates. *Journal of Multivariate Analysis*, **5** 59-77.

Engle, R. F., Granger, C.W.J., Rice,J. and Weiss, A. (1986). Semiparametric Estimates of the Relation Between Weather and Electricity Sales. *Journal of the American Statistical Association*, **81** 310-320.

Gao, J. T., Hong, S.Y. and Liang, H. (1995). Convergence rates of a class of estimates in partly linear models. *Acta Math. Sinica,* **38**, 658-669.

Härdle, W. (1990). *Applied Nonparametric Regression.* Cambridge University Press, New York.

Heckman, N.E. (1986). Spline Smoothing in Partly Linear Models. *Journal of the Royal Statistical Society, Series B*, **48** 244-248.

Heckman, N.E.(1988). Minimax Estimates in a Semiparametric Model. *Journal of the American Statistical Association*, **83** 1090-1096.

Robinson, P.M. (1988). Root-N-Consistent Semiparametric Regression. *Econometrica*, **56** 931-954.

Schick, A. (1993). On Efficient Estimation in Regression Models. *Annals of Statistics*, **21** 1486-1521.

Schick, A. (1996). Root-N Consistent Estimation in Partly Linear Regression Models. *Statistics & Probability Letters*, **28** 353-358.

Serfling, R.J. (1980).*Approximation Theorems of Mathematical Statistics.* John Wiley & Sons 95-96.

Speckman, P. (1988). Kernel Smoothing in Partial Linear Models. *Journal of the Royal Statistical Society, Series B*, **50** 413-436.