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Duality Theory for Optimal Investments under Model Uncertainty

Alexander Schied *
Ching-Tang Wu**

* Technische Universität Berlin, Germany
** National University of Kaohsiung, Taiwan

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Duality theory for optimal investments under model uncertainty

Alexander Schied*
Institut für Mathematik, MA 7-4
TU Berlin
Strasse des 17. Juni 136
10623 Berlin, Germany
e-mail: schied@math.tu-berlin.de

Ching-Tang Wu**
Department of Applied Mathematics
National University of Kaohsiung
700 Kaohsiung University Rd., Nan-Tzu Dist.
811 Kaohsiung, Taiwan
e-mail: ctwu@nuk.edu.tw

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Abstract: Robust utility functionals arise as numerical representations of investor preferences, when the investor is uncertain about the underlying probabilistic model and averse against both risk and model uncertainty. In this paper, we study the the duality theory for the problem of maximizing the robust utility of the terminal wealth in a general incomplete market model. We also allow for very general sets of prior models. In particular, we do not assume that that all prior models are equivalent to each other, which allows us to handle many economically meaningful robust utility functionals such as those defined by $AVaR_\lambda$, concave distortions, or convex capacities. We also show that dropping the equivalence of prior models may lead to new effects such as the existence of arbitrage strategies under the least favorable model.

1 Introduction

There is a vast literature on the construction of utility-maximizing investment strategies in complete and incomplete market models. An implicit assumption made in most papers on this subject is that the investor is in possession of a market model that accurately describes the probabilities for the future stock price evolution. In reality, however, the exact probabilities themselves are often unknown, i.e., the choice of an appropriate model is subject to *Knightian uncertainty*.

In the late 1980's, Gilboa and Schmeidler [8], [20], [9] and Yaari [21] formulated natural axioms which should be satisfied by a preference order on payoff profiles in order to account for aversion against both risk and Knightian uncertainty. They showed that such

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a preference order can be numerically represented by a *robust utility functional* of the form

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[\, U(X)\,]\,,$$

where $\mathcal{Q}$ is a set of probability measures and U is a utility function; see also [6, Section 2.5].

In a financial market model, a natural question is thus to construct dynamic investment strategies whose terminal wealth maximizes a given robust utility functional. Systematic approaches to this question were independently[1] given by M. Quenez [16] and the first author [18]. Quenez [16] gives two types of result. The first is a duality result in the spirit of Kramkov and Schachermayer [13, 14] under relatively strong assumptions on the prior set $\mathcal{Q}$; in particular it is assumed that

$$\text{all measures in (the closure of) } \mathcal{Q} \text{ are equivalent to a given reference measure } \mathbb{P}. \quad (1)$$

The second class of results in [16] deals with explicit examples that can be handled with BSDE techniques. This technique basically requires that $\mathcal{Q}$ consists of the class of all market models with a fixed volatility and a varying drift process that takes values in (possibly random) closed sets $C_t \subset \mathbb{R}^d$.

In [18], the focus is on determining explicit solutions for several classes of prior sets $\mathcal{Q}$ in complete market models. More precisely, it is shown that in numerous situations the set $\mathcal{Q}$ admits a measure $\widehat{Q}$ that is "least-favorable" in the sense that the robust problem becomes equivalent to the standard problem for $\widehat{Q}$, regardless of the choice of the utility function. For most examples in [18], the condition (1) is too restrictive. For instance, (1) cannot not hold if the set $\mathcal{Q}$ arises from coherent risk measures such as Average Value at Risk,

$$AVaR_\lambda(X) = \sup\Big\{ E_Q[\,-X\,]\,\big|\, Q \ll \mathbb{P} \text{ and } \frac{dQ}{dP} \le \frac{1}{\lambda}\Big\},$$

which typically coincides with the worst conditional expectation

$$WCE_\lambda(X) = \sup\big\{ \mathbb{E}[\,-X\,|\,A\,]\,|\,\mathbb{P}[\,A\,] > \lambda \big\}.$$

Condition (1) is also often violated if the prior set is the core of a concave distortion or, more generally, of a submodular capacity. These examples also play an important role in economics; see, e.g., Schmeidler [20] and Yaari [21]. The same is true for law-invariant robust utility functionals as considered in [19] and by Jouini et al. [12]. Another example for which (1) is not satisfied is provided by the case of Baudoin's [2] "weak information", where $\mathcal{Q}$ consists of all measures $Q \ll \mathbb{P}$ under which a given random variable has a fixed distribution. Nevertheless, in many of these case it is possible to construct solutions for the robust utility maximization problem, at least if the market model is complete; see [18]. We also refer to Cont [4] for a further discussion of (1) in the theory of model uncertainty.

In a more recent paper, Gundel [11] has extended the method of Goll and Rüschendorf [10] to obtain results in an incomplete market. This method works if U is defined on all

[1] In fact, a first version of the present paper was completed by us without knowledge of [16], and we are grateful to Martin Schweizer for informing us about this related work.

of $\mathbb{R}$ but may fail, e.g., for HARA utility functions. Gundel [11] also requires condition (1). For a recent extension of the BSDE approach, we refer to Müller [15].

In this note, we are interested in developing the duality theory for robust utility maximization in a very general framework. As for the conditions on the financial market model and the utility function, the benchmark has already been set in the work of Kramkov and Schachermayer [13, 14]. Here we will adopt their framework in assuming that the price process S is a general d-dimensional semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. As in [13, 14], we will assume that the model is 'arbitrage free' in the sense that there exists an equivalent martingale measure for all admissible value processes.

With the market model being fixed, we need to formulate natural conditions on the set $\mathcal{Q}$ from which our robust utility functional will be defined. First of all, it is necessary that each measure $Q \in \mathcal{Q}$ respects $\mathbb{P}$-nullsets, for otherwise a stochastic integral defined with respect to $\mathbb{P}$ might make no sense under Q. Thus, we assume that

(a) $Q \ll \mathbb{P}$ for all $Q \in \mathcal{Q}$.

Next, there is no loss of generality in assuming that

(b) $\mathcal{Q}$ is convex.

As mentioned above, a typical result in all previous papers on robust utility maximization is the existence of a measure $\widehat{Q}$ which is "least favorable" in the sense that the robust problem is equivalent to the standard problem for $\widehat{Q}$. If one wishes to get some control over $\widehat{Q}$ then it is natural to require that $\widehat{Q} \in \mathcal{Q}$. This requirement will be guaranteed by assuming that

(c) $\mathcal{Q}$ is closed in some reasonable topology such as total variation.

To obtain the existence of $\widehat{Q}$ in our general setup, one needs to assume that

(d) $\mathcal{Q}$ is relatively compact in a reasonable topology.

We finally add the assumption that our set $\mathcal{Q}$ is "sensitive" in the sense that

(e) $Q[\,A\,] = 0$ for all $Q \in \mathcal{Q}$ implies $\mathbb{P}[\,A\,] = 0$.

At first glance, this condition may seem less natural than the preceding ones. But note that, due to the Halmos-Savage theorem and the assumptions (a), (b), and (c), it is equivalent to the rather weak requirement that there exists *one* $Q \in \mathcal{Q}$ that is equivalent to $\mathbb{P}$. This latter requirement should be compared to the assumption (1), which would add to the set (a)–(e) the condition that "$\mathbb{P} \ll Q$ for all $Q \in \mathcal{Q}$".

Our aim in this paper is to establish a duality theory for robust utility maximization given the set of assumptions (a)–(e). On the one hand, our main results will be formulated within the above-mentioned paradigm: For each level of initial wealth there exists a measure $\widehat{Q}$ that is least favorable in the sense explained above. On the other hand, we will also challenge this paradigm at least partially: In our general setup, the measure $\widehat{Q}$ may no longer be equivalent to the reference measure $\mathbb{P}$. In fact, we will see in Example

2.5 that one may have to face the situation that $\widehat{Q}$ admits *arbitrage opportunities* when considered as a market model on its own. If this is the case, it will no longer be possible to apply the standard theory of utility maximization to the model with subjective measure $\widehat{Q}$. The failure of equivalence also creates some difficulties in our proofs and a number of open questions such as regularity properties of the value functions or the uniqueness of optimal strategies for the primal and dual problems.

2 Statement of main results

As in Kramkov and Schachermayer [13, 14], we assume that the utility function of the investor is a strictly increasing and strictly concave function $U : (0,\infty) \to \mathbb{R}$, which is also continuously differentiable and satisfies the Inada conditions

$$U'(0+) = +\infty \qquad \text{and} \qquad U'(\infty-) = 0.$$

Payoffs are modeled as random variables X on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Their utility shall be assessed in terms of a robust utility functional

$$X \longmapsto \inf_{Q\in\mathcal{Q}} E_Q[\,U(X)\,],$$

where $\mathcal{Q}$ is a set of probability measures on $(\Omega, \mathcal{F})$. We assume the following conditions:

Assumption 2.1
(i) $\mathcal{Q}$ *is convex.*
(ii) $\mathbb{P}[\,A\,] = 0$ *if and only if* $Q[\,A\,] = 0$ *for all* $Q \in \mathcal{Q}$
(iii) *The set* $\mathcal{Z} := \{dQ/d\mathbb{P} \,|\, Q \in \mathcal{Q}\,\}$ *is closed in* $L^0(\mathbb{P})$

Condition (ii) combines assumptions (a) and (e) as formulated in Section 1. Condition (iii) takes care of closedness (c) and compactness (d); see Lemma 3.1. We emphasize once more that (ii) is strictly weaker than the assumtion that all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$, which is assumed in [16], [11] and rules out many examples, which are explicitly solvable for complete models [18]. In fact, the Halmos-Savage theorem shows that, under condition (iii), condition (ii) is equivalent to the assumption

$$Q \ll \mathbb{P} \text{ for all } Q \in \mathcal{Q} \text{ and } \mathcal{Q}_e \neq \emptyset,$$

where $\mathcal{Q}_e$ denotes the set of measures in $\mathcal{Q}$ that are equivalent to $\mathbb{P}$.

We use the same setup as in [13, 14] also for the financial market model. The discounted price process of d assets is modeled by a stochastic process $S = (S_t)_{0\le t\le T}$. We assume that S is a d-dimensional semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a filtration $(\mathcal{F}_t)_{0\le t\le T}$. A self-financing trading strategy can be regarded as a pair (x, ξ), where $x \in \mathbb{R}$ is the initial investment and $\xi = (\xi_t)_{0\le t\le T}$ is a d-dimensional predictable and S-integrable process. The value process X associated with (x, ξ) is given by $X_0 = x$ and

$$X_t = X_0 + \int_0^t \xi_r \, dS_r\,, \qquad 0 \le t \le T\,.$$

For $x > 0$ given, we denote by $\mathcal{X}(x)$ the set of all such processes X with $X_0 \le x$ which are admissible in the sense that $X_t \ge 0$ for $0 \le t \le T$ and whose terminal wealth X_T has a well-defined robust utility in the sense that

$$\inf_{Q \in \mathcal{Q}} E_Q[\,U(X_T) \wedge 0\,] > -\infty\,. \tag{2}$$

We assume that our model is arbitrage-free in the sense that the $\mathcal{M} \neq \emptyset$, where $\mathcal{M}$ denotes the set of measures equivalent to $\mathbb{P}$ under which each $X \in \mathcal{X}(1)$ is a local martingale; see [13]. Thus, our main problem can be stated as follows:

$$\text{Maximize } \inf_{Q \in \mathcal{Q}} E_Q[\,U(X_T)\,] \text{ among all } X \in \mathcal{X}(x).$$

Consequently, the *value function of the robust problem* is defined as

$$u(x) := \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} E_Q[\,U(X_T)\,]$$

One of our first results will be the minimax identity

$$u(x) = \inf_{Q \in \mathcal{Q}} u_Q(x), \qquad \text{where} \qquad u_Q(x) := \sup_{X \in \mathcal{X}(x)} E_Q[\,U(X_T)\,]$$

is the value function of the optimal investment problem for an investor with subjective measure $Q \in \mathcal{Q}$. Next, we define as usual the convex conjugate function V of U by

$$V(y) := \sup_{x>0} \big(U(x) - xy\big), \qquad y > 0.$$

With this notation, it follows from Theorem 3.1 of [13] that, for $Q \in \mathcal{Q}_e$ with finite value function u_Q,

$$u_Q(x) = \inf_{y>0} \big(v_Q(y) + xy\big) \qquad \text{and} \qquad v_Q(y) = \sup_{x>0} (u_Q(x) - xy), \tag{3}$$

where the dual value function v_Q is given by

$$v_Q(y) = \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[\,V(Y_T)\,],$$

and the space $\mathcal{Y}_Q(y)$ is defined as

$$\mathcal{Y}_Q(y) = \big\{\, Y \ge 0 \,|\, Y_0 = y \text{ and } XY \text{ is a } Q\text{-supermartingale for all } X \in \mathcal{X}(1)\,\big\}.$$

We thus define the *dual value function of the robust problem* by

$$v(y) := \inf_{Q \in \mathcal{Q}_e} v_Q(y) = \inf_{Q \in \mathcal{Q}_e} \inf_{Y \in \mathcal{Y}_Q(y)} E_Q[\,V(Y_T)\,].$$

Theorem 2.2 *In addition to the above assumptions, let us assume that*

$$u_{Q_0}(x) < \infty \text{ for some } x > 0 \text{ and some } Q_0 \in \mathcal{Q}_e. \tag{4}$$

Then the value function u is concave, takes only finite values, and satisfies

$$u(x) = \sup_{X\in\mathcal{X}} \inf_{Q\in\mathcal{Q}} E_Q[\,U(X_T)\,] = \inf_{Q\in\mathcal{Q}} \sup_{X\in\mathcal{X}} E_Q[\,U(X_T)\,]. \tag{5}$$

Moreover, two value functions u and v are conjugate to another:

$$u(x) = \inf_{y>0} \big(v(y) + xy\big) \qquad \text{and} \qquad v(y) = \sup_{x>0} \big(u(x) - xy\big). \tag{6}$$

In particular, v is convex. The derivatives of u and v satisfy

$$u'(0+) = \infty \qquad \text{and} \qquad v'(\infty-) = 0. \tag{7}$$

Remark 2.3 It will turn out in the proof of this theorem that the value function u and its dual v can be defined via the smaller set

$$\mathcal{Q}_e^f := \big\{Q \in \mathcal{Q}_e \,|\, u_Q(x) < \infty \text{ for some } x > 0\big\} = \big\{Q \in \mathcal{Q}_e \,|\, u_Q(x) < \infty \text{ for all } x > 0\big\},$$

i.e.,

$$u(x) = \inf_{Q\in\mathcal{Q}_e^f} u_Q(x) \qquad \text{and} \qquad v(y) = \inf_{Q\in\mathcal{Q}_e^f} v_Q(y).$$

Also note that (4) can be restated as $\mathcal{Q}_e^f \neq \emptyset$. ◊

The situation becomes much simpler if we assume that all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. In this case, we also get some additional results, which generalize those in Quenez [16], where additional assumptions on $\mathcal{Q}$ are required.

Corollary 2.4 *In addition to the assumptions of Theorem 2.2 suppose that all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$. Then the value function u is continuously differentiable, the dual value function v is strictly convex, and for each $y > 0$ such that $v(y) < \infty$ there exists $\widehat{Q} \in \mathcal{Q}$ and $\widehat{Y} \in \mathcal{Y}_{\widehat{Q}}(y)$ such that $v(y) = E_{\widehat{Q}}[\,V(\widehat{Y}_T)\,]$. Moreover, $\widehat{Y}$ is unique: any other optimal pair $(Q', Y') \in \{(Q,Y)\,|\,Q \in \mathcal{Q}, Y \in \mathcal{Y}_Q(y)\,\}$ satisfies $Y' = \widehat{Y}$ $\mathbb{P}$-a.s.*

We now come to the existence of optimal strategies. The following simple example illustrates some of the difficulties one might meet if $\mathcal{Q}$ contains measures that are not equivalent to $\mathbb{P}$.

Example 2.5 Consider a one-period model in discrete time ($t = 0, 1$) with two assets S^1, S^2 satisfying $S^1_0 = S^2_0 = 1$. Under the measure Q_1, the first asset has, at time 1, the distribution

$$Q_1[\,S^1_1 = 2\,] =: q = 1 - Q_1[\,S^1_1 = 0\,],$$

where $1/2 < q < 1$. The second asset S^2_1 is independent of S^1_1 under Q_1, has support $\{0, 1, \dots\}$, and finite expected value $E_{Q_1}[\,S^2_1\,] > S^2_0 = 1$. We take $\mathbb{P} := Q_1$ as our reference measure. We introduce another measure $Q_0 \ll \mathbb{P}$ by requiring that

$$Q_0[\,S^1_1 = 2\,] = Q_0[\,S^1_1 = 0\,] = 1/2 \qquad \text{and} \qquad Q_0[\,S^2_1 = 0\,] = 1.$$

For $\mathcal{Q}$ we take the set of all convex combinations $Q_\alpha := \alpha Q_1 + (1-\alpha)Q_0$, $0 \le \alpha \le 1$.

Note first that a trading strategy can only be admissible for $\mathbb{P}$ if it does not contain short positions in the second asset, because S_1^2 is unbounded. Let us now look at the optimal strategy under Q_α. It is well known that the optimal portfolio will contain no long positions in the i^{th} asset if and only if $E_{Q_\alpha}[\,S_1^i\,] \le 1$ (e.g., Proposition 2.41 in [6]). Thus, there exists some $\alpha_0 \in (0,1)$ such that there will be no investment, long or short, into the second asset for $\alpha \le \alpha_0$, because our admissibility assumption excludes short positions. Next, it will be optimal to allocate some investment into the first asset for all $\alpha > 0$. It follows that $u_{Q_\alpha}(x) > U(x)$ for all $\alpha > 0$. On the other hand, under Q_0 it is not optimal to allocate any admissible investment, long or short, to either of the risky assets, and it follows that $u_{Q_0}(x) = U(x)$.

Thus, $\widehat{Q} := Q_0$ is the unique measure in $\mathcal{Q}$ such that $u_{\widehat{Q}}(x) = u(x) = \inf_{Q\in\mathcal{Q}} u_Q(x)$, and in order to determine the optimal strategy for the robust problem, we must look for the optimal strategy for the model $\widehat{Q}$. This task is straightforward in this simple example: just put everything into the bond. However, it would create difficulties if we would try to apply the general theory of utility maximization, because $\widehat{Q} = Q_0$ is *not* equivalent to the martingale measures in our market model. To make things worse, Q_0 considered as a market model on its own has not the same admissible strategies than $\mathbb{P}$, since short selling the second asset is admissible in the model Q_0. In fact, such short sales even creates *arbitrage opportunities* under Q_0. $\diamondsuit$

Our next aim is to get existence results for optimal strategies despite the difficulties displayed by the preceding example. Even for the classical case $\mathcal{Q} = \{Q\}$ additional assumptions are needed to guarantee the existence of optimal strategies for each initial capital: It was shown in [14] that a necessary and sufficient condition is the finiteness of the dual value function v_Q. This condition translates as follows to our robust setting:

$$v_Q(y) < \infty \qquad \text{for all } y > 0 \text{ and each } Q \in \mathcal{Q}_e. \tag{8}$$

Recall from [14, Note 2] that (8) holds as soon as u_Q is finite for all $Q \in \mathcal{Q}_e$ and the asymptotic elasticity of the utility function U is strictly less than one:

$$AE(U) = \limsup_{x\uparrow\infty} \frac{xU'(x)}{U(x)} < 1. \tag{9}$$

While it is sufficient to assume (8) when all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$, we need to assume (9) to get some regularity results in the general case.

Theorem 2.6 *In addition to Assumption 2.1 let us assume* (8). *Then both value functions u and v take only finite values and satisfy*

$$u'(\infty-) = 0 \qquad \textit{and} \qquad v'(0+) = -\infty. \tag{10}$$

For any $x > 0$ there exists an optimal strategy $\widehat{X} \in \mathcal{C}(x)$ and a measure $\widehat{Q} \in \mathcal{Q}$ such that

$$u(x) = \inf_{Q\in\mathcal{Q}} E_Q[\,U(\widehat{X}_T)\,] = E_{\widehat{Q}}[\,U(\widehat{X}_T)\,] = u_{\widehat{Q}}(x).$$

In particular, the suprema and infima in (5) *are attained. There also exists some* $\widehat{y}$ *in the supergradient of* $u(x)$ *and some* $Y \in \mathcal{Y}_{\mathbb{P}}(\widehat{y})$ *such that,*

$$v(\widehat{y}) = \mathbb{E}\Big[\,\widehat{Z}V\Big(\frac{Y_T}{\widehat{Z}}\Big)\Big], \qquad and \qquad \widehat{X}_T = I\Big(\frac{Y_T}{\widehat{Z}}\Big) \qquad \widehat{Q}\text{-a.s.}, \tag{11}$$

where $\widehat{Z} = d\widehat{Q}/d\mathbb{P}$ *and* $I = -V'$. *Furthermore,* $\widehat{X}Y$ *is a martingale under* $\mathbb{P}$, *and the dual value function satisfies*

$$v(y) = \inf_{P^* \in \mathcal{M}} \inf_{Q \in \mathcal{Q}_e} E_Q\Big[\,V\Big(y\frac{dP^*}{dQ}\Big)\Big]. \tag{12}$$

If in addition $AE(U) < 1$ *holds, then* u *is strictly concave and* v *is continuously differentiable. Moreover,* $\widehat{X}_T Y_T$ *is supported by* $\{\widehat{Z} > 0\}$, *i.e.,*

$$\{\widehat{X}_T Y_T > 0\} = \{\widehat{Z} > 0\} \qquad \mathbb{P}\text{-a.s.} \tag{13}$$

Remark: The identity (13) shows that the duality relation (11) cannot be extended beyond the support of $\widehat{Q}$. This fact challenges the paradigm of solving the robust problem via determining a least favorable measure. On the other hand, if S is *continuous* and

$$\widehat{X} = x + \int_0^{\cdot} \widehat{\xi}_t \, dS_t$$

is known under $\widehat{Q}$, then this strategy can be extended to all of Ω by replacing $\widehat{\xi}$ by $\widehat{\xi}_t \mathrm{I}_{\{\zeta > t\}}$, where $\zeta := \inf\{t \geq 0 \,|\, \mathbb{E}[\,\widehat{Z} \,|\, \mathcal{F}_t\,] = 0\}$. $\diamondsuit$

We get some additional results if all measures in $\mathcal{Q}$ are equivalent to $\mathbb{P}$:

Corollary 2.7 *In addition to the assumptions of Theorem 2.2 let us assume* (8) *and that all measures in* $\mathcal{Q}$ *are equivalent to* $\mathbb{P}$. *Then both value functions* u *and* v *take only finite values, are continuously differentiable on* $(0, \infty)$, *and their derivatives* u' *and* v' *are strictly decreasing and increasing, respectively. For any* $x > 0$, *the optimal solution* $\widehat{X} \in \mathcal{X}(x)$ *is unique and it is given by*

$$\widehat{X}_T = I(\widehat{Y}_T),$$

where I *is the inverse function of* U' *and* $\widehat{Y}$ *is as in Corollary 2.4 for* $\widehat{y} := u'(x)$. *If* $\widehat{Q}$ *is as in Corollary 2.4, then it satisfies all the properties of the measure* $\widehat{Q}$ *in Theorem 2.6.*

3 The duality of the value functions

As in [13, 14], we obtain "abstract versions" of our theorems if we replace the spaces $\mathcal{X}(x)$ and $\mathcal{Y}_Q(y)$ by the respective spaces

$$\mathcal{C}(x) = \big\{\, g \in L^0_+(\Omega, \mathcal{F}_T, \mathbb{P}) \,|\, 0 \leq g \leq X_T \text{ for some } X \in \mathcal{X}(x) \,\big\}.$$

and, for $Q \in \mathcal{Q}_e$,

$$\mathcal{D}_Q(y) = \big\{\, h \in L^0_+(\Omega, \mathcal{F}_T, \mathbb{P}) \,|\, 0 \le h \le Y_T \text{ for some } Y \in \mathcal{Y}_Q(y) \,\big\}.$$

It is easy to see that this substitution does not affect the values of our value functions, i.e., we have $u_Q(x) = \sup_{g \in \mathcal{C}(x)} E_Q[\,U(g)\,]$ and $v_Q(y) = \inf_{h \in \mathcal{D}_Q(y)} E_Q[\,V(h)\,]$. Moreover, any optimal g or h must clearly be the terminal value of some process $X \in \mathcal{X}(x)$ or $Y \in \mathcal{Y}_Q(y)$. We note next that the spaces $\mathcal{Y}_Q(y)$ and $\mathcal{D}_Q(y)$ can easily be related to $\mathcal{Y}(y) := \mathcal{Y}_{\mathbb{P}}(y)$ and $\mathcal{D}(y) := \mathcal{D}_{\mathbb{P}}(y)$: if $(Z^Q_t)_{0 \le t \le T}$ is the density process of $Q \in \mathcal{Q}_e$ with respect to $\mathbb{P}$, then

$$\mathcal{Y}_Q(y) = \{\, Y/Z^Q \,|\, Y \in \mathcal{Y}(y) \,\} \quad \text{and} \quad \mathcal{D}_Q(y) = \{\, h/Z^Q_T \,|\, h \in \mathcal{D}(y) \,\},$$

as can be seen easily by the Bayes formula for conditional expectations. Hence, the dual value function satisfies

$$v(y) = \inf_{Q \in \mathcal{Q}_e} v_Q(y) = \inf_{Z \in \mathcal{Z}_e} \inf_{h \in \mathcal{D}(y)} \mathbb{E}\Big[\, Z V\Big(\frac{h}{Z}\Big) \Big], \tag{14}$$

where

$$\mathcal{Z} = \Big\{\, \frac{dQ}{d\mathbb{P}} \,\Big|\, Q \in \mathcal{Q} \Big\} \qquad \text{and} \qquad \mathcal{Z}_e := \Big\{\, \frac{dQ}{d\mathbb{P}} \,\Big|\, Q \in \mathcal{Q}_e \Big\}.$$

The formula (14) is convenient, since the infimum is now taken over two sets that are no longer related to another. Also, recall from [13] that for $Q \in \mathcal{Q}_e$

$$\begin{aligned} g \in \mathcal{C}(x) \quad &\Longleftrightarrow \quad g \ge 0 \text{ and } \sup_{h \in \mathcal{D}_Q(y)} E_Q[\,hg\,] \le xy \\ h \in \mathcal{D}_Q(y) \quad &\Longleftrightarrow \quad h \ge 0 \text{ and } \sup_{g \in \mathcal{C}(x)} E_Q[\,hg\,] \le xy. \end{aligned} \tag{15}$$

Let us note next that the function u is concave. In particular, (4) implies that u takes only finite values and is continuous on $(0, \infty)$. Indeed, the concavity of U easily implies that $g \mapsto \inf_{Q \in \mathcal{Q}} E_Q[\,U(g)\,]$ is a concave functional on $\mathcal{C}(x)$ for each x. Hence, the concavity of u follows from the fact that

$$\{\, \alpha g + (1-\alpha) g' \,|\, g \in \mathcal{C}(x),\, g' \in \mathcal{C}(x') \,\} \subset \mathcal{C}(\alpha x + (1-\alpha) x').$$

The following lemma is certainly well known; we include a short proof for the convenience of the reader.

Lemma 3.1 *Suppose parts* (i) *and* (ii) *of Assumption 2.1 hold. Then part (iii) of Assumption 2.1 holds if and only if $\mathcal{Z}$ is weakly compact in $L^1(\mathbb{P})$.*

Proof: Assume (iii), take $F \in L^\infty_+(\mathbb{P})$, and let (Z_n) be a sequence in $\mathcal{Z}$ such that $\mathbb{E}[\,Z_n F\,]$ tends to $\inf_{Z \in \mathcal{Z}} \mathbb{E}[\,ZF\,]$. By the standard Komlos-type argument [5, Lemma A1.1], there exists a sequence of convex combinations $\widetilde{Z}_n \in \operatorname{conv}\{Z_n, Z_{n+1}, \dots\} \subset \mathcal{Z}$ converging $\mathbb{P}$-a.s. to some random variable $Z_0 \in \mathcal{Z}$. Thus, for every $F \in L^\infty_+(\mathbb{P})$ there exists $Z_0 \in \mathcal{Z}$ such that $\mathbb{E}[\,Z_0 F\,] \le \mathbb{E}[\,ZF\,]$ for all $Z \in \mathcal{Z}$. Since all members of $\mathcal{Z}$ are probability densities, the same is true for arbitrary $F \in L^\infty$, and weak compactness follows from James' theorem (see, e.g., [7]).

Conversely, suppose (Z_n) is a sequence in $\mathcal{Z}$ converging $\mathbb{P}$-a.s. to some Z_0. Weak compactness of $\mathcal{Q}$ gives $\mathbb{E}[\,\widetilde{Z}_n 1\,] \to \mathbb{E}[\,Z_0 1\,]$, and it follows that $\widetilde{Z}_n \to Z_0$ in $L^1(\mathbb{P})$. Since $\mathcal{Z}$ is closed in $L^1(\mathbb{P})$ due to part (i) we get $Z_0 \in \mathcal{Z}$. $\square$

Recall that $\mathcal{Q}^f$ denotes the set of $Q \in \mathcal{Q}$ such that $u_Q(x) < \infty$ for some and hence all $x > 0$.

Lemma 3.2 *For Q_0, $Q_1 \in \mathcal{Q}^f$ and $0 \le t \le 1$ let $Q_t := tQ_1 + (1-t)Q_0 \in \mathcal{Q}$. Then $t \mapsto u_{Q_t}(x)$ is a continuous function for each $x > 0$.*

Proof: On the one hand, $f(t) := u_{Q_t}(x)$ takes only finite values and is convex as the supremum, taken over $g \in \mathcal{C}(x)$, of the affine functions $t \mapsto E_{Q_t}[\,U(g)\,]$. Hence f is continuous on $(0,1)$ and upper semicontinuous on $[0,1]$. On the other hand, $U(\cdot + \varepsilon)$ is bounded from below for any $\varepsilon \in (0,x)$, and so

$$t \mapsto E_{Q_t}[\,U(\varepsilon + g)\,] = \sup_n E_{Q_t}[\,U(\varepsilon + g) \wedge n\,]$$

is lower semicontinuous for each $g \in \mathcal{C}(x)$. Moreover, $g + \varepsilon \in \mathcal{C}(x)$ for each $g \in \mathcal{C}(x - \varepsilon)$ and hence

$$\liminf_{t \downarrow 0} u_{Q_t}(x) \ge \liminf_{t \downarrow 0} \sup_{g \in \mathcal{C}(x-\varepsilon)} E_{Q_t}[\,U(g+\varepsilon)\,] \ge \sup_{g \in \mathcal{C}(x-\varepsilon)} E_{Q_0}[\,U(g+\varepsilon)\,] \ge u_{Q_0}(x-\varepsilon).$$

Sending $\varepsilon \downarrow 0$ and using the continuity of u_{Q_0} as a concave function, we get that f is also lower semicontinuous at $t = 0$. The proof for $t = 1$ is identical. □

Lemma 3.3 *We have*

$$u(x) = \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} E_Q[\,U(g)\,] = \inf_{Q \in \mathcal{Q}} \sup_{g \in \mathcal{C}(x)} E_Q[\,U(g)\,] \tag{16}$$

$$= \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}_e} E_Q[\,U(g)\,] = \inf_{Q \in \mathcal{Q}_e} \sup_{g \in \mathcal{C}(x)} E_Q[\,U(g)\,] \tag{17}$$

Proof: To prove that supremum and infimum may be interchanged, take $\varepsilon > 0$ and note that

$$u(x+\varepsilon) \ge \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} E_Q[\,U(\varepsilon + g)\,] = \sup_{g \in \mathcal{C}(x)} \inf_{Z \in \mathcal{Z}} \mathbb{E}[\,ZU(\varepsilon + g)\,].$$

As in the proof of Lemma 3.2, we see that $Z \mapsto \mathbb{E}[\,ZU(\varepsilon + g)\,]$ is, for each $g \in \mathcal{C}(x)$, a weakly lower semicontinuous affine functional defined on the weakly compact convex set $\mathcal{Z}$. Moreover, for each $Z \in \mathcal{Z}$, $g \mapsto \mathbb{E}[\,ZU(\varepsilon + g)\,]$ is a concave functional defined on the convex set $\mathcal{C}(x)$. Thus, the conditions of the lop sided minimax theorem [1, Chapter 6, p. 295] are satisfied, and so

$$\sup_{g \in \mathcal{C}(x)} \min_{Z \in \mathcal{Z}} \mathbb{E}[\,ZU(\varepsilon + g)\,] = \min_{Z \in \mathcal{Z}} \sup_{g \in \mathcal{C}(x)} \mathbb{E}[\,ZU(\varepsilon + g)\,].$$

Hence, we arrive at

$$u(x+\varepsilon) \ge \min_{Q \in \mathcal{Q}} \sup_{g \in \mathcal{C}(x)} E_Q[\,U(\varepsilon + g)\,] \ge \inf_{Q \in \mathcal{Q}} \sup_{g \in \mathcal{C}(x)} E_Q[\,U(g)\,] \ge \sup_{g \in \mathcal{C}(x)} \inf_{Q \in \mathcal{Q}} E_Q[\,U(g)\,] = u(x).$$

Sending $\varepsilon \downarrow 0$ and using the continuity of u yields the first part of the lemma.

We still have to show that $\mathcal{Q}$ may be replaced by $\mathcal{Q}_e$. We obtain from Lemma 3.2 that $u(x) = \inf_{Q\in\mathcal{Q}_e} u_Q(x)$. Hence

$$\begin{aligned} u(x) &= \inf_{Q\in\mathcal{Q}_e} u_Q(x) = \inf_{Q\in\mathcal{Q}_e} \sup_{g\in\mathcal{C}(x)} E_Q[\,U(g)\,] \geq \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}_e} E_Q[\,U(g)\,] \\ &\geq \sup_{g\in\mathcal{C}(x)} \inf_{Q\in\mathcal{Q}} E_Q[\,U(g)\,] = u(x). \end{aligned}$$

□

A key observation for our future analysis is the convexity of the function $(x, y) \mapsto xV(y/x)$. A proof of this observation goes as follows. For $x_0, x_1, y_0, y_1 \in (0, \infty)$ let $\alpha := x_1/(x_0 + x_1) \in (0, 1)$. Then

$$\alpha\frac{y_1}{x_1} + (1-\alpha)\frac{y_0}{x_0} = \frac{y_0 + y_1}{x_0 + x_1} = \frac{y_{\frac{1}{2}}}{x_{\frac{1}{2}}},$$

where $x_{\frac{1}{2}} := (x_0 + x_1)/2$ and $y_{\frac{1}{2}} := (y_0 + y_1)/2$. It follows that

$$x_{\frac{1}{2}}V\Big(\frac{x_{\frac{1}{2}}}{y_{\frac{1}{2}}}\Big) \leq x_{\frac{1}{2}}\Big[(1-\alpha)V\Big(\frac{y_0}{x_0}\Big) + \alpha V\Big(\frac{y_1}{x_1}\Big)\Big] = \frac{1}{2}x_0V\Big(\frac{x_0}{y_0}\Big) + \frac{1}{2}x_1V\Big(\frac{x_1}{y_1}\Big). \tag{18}$$

Note that the inequality is strict if $y_0/x_0 \neq y_1/x_1$.

We will show next that in (14) the set $\mathcal{Z}_e$ can be replaced by the larger set $\mathcal{Z}$ or by the smaller set $\mathcal{Z}_e^f$, where $\mathcal{Z}^f$ and $\mathcal{Z}_e^f$ correspond to the densities of measures in $\mathcal{Q}^f$ and $\mathcal{Q}_e^f$. If $Z = dQ/d\mathbb{P}$, we will also write u_Z and v_Z for u_Q und v_Q, respectively.

Lemma 3.4 *The dual value function of the robust problem satisfies*

$$v(y) = \inf_{Q\in\mathcal{Q}_e^f} v_Q(y) = \inf_{Z\in\mathcal{Z}} \inf_{h\in\mathcal{D}(y)} \mathbb{E}\Big[\,ZV\Big(\frac{h}{Z}\Big)\,\Big].$$

Proof: First we show that

$$v(y) = \inf_{Q\in\mathcal{Q}_e} v_Q(y) = \inf_{Z\in\mathcal{Z}} \inf_{h\in\mathcal{D}(y)} \mathbb{E}\Big[\,ZV\Big(\frac{h}{Z}\Big)\,\Big]. \tag{19}$$

To this end, take $Z_0 \in \mathcal{Z}_e$, $Z_1 \in \mathcal{Z}\backslash\mathcal{Z}_e$, and define $Z_t := tZ_1 + (1-t)Z_0 \in \mathcal{Z}_e$ for $0 \leq t < 1$. Then the function $t \mapsto \mathbb{E}[\,Z_tV(h/Z_t)\,]$ is convex and hence upper semicontinuous for each $h \in \mathcal{D}(y)$. Consequently, the function

$$t \longmapsto \inf_{h\in\mathcal{D}(y)} \mathbb{E}\Big[\,Z_tV\Big(\frac{h}{Z_t}\Big)\,\Big] =: v_{Z_t}(y), \qquad 0 \leq t \leq 1,$$

is also upper semicontinuous on $[0, 1]$, so that we get $v_{Z_1}(y) \geq \limsup_{t\uparrow 1} v_{Z_t}$. This proves our claim (19).

Now we will follow the proof of Lemma 3.4 in [13] to show that $v_Q(y) = \infty$ for $Q \in \mathcal{Q}_e\backslash\mathcal{Q}_e^f$. This fact will complete the proof. With $\mathcal{B}_n := \{g \,|\, 0 \leq g \leq n\}$, we get as in [13, Lemma 3.4] that, for $V^n(y) := \sup_{0<x\leq n}(U(x) - xy)$,

$$v_Q^n(y) := \inf_{h\in\mathcal{D}(y)} E_Q[\,V^n(h)\,] = \sup_{g\in\mathcal{B}_n} \inf_{h\in\mathcal{D}(y)} E_Q[\,U(g) - gh\,] = \inf_{h\in\mathcal{D}(y)} \sup_{g\in\mathcal{B}_n} E_Q[\,U(g) - gh\,].$$

We also get that $v_Q(y) \geq v_Q^n(y)$ and that

$$v_Q^n(y) \nearrow \sup_{x>0} \big(u_Q(x) - xy\big) = \infty.$$

This proves our claim. □

Lemma 3.5 *With V^- denoting the negative part of V, the set of random variables*

$$\Big\{\, ZV^-\Big(\frac{h}{Z}\Big) \,\Big|\, Z \in \mathcal{Z},\, h \in \mathcal{D}(y) \,\Big\}$$

is uniformly integrable with respect to $\mathbb{P}$.

Proof: The set $\mathcal{Z}$ is uniformly integrable according to Lemma 3.1 and the Dunford-Pettis theorem. Hence, there is nothing to show if V is bounded from below. If V is unbounded from below, let ϕ denote the inverse function of $-V$ and $y_0 := \phi(0)$. We have

$$\mathbb{E}\Big[\, Z\phi\Big(V^-\Big(\frac{h}{Z}\Big)\Big)\,\Big] \leq \mathbb{E}\Big[\, Z\phi\Big(-V\Big(\frac{h}{Z}\Big)\Big)\,\Big] + y_0 \leq \mathbb{E}[\, h\,] + y_0 \leq y + y_0 =: M \tag{20}$$

for all $Z \in \mathcal{Z}$ and $h \in \mathcal{D}(y)$. It was shown in Lemma 3.2 of [13] that $\phi(t)/t \to \infty$ as $t \uparrow \infty$. Hence, for every $a > 0$ there exists $c(a) > 0$ such that $\phi(t) \geq at$ for all $t \geq c(a)$. Let us write F_Z^h for $V^-(h/Z)$. Then (20) implies that

$$\mathbb{E}\big[\, ZF_Z^h \mathrm{I}_{\{F_Z^h \geq c(a)\}}\,\big] \leq \frac{1}{a}\mathbb{E}\big[\, Z\phi(F_Z^h)\,\big] \leq \frac{M}{a}, \tag{21}$$

uniformly in $Z \in \mathcal{Z}$ and $h \in \mathcal{D}(y)$. Now suppose $\varepsilon > 0$ is given. Take $c := c(2M/\varepsilon)$ and let $\eta := \varepsilon/(2c)$. Then $\mathbb{E}[\, Z;\, A\,] \leq \eta$ implies that

$$\mathbb{E}\big[\, ZF_Z^h \cdot \mathrm{I}_A\,\big] \;=\; \mathbb{E}\big[\, ZF_Z^h \cdot \mathrm{I}_{A\cap\{F_Z^h \geq c\}}\,\big] + \mathbb{E}\big[\, ZF_Z^h \cdot \mathrm{I}_{A\cap\{F_Z^h < c\}}\,\big] \leq \frac{\varepsilon}{2} + c \cdot \mathbb{E}[\, Z;\, A\,] \leq \varepsilon.$$

Finally, the uniform integrability of $\mathcal{Z}$ yields the existence of some $\delta > 0$ such that $\mathbb{E}[\, Z;\, A\,] \leq \eta$ as soon as $\mathbb{P}[\, A\,] \leq \delta$, and the proof is complete. □

Lemma 3.6 *If $v(y) < \infty$, then there exist $\widehat{Z} \in \mathcal{Z}$ and $\widehat{h} \in \mathcal{D}(y)$ such that*

$$v(y) = \mathbb{E}[\, \widehat{Z}V(\widehat{h}/\widehat{Z})\,].$$

Proof: In a first step, we show that the function

$$\mathcal{Z} \times \mathcal{D}(y) \ni (Z,h) \longmapsto \mathbb{E}\big[\, ZV\big(h/Z\big)\,\big]$$

is lower semicontinuous with respect to $\mathbb{P}$-a.s. convergence. Without loss of generality, we may assume $V(0) > 0$. Suppose that $Z_n \to Z$ and $h_n \to h$. The positive and negative parts of V are continuous and hence $Z_n V^{\pm}(h_n/Z_n) \to ZV^{\pm}(hZ)$. Now Fatou's lemma yields $\mathbb{E}[\, ZV^{+}(h/Z)\,] \le \liminf_{n\uparrow\infty} \mathbb{E}[\, Z_n V^{+}(h_n/Z_n)\,]$, while Lemma 3.5 implies that $\mathbb{E}[\, Z_n V^{-}(h_n/Z_n)\,] \to \mathbb{E}[\, ZV^{-}(h/Z)\,]$. Combining these two facts gives lower semicontinuity.

Now let $(Z_n, h_n) \in \mathcal{Z} \times \mathcal{D}(y)$ be a sequence such that $\mathbb{E}[\, Z_n V(h_n/Z_n)\,] \to v(y)$. Applying twice the standard Komlos-type argument of Lemma A1.1 in [5], we obtain a sequence

$$(\widetilde{Z}_n, \widetilde{h}_n) \in \mathrm{conv}\{(Z_n,h_n),(Z_{n+1},h_{n+1}),\dots\} \subset \mathcal{Z} \times \mathcal{D}(y)$$

that converges $\mathbb{P}$-a.s. to some $(\widehat{Z}, \widehat{h})$. We have $\widehat{Z} \in \mathcal{Z}$ by Lemma 3.1. Moreover, $\mathcal{D}(y)$ is closed in L^0 by [13, Proposition 3.1], and we get $\widehat{h} \in \mathcal{D}(y)$.

By the convexity of $(x,z) \mapsto zV(x/z)$ and step one of this proof we get

$$\mathbb{E}[\, \widehat{Z}V(\widehat{h}/\widehat{Z})\,] \le \liminf_{n\uparrow\infty} \mathbb{E}[\, \widetilde{Z}_n V(\widetilde{h}_n/\widetilde{Z}_n)\,] \le \liminf_{n\uparrow\infty} \mathbb{E}[\, Z_n V(h_n/Z_n)\,] = v(y).$$

Lemma 3.4 then shows that the pair $(\widehat{h}, \widehat{Z})$ is optimal. $\square$

Proof of Theorem 2.2: By Lemma 3.3, (3), and Lemma 3.4,

$$u(x) = \inf_{Q\in\mathcal{Q}_e} u_Q(x) = \inf_{Q\in\mathcal{Q}_e^f} u_Q(x) = \inf_{Q\in\mathcal{Q}_e^f}\inf_{y>0}\big(v_Q(y)+xy\big) = \inf_{y>0}\big(v(y)+xy\big),$$

which is the first identity in (6).

To prove the second one, note first that, by Lemma 3.4 and (3),

$$v(y) = \inf_{Q\in\mathcal{Q}_e^f} v_Q(y) = \inf_{Q\in\mathcal{Q}_e^f}\sup_{x>0}\big(u_Q(x)-xy\big). \tag{22}$$

We will show next that

$$\inf_{Q\in\mathcal{Q}_e^f}\sup_{x>0}\big(u_Q(x)-xy\big) = \inf_{Q\in\mathcal{Q}}\sup_{x>0}\big(u_Q(x)-xy\big). \tag{23}$$

The corresponding argument is similar to the one in the proof of Lemma 3.3. Choose $Q_1 \in \mathcal{Q}\backslash\mathcal{Q}_e^f$ and let $Q_t := tQ_1 + (1-t)Q_0$ where $Q_0 \in \mathcal{Q}_e^f$ is as in (4). There is nothing to show if $u_{Q_1}(x) = \infty$ for all x, so we may assume that $Q_1 \in \mathcal{Q}^f$. Then the function $t \mapsto u_{Q_t}(x)$ is convex as the supremum of the affine functions $t \mapsto E_{Q_t}[\, U(g)\,]$, and it follows that $u_{Q_t}(x) < \infty$ for all t and x. Hence, $Q_t \in \mathcal{Q}_e^f$ for all $t < 1$. Next, the function

$$t \longmapsto u^*_{Q_t}(y) := \sup_{x>0}\big(u_{Q_t}(x) - xy\big)$$

is also convex as the supremum of convex functions. In particular, $u^*_{Q_t}(y)$ is upper semicontinuous in t, and we obtain $u^*_{Q_1}(y) \geq \limsup_{t\uparrow 1} u^*_{Q_t}(y)$. This proves (23).

In the next step, we show that

$$\inf_{Q\in\mathcal{Q}}\sup_{x>0}\big(u_Q(x)-xy\big)=\sup_{x>0}\inf_{Q\in\mathcal{Q}}\big(u_Q(x)-xy\big)=\sup_{x>0}\big(u(x)-xy\big). \tag{24}$$

Combining this identity with (23) and (22) will complete the proof of the duality formula for v. We have for $\varepsilon > 0$

$$\inf_{Q\in\mathcal{Q}}\sup_{x>0}\big(u_Q(x)-xy\big)\leq\inf_{Q\in\mathcal{Q}}\sup_{x>0}\sup_{g\in\mathcal{C}(x)}\big(E_Q[\,U(\varepsilon+g)\,]-xy\big).$$

On the one hand, the function $x\mapsto\sup_{g\in\mathcal{C}(x)}(E_Q[\,U(\varepsilon+g)\,]-xy)$ is concave. On the other hand, the functional $Q\mapsto\sup_{g\in\mathcal{C}(x)}(E_Q[\,U(\varepsilon+g)\,]-xy)$ is convex and weakly lower semicontinuous, as can be seen as in the proof of Lemma 3.3. By Lemma 3.1, we may thus apply the lop sided minimax theorem [1, Chapter 6, p. 295] and obtain

$$\begin{aligned}\inf_{Q\in\mathcal{Q}}\sup_{x>0}\sup_{g\in\mathcal{C}(x)}\big(E_Q[\,U(\varepsilon+g)\,]-xy\big)&=\sup_{x>0}\inf_{Q\in\mathcal{Q}}\sup_{g\in\mathcal{C}(x)}\big(E_Q[\,U(\varepsilon+g)\,]-xy\big)\\&\leq\sup_{x>0}\inf_{Q\in\mathcal{Q}}\big(u_Q(x+\varepsilon)-xy\big)\\&\leq\sup_{x>0}\inf_{Q\in\mathcal{Q}}\big(u_Q(x)-xy\big)+\varepsilon y.\end{aligned}$$

Sending $\varepsilon\downarrow 0$ thus yields (24). The identities in (7) can be proved as in [13, Lemma 3.5]. □

Proof of Corollary 2.4: The uniqueness of $\widehat{Y}$ follows from the strict convexity of V and the fact that the inequality (18) is strict if $y_0/x_0\neq y_1/x_1$. This also yields the strict convexity of v and in turn the differentiability of u; see, e.g., [17, Theorem V.26.3]. □

4 The existence of optimal strategies

Lemma 4.1 *For any $x_0>0$, there exists some $\widehat{Z}\in\mathcal{Z}$, $\widehat{y}>0$, $\widehat{g}\in\mathcal{C}(x_0)$, and $\widehat{h}\in\mathcal{D}(\widehat{y})$ such that*

(a) $u(x_0)=u_{\widehat{Z}}(x_0)=\mathbb{E}[\,\widehat{Z}U(\widehat{g})\,]$,

(b) $v(\widehat{y})=v_{\widehat{Z}}(\widehat{y})=\mathbb{E}[\,\widehat{Z}V(\widehat{h}/\widehat{Z})\,]$,

(c) $u(x_0)=v(\widehat{y})+x_0\widehat{y}$,

Proof: Let (Z_n) be any sequence in $\mathcal{Z}_e$ such that $u_{Z_n}(x_0) \to u(x_0)$. Such sequences exist due to Lemma 3.3. In the first step, we show that

$$u'_+(x_0) \le \liminf_{n\uparrow\infty} u'_{Z_n}(x_0) \le \limsup_{n\uparrow\infty} u'_{Z_n}(x_0) \le u'_-(x_0),$$

where $u'_\pm(x_0)$ are the left- and right-hand derivatives of u in x_0. Indeed, the concavity of u_{Z_n} implies that for $x_1 \in (0, x_0)$

$$u'_{Z_n}(x_0) \le \frac{u_{Z_n}(x_0) - u_{Z_n}(x_1)}{x_0 - x_1} \le \frac{u_{Z_n}(x_0) - u(x_1)}{x_0 - x_1}.$$

Sending first $n \uparrow \infty$ and then $x_1 \uparrow x_0$ yields $\limsup_n u'_{Z_n}(x_0) \le u'_-(x_0)$. To get the lower bound, use a similar argument with $x_2 > x_0$.

In the next step, we use the standard Komlos-type argument to obtain a sequence (Z_n) in $\mathcal{Z}_e$ such that both $u_{Z_n}(x_0) \to u(x_0)$ and $Z_n \to \widehat{Z}$ $\mathbb{P}$-a.s., which is possible due to the convexity of the functional $Z \mapsto u_Z(x_0)$. Moreover, we have for any $\varepsilon > 0$

$$\begin{aligned} u_{\widehat{Z}}(x_0) &\le \sup_{g\in\mathcal{C}(x_0)} \mathbb{E}[\,\widehat{Z}U(g+\varepsilon)\,] \le \liminf_{n\uparrow\infty} \sup_{g\in\mathcal{C}(x_0)} \mathbb{E}[\,Z_nU(g+\varepsilon)\,] \\ &\le \liminf_{n\uparrow\infty} u_{Z_n}(x_0+\varepsilon) \le \liminf_{n\uparrow\infty} \big(u_{Z_n}(x_0) + \varepsilon u'_{Z_n}(x_0)\big) \\ &\le u(x_0) + \varepsilon u'_-(x_0). \end{aligned}$$

Taking $\varepsilon \downarrow 0$ gives $u(x_0) = u_{\widehat{Z}}(x_0)$.

Let $y_n := u'_{Z_n}(x_0)$. By passing to a subsequence if necessary, we may assume that (y_n) converges to some $\widehat{y} \in [u'_+(x_0), u'_-(x_0)]$. Since u is concave and strictly increasing, we have $\widehat{y} > 0$. Applying the results of [13, 14] for each n, we get

$$v_{Z_n}(y_n) = u_{Z_n}(x_0) - x_0y_n \longrightarrow u(x_0) - x_0\widehat{y} = v(\widehat{y}),$$

where we have used the duality relation (6) and the fact that $\widehat{y}$ is in the supergradient of u. Due to the results in [13, 14], there exit $h_n \in \mathcal{D}(y_n)$ such that $v_{Z_n}(y_n) = \mathbb{E}[\,Z_nV(h_n/Z_n)\,]$. As in the proof of Lemma 3.6, we obtain a sequence

$$(Z'_n, h'_n) \in \operatorname{conv}\{(Z_n, h_n), (Z_{n+1}, h_{n+1}), \dots\}$$

that converges $\mathbb{P}$-a.s. to $(\widehat{Z}, \widehat{h})$, where $\widehat{h} \in \mathcal{D}(\widehat{y})$. As in Lemma 3.6, we obtain $\mathbb{E}[\,\widehat{Z}V(\widehat{h}/\widehat{Z})\,] = v(\widehat{y})$. □

Lemma 4.2 *If $AE(U) < 1$ then $\{\widehat{Z} > 0\} \subseteq \{\widehat{h} > 0\}$.*

Proof: Suppose by way of contradiction that $B := \{\widehat{Z} > 0\} \cap \{\widehat{h} = 0\}$ satisfies $\mathbb{P}[\,B\,] > 0$. If $V(0) = \infty$, then $\mathbb{P}[\,B\,] > 0$ would contradict the fact that $v(\widehat{y}) < \infty$. Now we consider the case $V(0) < \infty$. Take $h \in \mathcal{D}(\widehat{y})$ such that $\mathbb{P}[\,h > 0\,] = 1$ (e.g., we can take $ydP^*/d\mathbb{P}$ where $P^* \in \mathcal{M}$), and let $h_t := (1-t)\widehat{h} + th\mathrm{I}_B \in \mathcal{D}(\widehat{y})$ for $0 \le t \le 1$. Then $\widehat{Z}V(h_t/\widehat{Z}) \in L^1(\mathbb{P})$, due to Lemma 3.5.

Next, as $t \downarrow 0$, $\frac{1}{t}\widehat{Z}V(h_t/\widehat{Z})$ decreases to

$$V'(0+)h \cdot \mathrm{I}_B - V'\Big(\frac{\widehat{h}}{\widehat{Z}}\Big)\widehat{h} \cdot \mathrm{I}_{B^c \cap \{\widehat{Z} > 0\}}.$$

Due to our assumtion $AE(U) < 1$ and [13, Lemma 6.3 (iv)], there exist constants $c, y_0 > 0$ such that $-V'(y) \le cV(y)/y$ for $0 < y \le y_0$. This implies that

$$-\mathbb{E}\Big[\,V'\Big(\frac{\widehat{h}}{\widehat{Z}}\Big)\widehat{h} \cdot \mathrm{I}_{B^c \cap \{\widehat{Z} > 0\}}\,\Big] < \infty.$$

On the other hand, $V'(0+) = -\infty$, and so monotone convergence guarantees that

$$\frac{1}{t}\mathbb{E}\Big[\,\widehat{Z}V\Big(\frac{h_t}{\widehat{Z}}\Big)\,\Big] \longrightarrow -\infty \qquad \text{as } t \downarrow 0.$$

But this contradicts the optimality of $\widehat{h}$. □

Proof of Theorem 2.6: Due to our assumption (8), we have

$$\frac{u_Q(x)}{x} \longrightarrow 0 \qquad \text{as } x \uparrow \infty \tag{25}$$

for each $Q \in \mathcal{Q}_e$; see [14, Note 1]. Hence it follows from the proof of [14, Eq. (25)] that the mapping $\mathcal{C}(x) \ni g \mapsto E_Q[\,U(g)\,]$ is upper semicontinuous with respect to almost-sure convergence (note that the proof of Eq. (25) in [14] does not use the assumption that (g^n) is a maximizing sequence). Hence, $\mathcal{C}(x) \ni g \mapsto \inf_{Q \in \mathcal{Q}_e} E_Q[\,U(g)\,]$ is also upper semicontinuous with respect to almost-sure convergence. Now let $(\widetilde{g}_n)$ be a maximizing sequence in $\mathcal{C}(x)$. By the usual Komlos-type argument there is a sequence $g_n \in \mathrm{conv}\{\widetilde{g}_n, \widetilde{g}_{n+1}, \dots\}$ converging $\mathbb{P}$-a.s. to some $\widehat{g} \ge 0$. We have $\widehat{g} \in \mathcal{C}(x)$ due to (15). Moreover, the concavity of the functional $g \mapsto \inf_{Q \in \mathcal{Q}_e} E_Q[\,U(g)\,]$ implies that (g_n) is again a maximizing sequence, while its upper semicontinuity yields that $\inf_{Q \in \mathcal{Q}_e} E_Q[\,U(\widehat{g})\,] \ge u(x)$. In fact, we even have $\inf_{Q \in \mathcal{Q}} E_Q[\,U(\widehat{g})\,] \ge u(x)$. To see this, note first that the set $\{Q \in \mathcal{Q} \,|\, E_Q[\,U(\widehat{g})\,] = -\infty\}$ must be empty, for otherwise it would have a non-void intersection with $\mathcal{Q}_e$. Hence, for $Q \in \mathcal{Q} \backslash \mathcal{Q}_e$ and $Q_0 \in \mathcal{Q}_e$, $E_Q[\,U(\widehat{g})\,]$ is the limit as $t \uparrow 1$ of $E_{Q_t}[\,U(\widehat{g})\,]$ with $Q_t := tQ + (1-t)Q_0 \in \mathcal{Q}_e$.

Next, for $\widehat{Z}$ as in Lemma 4.1, we get

$$u(x) = u_{\widehat{Z}}(x) \ge \mathbb{E}[\,\widehat{Z}U(\widehat{g})\,] \ge \inf_{Q \in \mathcal{Q}} E_Q[\,U(\widehat{g})\,] \ge u(x),$$

so all inequalities are in fact identities, and $\widehat{g}$ is optimal.

Next, we show that the optimal $\widehat{g}$ coincides $d\widehat{Q} := \widehat{Z}\, d\mathbb{P}$-a.s. with $I(\widehat{h}/\widehat{Z})$, where $\widehat{y}$ and $\widehat{h}$ are as in Lemma 4.1 for $x_0 := x$. We have $0 \le V(\widehat{h}/\widehat{Z}) + \widehat{g}\widehat{h}/\widehat{Z} - U(\widehat{g})$ and

$$\begin{aligned} E_{\widehat{Q}}[\, V(\widehat{h}/\widehat{Z}) + \widehat{g}\widehat{h}/\widehat{Z} - U(\widehat{g})\,] &= v(\widehat{y}) + \mathbb{E}[\,\widehat{g}\widehat{h};\, \widehat{Z} > 0\,] - u(x) \\ &\le v(\widehat{y}) + x\widehat{y} - u(x) = 0. \end{aligned}$$

Thus, $0 = V(\widehat{h}) + \widehat{g}\widehat{h}/\widehat{Z} - U(\widehat{g})$ and in turn $\widehat{g} = I(\widehat{h}/\widehat{Z})$ $\widehat{Q}$-a.s. We also get $\mathbb{E}[\,\widehat{g}\widehat{h};\, \widehat{Z} > 0\,] = x\widehat{y}$, which in view of the *a priori* bound $\mathbb{E}[\,\widehat{g}\widehat{h}\,] \le x\widehat{y}$ implies that

$$\{\widehat{g}\widehat{h} > 0\} \subset \{\widehat{Z} > 0\}. \tag{26}$$

Clearly, $\widehat{g} = \widehat{X}_T$ and $\widehat{h} = Y_T$ for some $\widehat{X} \in \mathcal{X}(x)$ and $Y \in \mathcal{Y}(\widehat{y})$. Their product forms a martingale under $\mathbb{P}$ since $\mathbb{E}[\,\widehat{X}_T Y_T\,] = x\widehat{y}$. The identity (12) follows from the definition of the dual value function and the corresponding identity in [13, 14]. The assertion that $u'(\infty-) = 0$ follows from the fact that $u(x)/x \to 0$ as $x \uparrow \infty$, which is itself a consequence of [14, Note 1]. The second identity in (10) follows from the first and the duality relations between u and v.

Next, suppose that $AE(U) < 1$. The identity (13) follows from (26) and Lemma 4.2. We now prove the strict concavity of u, which will in turn imply the differentiability of v, due to the duality relations and general principles. Taking $Z_1 \in \mathcal{Z}_e$ and letting $Z_t := tZ_1 + (1-t)\widehat{Z} \in \mathcal{Z}_e$, we get from Lemma 3.2 and (3) that

$$u_{\widehat{Z}}(x) = \lim_{t\downarrow 0} u_{Z_t}(x) = \lim_{t\downarrow 0} \inf_{y>0} \big(v_{Z_t}(y) + xy\big) \le \inf_{y>0} \big(v_{\widehat{Z}}(y) + xy\big), \tag{27}$$

where we have used in the last step that $t \mapsto v_{Z_t}(y)$ is upper semicontinuous as the infimum, taken over h, of the convex functions $t \mapsto \mathbb{E}[\, Z_t V(h/Z_t)\,]$. Since $v_{\widehat{Z}}(\widehat{y}) < \infty$, it follows as in [14, Note 2] that $v_{\widehat{Z}}(y) < \infty$ for all $y > 0$ provided that $AE(U) < 1$ holds. Hence (27) implies that (25) holds for $Q := \widehat{Q}$. It follows from the proof of [14, Lemma 1] that for each $\xi > 0$ there is some $g \in \mathcal{C}(\xi)$ such that $u_{\widehat{Z}}(\xi) = \mathbb{E}[\,\widehat{Z}U(g)\,]$. This g must be $\widehat{Q}$-a.s. unique, and we obtain the strict concavity of the function $u_{\widehat{Z}}$. From here we get the strict concavity of the robust value function u: Take $x_1, x_2 > 0$, $x := (x_1 + x_2)/2$, and let $\widehat{Z}$ be as above, then

$$\frac{1}{2}\big(u(x_1) + u(x_2)\big) - u(x) \le \frac{1}{2}\big(u_{\widehat{Z}}(x_1) + u_{\widehat{Z}}(x_2)\big) - u_{\widehat{Z}}(x) < 0.$$

□

Proof of Corollary 2.7: Let us suppose that $\widehat{g}_i \in \mathcal{C}(x_i)$, $i = 1, 2$, are such that $u(x_i) = \inf_{Q\in\mathcal{Q}} E_Q[\,U(\widehat{g}_i)\,]$. Due to Lemma 3.1 and the weak lower semicontinuity of $Z \mapsto \mathbb{E}[\,ZF\,]$ for $F \ge 0$, there exists some $\widetilde{Q} \in \mathcal{Q}$ such that

$$\begin{aligned} u\Big(\frac{x_1 + x_2}{2}\Big) - \frac{u(x_1) + u(x_2)}{2} &\ge \inf_{Q\in\mathcal{Q}} E_Q\Big[\,U\Big(\frac{\widehat{g}_1 + \widehat{g}_2}{2}\Big) - \frac{U(\widehat{g}_1) + U(\widehat{g}_2)}{2}\,\Big] \\ &= E_{\widetilde{Q}}\Big[\,U\Big(\frac{\widehat{g}_1 + \widehat{g}_2}{2}\Big) - \frac{U(\widehat{g}_1) + U(\widehat{g}_2)}{2}\,\Big], \end{aligned}$$

and the last term is strictly positive as soon as $\mathbb{P}[\,\widehat{g}_1 \neq \widehat{g}_2\,] > 0$. With $x_1 = x_2$, this gives the uniqueness of the optimal $\widehat{g}$, for $x_1 \neq x_2$ we then obtain the strict concavity of u. The asserted properties of v now follow by general principles (e.g., [17, Theorem V.26.3]) from the duality relation (6). The remaining assertions follow from the preceding results. □

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