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On Local Times of Ranked Continuous Semimartingales; Application to Portfolio Generating Functions

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On Local Times of Ranked Continuous Semimartingales; Application to Portfolio Generating Functions

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We derive the decomposition of the ranked continuous semimartingales i.e. order-statistics processes. We apply it to portfolios generated by functions of the ranked market weights. Thus we generalize recent results of Fernholz.

Key words and phrases: Portfolio-generating function, continuous semimartingale, local time, ranked processes.

MSC2000: 60H05, 60J65.

1 Introduction

The distribution of capital, i.e., the family of ranked market weights, starting with the largest weight and going to the smallest, is of central importance in stochastic portfolio theory, as are functionally generated portfolios. Usually we identify stocks by their *names*, i.e., their subscripts, X_1, X_2, X_3, etc. However, with regards to the distribution of capital, it is advantageous to identify the stocks by their *ranks* rather than their names. Portfolios generated by functions of the ranked market weights first appeared in Fernholz ([2]). Fernholz derivations require that the processes $X_1, \cdots, X_n$ be *pathwise mutually nondegenerate*. Here, we shall extend Fernholz's results in its full generality. The relative return of a functionally generated portfolio satisfies a stochastic differential equation wich depends on the *local times* associated with the changes in rank among the stocks.

Section 2 of the paper contains the decomposition of the ranked continuous semimartingales i.e. order-statistics processes. As an interesting byproduct, we obtain an extension of Ouknine's formula [5, 6, 7]. In section 3 we use these decompositions to portfolio generated by functions of the ranked market weights. Thus we generalize recent results of Fernholz [2, 3].

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2 Decomposition of Ranked Continuous Semimartingales

Definition 2.1 *Let $X_1, \cdots, X_n$ be continuous semimartingales. For $1 \leq k \leq n$, the k-th rank process of $X_1, \cdots, X_n$ is defined by*

$$X^{(k)} = \max_{i_1 < \cdots < i_k} \min(X_{i_1}, \cdots, X_{i_k})$$

where $1 \leq i_1$ and $i_k \leq n$.

Note that, according to Definition 2.1, for $t \in \mathbb{R}^+$,

$$\max_{1 \leq i \leq n} X_i(t) = X^{(1)}(t) \geq X^{(2)}(t) \geq \cdots \geq X^{(n)}(t) = \min_{1 \leq i \leq n} X_i(t)$$

so that at any given time, the values of the rank processes represent the values of the original processes arranged in descending order (i.e. the (reverse) order statistics).

The following proposition shows that the rank processes derived from continuous semimartingales can be expressed in terms of the original processes, adjusted by local times.

Proposition 2.1 *Let $X_1, \cdots, X_n$ be continuous semimartingales. For $k \in \{1, 2, \cdots, n\}$, Let $u(k) = (u_t(k),\, t \geq 0) : \Omega \times [0, \infty[\to \{1, 2, \cdots, n\}$ be any predictable process with the property:*

$$X_t^{(k)} = X_{u_t(k)}(t)$$

Then the k-th rank processes $X^{(k)}, k = 1, \cdots, n$, are continuous semimartingales and we have:

$$\begin{aligned} X^{(k)}(t) = & X^{(k)}(0) + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, dX_i(s) \;+\; \frac{1}{2} \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L_s^0((X^{(k)} - X_i)^+) \\ & - \frac{1}{2} \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L_s^0((X^{(k)} - X_i)^-). \end{aligned}$$

where $L_t^0(X)$ is the local time of the continuous semimartingale X at 0.

Proof: We adapt here the proof given by Chitashvili and Mania ([1]) for the decomposition of the maximum of semimartingales (i.e. $k = 1$), we have,

$$X_t^{(k)} - X_0^{(k)} = \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, dX_s^i \;+\; \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d\,(X_s^{(k)} - X_s^i)$$

Where we used the property $\sum_{i=1}^{n} 1_{\{u_s(k)=i\}} = 1$ it follows,

$$X_t^{(k)} - X_0^{(k)} = \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, dX_s^i + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d\,(X_s^{(k)} - X_s^i)^+ - \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, d\,(X_s^{(k)} - X_s^i)^-$$

We note the fact:

$$\{u_s(k) = i\} \subset \{X_s^{(k)} = X_i(s)\}$$

Therefore, using the following formula

$$\frac{1}{2}\, L_t^0(X) = \int_0^t 1_{\{(X_s = 0)\}}\, d\, X_s$$

which is valid for non-negative continuous semimartingales X.

$\square$

We can give a more explicit decomposition as follows:

Corollary 2.1 *Let $X_1, \cdots, X_n$ be continuous semimartingales. Then the k-th rank processes $X^{(k)}, k = 1, \cdots, n$, are continuous semimartingales and we have:*

$$X^{(k)}(t) = X^{(k)}(0) + \sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\}}\, dX_i(s) + \frac{1}{2} \sum_{i=k+1}^{n} L_t^0(X^{(k)} - X^{(i)}) - \frac{1}{2} \sum_{i=1}^{k-1} L_t^0(X^{(i)} - X^{(k)}).$$

where $L_t^0(X)$ is the local time of the continuous semimartingale X at 0.

Proof: We fixe $i = 1, \cdots, n$ and we deal with $\int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L_s^0((X^{(k)} - X_i)^+)$

$$\begin{aligned}\int_0^t 1_{\{u_s(k)=i\}}\, d_s\, L_s^0((X^{(k)} - X_i)^+) &= \sum_{j=1}^{n} \int_0^t 1_{\{u_s(k)=i\} \cap \{X_i = X^{(j)}\}}\, d_s\, L_s^0((X^{(k)} - X_i)^+) \\ &= \sum_{j=1}^{n} \int_0^t 1_{\{u_s(k)=i\} \cap \{X_i = X^{(j)}\}}\, d_s\, L_s^0((X^{(k)} - X^{(j)})^+)\end{aligned}$$

We, conclude by noting:

$$\sum_{i=1}^{n} \int_0^t 1_{\{u_s(k)=i\} \cap \{X_i = X^{(j)}\}}\, d_s\, L_s^0((X^{(k)} - X^{(j)})^+) = L_t^0((X^{(k)} - X^{(j)})^+)$$

Indeed,

$$\begin{aligned}&\int_0^t 1_{\{u_s(k)=i\} \cap \{X_i = X^{(j)}\}}\, d_s\, L_s^0((X^{(k)} - X^{(j)})^+) \\ &= \int_0^t 1_{\{u_s(k)=i\} \cap \{X_i = X^{(j)}\}} \times 1_{\{X^{(k)} - X^{(j)} = 0\}}\, d_s\, L_s^0((X^{(k)} - X^{(j)})^+) \\ &= \int_0^t 1_{\{u_s(k)=i\} \cap \{X_i = X^{(k)}\}} \times 1_{\{X^{(k)} - X^{(j)} = 0\}}\, d_s\, L_s^0((X^{(k)} - X^{(j)})^+) \\ &= \int_0^t 1_{\{u_s(k)=i\}} d_s\, L_s^0((X^{(k)} - X^{(j)})^+)\end{aligned}$$

$\square$

In the particular case considered by Fernholz (see [2]) of rank processes derived from pathwise mutually nondegenerate absolutely continuous semimartingales

Corollary 2.2 (**Fernholz**)
Let $X_1, \cdots, X_n$ be pathwise mutually nondegenerate absolutely continuous semimartingales, and for $t \in [0, T]$, let p_t be the random permutation of $\{1., n\}$ such that for $k = 1, \cdots, n$,

$$X_{p_t(k)(t)} = X^{(k)}(t), \qquad \textit{and} \qquad p_t(k) < p_t(k+1) \quad \textit{if} \quad X^{(k)}(t) = X^{(k+1)}(t).$$

Then the k-th rank processes $X^{(k)}, k=1,\cdots,n$, *are continuous semimartingales and we have:*

$$X^{(k)}(t) = X^{(k)}(0) + \sum_{i=1}^{n} \int_0^t 1_{\{p_s(k)=i\}}\, dX_i(s) + \frac{1}{2} L_t^0(X^{(k)} - X^{(k+1)}) - \frac{1}{2} L_t^0(X^{(k-1)} - X^{(k)}).$$

where $L_t^0(X)$ *is the local time of the continuous semimartingale* X *at* 0.

Now, we state the solution to Problem 4.1.13 stated in Fernholz's book:

Corollary 2.3 *Let* $X_1,\cdots,X_n$ *be continuous semimartingales. Then the k-th rank processes* $X^{(k)}, k=1,\cdots,n$, *are continuous semimartingales and we have:*

$$\begin{aligned} X^{(k)}(t) =& X^{(k)}(0) + \sum_{i=1}^{n} \int_0^t 1_{\{p_s(k)=i\}}\, dX_i(s) + \frac{1}{2} L_t^0(X^{(k)} - X^{(k+2)}) + \frac{1}{2} L_t^0(X^{(k)} - X^{(k+1)}) \\ & - \frac{1}{2} L_t^0(X^{(k-2)} - X^{(k)}) - \frac{1}{2} L_t^0(X^{(k-1)} - X^{(k)}). \end{aligned}$$

where $L_t^0(X)$ *is the local time of the continuous semimartingale* X *at* 0.

Theorem 2.1 *Let* $X_1,\cdots,X_n$ *be continuous semimartingales. Then we have:*

$$\sum_{i=1}^{n} L_t^0(X^{(i)}) = \sum_{i=1}^{n} L_t^0(X_i)$$

where $L_t^0(X)$ *is the local time of the continuous semimartingale* X *at* 0.

Proof:

We recall first that $L_t^0(Z) = L_t^0(Z^+)$ for every semimartingale Z. Hence, it is enough to consider the case where $X_1,\cdots,X_n$ are non-negatives continuous semimartingales. We need to check the following equality holds:

$$\sum_{i=1}^{n} 1_{\{X^{(i)}=0\}}\, dX^{(i)} = \sum_{i=1}^{n} 1_{\{X_i=0\}}\, dX_i$$

From Corollary 2.1, we have

$$d\,X^{(k)} = \sum_{i=1}^{n} 1_{\{u_t(k)=i\}}\, dX_i + \sum_{i=k+1}^{n} 1_{\{X^{(k)}-X^{(i)}=0\}} d\,(X^{(k)}-X^{(i)}) - \sum_{i=1}^{k-1} 1_{\{X^{(i)}-X^{(k)}=0\}} d\,(X^{(i)}-X^{(k)}).$$

It follows:

$$\begin{aligned} 1_{\{X^{(k)}=0\}}\, d\,X^{(k)} =& \sum_{i=1}^{n} 1_{\{u_t(k)=i\}}\, 1_{\{X^{(k)}=0\}}\, 1_{\{X_i=X^{(k)}\}} dX_i + \sum_{i=k+1}^{n} 1_{\{X^{(k)}-X^{(i)}=0\}} 1_{\{X^{(k)}=0\}}\, d\,(X^{(k)}-X^{(i)}) \\ & - \sum_{i=1}^{k-1} 1_{\{X^{(i)}-X^{(k)}=0\}} 1_{\{X^{(k)}=0\}}\, d\,(X^{(i)}-X^{(k)}). \\ =& \sum_{i=1}^{n} 1_{\{u_t(k)=i\}}\, 1_{\{X_i=0\}} dX_i + \sum_{i=k+1}^{n} 1_{\{X^{(i)}=0\}} 1_{\{X^{(k)}=0\}}\, d\,(X^{(k)}-X^{(i)}) \\ & - \sum_{i=1}^{k-1} 1_{\{X^{(i)}=0\}} 1_{\{X^{(k)}=0\}}\, d\,(X^{(i)}-X^{(k)}). \end{aligned}$$

Now we may choose our $u_t(\cdot)$ as a bijection on $\{1, \cdots, n\}$, we denote its inverse by $v_t(\cdot)$

$$1_{\{X^{(k)}=0\}}\, d\, X^{(k)} = \sum_{i=1}^{n} 1_{\{v_t(i)=k\}}\, 1_{\{X_i=0\}} dX_i \;+\; \sum_{i=k+1}^{n} 1_{\{X^{(i)}=0\}} 1_{\{X^{(k)}=0\}}\, d\,(X^{(k)} - X^{(i)}) - \sum_{i=1}^{k-1} 1_{\{X^{(i)}=0\}} 1_{\{X^{(k)}=0\}}\, d\,(X^{(i)} - X^{(k)}).$$

By summation over the index $k = 1, \cdots, n$ we obtain our result.

□

In particular,

Corollary 2.4 Ouknine's formula
Let X and Y be semimartingales. It is shown that

$$L_t^0(X \vee Y) + L_t^0(X \wedge Y) = L_t^0(X) + L_t^0(Y)$$

where $L_t^0(X)$ $(t \geq 0)$ denotes the local time at 0 of X.

3 Portfolio Generating Functions

Theorem 3.1 *Let $\mathcal{M}$ be a market of stocks $X_1, \cdots, X_n$. Let $\mathbf{S}$ be a function defined on a neighborhood U of Δ^n. Suppose that there exists a positive C^2 function S defined on U . Then $\mathbf{S}$ generates the portfolio π such that for $k = 1, \cdots, n$,*

$$\pi_{p_t(k)}(t) \;=\; \big(D_k \log S(\mu_{(\cdot)}(t)) + 1 - \sum_{j=1}^{n} \mu_{(j)}(t)\, D_j \log S(\mu_{(\cdot)}(t))\big)\mu_{(k)}(t),$$

for all $t \in [0, T]$, a.s., with a drift process Θ that satisfies

$$d\Theta(t) = \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^{n} D_{ij} S(\mu_{(\cdot)}(t))\, \mu_{(i)}(t) \mu_{(j)}(t) \tau_{(ij)}(t) dt - \frac{1}{2} \sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=k+1}^{n} dL_t^0((\log(\mu_{(k)}) - \log(\mu_{(i)}))) + \frac{1}{2} \sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{k-1} dL_t^0((\log(\mu_{(i)}) - \log(\mu_{(k)})))$$

Hence,

$$
\begin{aligned}
d\,\log(Z_\pi(t)/Z_\mu(t)) &= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t) \\
&- \frac{1}{2}\sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t)) \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)dt \\
&- \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^+) \\
&+ \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^-)
\end{aligned}
$$

Remark 3.1 *The theorem above extend Fernholz's theorem where the assumption for $X_1, \cdots, X_n$ of pathwise mutually nondegenerate was needed.*

Proof :
Itô's formula implies that:

$$
\begin{aligned}
d \log \mathbf{S}(\mu_{(\cdot)}(t)) &= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) d\,\mu_{(i)}(t) \\
&+ \frac{1}{2\,\mathbf{S}(\mu_{(\cdot)}(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}(\mu_{(\cdot)}(t))\, d < \mu_i, \mu_j >_t \\
&- \frac{1}{2}\sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t))\, d < \mu_i, \mu_j >_t
\end{aligned}
$$

Now let us consider the relative return process $\log(Z_\pi(t)/Z(t))$. We have

$$
\begin{aligned}
d\,\log(Z_\pi(t)/Z_\mu(t)) &= \sum_{i=1}^{n} \pi_i(t) d \log \mu_i(t) + \gamma_\pi^\star\, dt \\
&= \sum_{k=1}^{n} \pi_{p_t(k)}(t)\, d\, \log \mu_{(k)}(t) + \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^+) \\
&- \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^-) + \gamma_\pi^\star\, dt \\
&= \sum_{k=1}^{n} \frac{\pi_{p_t(k)}(t)}{\mu_{(k)}(t)}\, d\,\mu_{(k)}(t) + \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^+) \\
&- \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^-) + \gamma_\pi^\star\, dt \\
&- \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) d < \log \mu_k(\cdot) >_t
\end{aligned}
$$

Let assume that the weights $\pi_i, i = 1, \cdots, n$ satisfy:

$$
\frac{\pi_{p_t(k)}(t)}{\mu_{(k)}(t)} = D_k \log \mathbf{S}(\mu_{(\cdot)}(t)) + \varphi(t)
$$

In this case:

$$\sum_{i=1}^{n} \frac{\pi_{p_t(i)}(t)}{\mu_{(i)}(t)}\, d\mu_{(i)}(t) = \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t) + \varphi(t) \sum_{i=1}^{n} d\mu_{(i)}(t)$$
$$= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t)$$

Now we consider the last summation $\gamma_\pi^\star\, dt - \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) d < \log \mu_k(\cdot) >_t$. It follows

$$\gamma_\pi^\star\, dt - \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) d < \log \mu_k(\cdot) >_t = \sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t)) \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)$$

Finally,

$$\begin{aligned} d\, \log(Z_\pi(t)/Z_\mu(t)) &= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t) \\ &- \frac{1}{2}\sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t)) \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t) dt \\ &+ \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^+) \\ &- \frac{1}{2}\sum_{k=1}^{n} \pi_{p_t(k)}(t) \sum_{i=1}^{n} 1_{\{u_t(k)=i\}} dL_t^0((\log(\mu_{(k)}) - \log(\mu_i))^-) \end{aligned}$$

□

Corollary 3.1 *In the particular case studied by Fernholz, where we assume the stocks $X_1, \cdots, X_n$ are pathwise mutually nondegerate, we have:*

$$\begin{aligned} d\, \log(Z_\pi(t)/Z_\mu(t)) &= \sum_{i=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t))\, d\mu_{(i)}(t) \\ &- \frac{1}{2}\sum_{i,j=1}^{n} D_i \log \mathbf{S}(\mu_{(\cdot)}(t)) D_j \log \mathbf{S}(\mu_{(\cdot)}(t)) \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t) dt \\ &+ \frac{1}{2}\sum_{k=1}^{n-1} (\pi_{p_t(k+1)}(t) - \pi_{p_t(k)}(t)) dL_t^0((\log(\mu_{(k)}) - \log(\mu_{(k+1)})) \end{aligned}$$

3.1 Examples of Rank-Dependent Portfolios

As considered by Fernholz, we have more generally:

Example. (The biggest stock) Let $\mathbf{S} = x_{(1)}$.

$$d\,\log(Z_\pi(t)/Z_\mu(t)) = \frac{1}{(\mu_{(1)}(t))}\,d\mu_{(1)}(t) - \frac{1}{2}\tau_{(11)}(t)\,dt - \frac{1}{2}\pi_{p_t(1)}(t)\sum_{i=1}^{n} 1_{\{u_t(1)=i\}} dL_t^0((\log(\mu_{(1)}) - \log(\mu_i)))$$

Hence,

$$d\,\log(Z_\pi(t)/Z_\mu(t)) = d\,\log\mu_{(1)}(t) - \frac{1}{2}\pi_{p_t(1)}(t)\sum_{i=1}^{n} 1_{\{u_t(1)=i\}} dL_t^0((\log(\mu_{(1)}) - \log(\mu_i)))$$

Equivalently,

$$d\,\log(Z_\pi(t)/Z_\mu(t)) = d\,\log\mu_{(1)}(t) - \frac{1}{2}\pi_{p_t(1)}(t)\sum_{i=1}^{n} dL_t^0((\log(\mu_{(1)}) - \log(\mu_{(i)})))$$

Since the local time component of the drift process is decreasing, the long-term relative performance of π will suffer if there are many changes of leadership in the market.

Example. (The size effect)
The *size effect* is the observed tendency of small stocks to have higher long-term returns than large stocks. Let $1 < m < n$ and suppose

$$\mathbf{S}_L(x) = x_{(1)} + \cdots + x_{(m)}.$$

Then the drift $d\Theta$ is given by:

$$d\Theta_t := -\frac{1}{2}\sum_{k=1}^{m}\frac{\mu_{(k)}}{S_L}\sum_{i=m+1}^{n} d\,L_t^0(\log\mu_{(k)} - \log\mu_{(i)})$$

If we specify, the Fernholz's case, we obtain:

$$d\Theta_t := -\frac{1}{2}\frac{\mu_{(m)}}{S_L}\,d\,L_t^0(\log\mu_{(m)} - \log\mu_{(m+1)})$$

Similarly,

$$\mathbf{S}_S(x) = x_{(m+1)} + \cdots + x_{(n)}.$$

Then the drift $d\Theta$ is given by:

$$d\Theta_t := +\frac{1}{2}\sum_{k=m+1}^{n}\frac{\mu_{(k)}}{S_L}\sum_{i=1}^{m-1} d\,L_t^0(\log\mu_{(i)} - \log\mu_{(k)})$$

Since the drift process is monotonically increasing, it is likely that the return on small-stock index will eventually be greater than that of the large-stock index. Hence, the higher long-term return of small-stock index is due to the increasing drift process, and the relative level of small-stock risk is irrelevant.

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