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On σ−additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model

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On $\sigma-$additive robust representation of convex risk measures for unbounded financial positions in the presence of uncertainty about the market model

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Abstract

Recently, Frittelli and Scandolo ([9]) extend the notion of risk measures, originally introduced by Artzner, Delbaen, Eber and Heath ([1]), to the risk assessment of abstract financial positions, including pay offs spread over different dates, where liquid derivatives are admitted to serve as financial instruments. The paper deals with $\sigma-$additive robust representations of convex risk measures in the extended sense, dropping the assumption of an existing market model, and allowing also unbounded financial positions. The results may be applied for the case that a market model is available, and they encompass as well as improve criteria obtained for robust representations of the original convex risk measures for bounded positions ([4], [7], [16]).

KEYWORDS: Convex risk measures, model uncertainty, $\sigma-$additive robust representation, Fatou property, non-sequential Fatou property, strong $\sigma-$additive robust representation, Krein-Smulian theorem, Greco theorem, inner Daniell stone theorem, general Dini theorem, Simons' lemma.

JEL CLASSIFICATION G10

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0 Introduction

The notion of risk measures has been introduced by Artzner, Delbaen, Eber and Heath (cf. [1]) as the key concept to found an axiomatic approach for risk assessment of fincancial positions. Technically, risk measures are functionals defined on sets of financial positions, satisfying some basic properties to qualify riskiness consistently. An outcome of such a functional, that means the risk of a position, is usually interpreted as the capital requirement of the position to become an acceptable one. Genuinely, risk measures has been defined for one-period positions. Recently Fritelli and Scandolo ([9]) provide a general framework which extends considerations to abstract financial positions including pay off streams with liquid derivatives as hedging positions. Applied to the risk assessment of pay off streams such general risk measures are used for an a priori qualification, which means to take the static perspective. In contrary the dynamic risk assessment take into account adjustments time after time. Readers who interested in this topic are referred to e.g. [8], [18], [21].

The main goal of this paper is to investigate risk measures ρ which admit a robust representation of the form

$$\rho(X) = \sup_{\Lambda}(-\Lambda(X) - \beta(\Lambda)),$$

where X denotes a financial position, Λ a linear form on the set of financial positions, and β stands for a penalty function on the set of linear forms. Special attention will be paid to the problem when these representing linear forms may in turn be represented by ($\sigma-$additive) probability measures. We shall speak of a robust representation of ρ by probability measures or a $\sigma-$additive robust representation. Necessarily, only so-called convex risk measures, that means risk measures which are convex mappings, may have such a robust representation. The basic assumption of this paper is that the investors are uncertain about the market model underlying the outcomes of the financial positions. Within this setting a robust representation by probability measures offers an additional economic interpretation of the risk measures. As suggested by Föllmer and Schied (cf. [7]) such a representation means that an investor has a set of possible models for the outcomes of the financial positions in mind, and evaluates the worst expected losses together with some penalty costs for misspecification w.r.t. these models.

The problem of $\sigma-$additive robust representations of convex risk measures in the genuine sense has been completely solved in the case that the investors have market models at hand. Ruszczynski and Shapiro showed that convex risk measures always admit robust representations by probability measures if for any real $p > 1$ every integrable mapping of order p is available (cf. [19]). However the used methods can not be applied to essentially bounded positions. Drawing on methods from functional analysis, Delbaen as well as Föllmer and Schied succeeded in giving a full characterization (cf. [4], [7]) by the so-called Fatou property. As pointed out by Delbaen, the Fatou property fails to be sufficient in general when the investor is faced with model uncertainty. Moreover, the problem of $\sigma-$additive robust representation is still open when a market model is not available. Restricting considerations on bounded one-period positions, Föllmer and Schied (in [7]) suggested a strict sufficient criterion, Krätschmer showed that it is in some sense also necessary, and he adds some more general conditions ([16]).

This paper may be viewed as a continuation of the studies in [7] as well as in [16]. The generalizations will be proceeded in several directions. First of all multiperiod positions and liquid hedging instruments will be allowed. Secondly we shall drop the assumptions that only bounded positions are traded. This is in accordance with empirical evidences that the distributions of risky assets often show heavy tails. Thirdly we want to investigate the issue of strong robust representations by probability measures in the sense that the optimization involved in the $\sigma-$additive robust representation has a solution. This is a quite important technical issue from the practical point of view. In many cases the calculation of outcomes of risk measures has to be employed by numerical optimization algorithms, and the most customary ones assume the existence of solutions. In presence of a market model, Jouini, Schachermayer and Touzi (cf. [11]) have given a full solution to the problem of strong robust representations.

Finally, the criteria should encompass the results already derived within a fixed market model.

The paper is organized as follows. Section 1 introduces the concept of Frittelli and Scandolo to define risk measures in general, and some representation results of risk measures will be presented as starting points for the investigations afterwards. The following section 2 deals with the question when the Fatou property might be used as a sufficient condition. In general, as a rule a nonsequential counterpart is more suitable unless in some special cases. However, it also seems that even the nonsequential Fatou property is appropriate in quite exceptional situations only. Therefore an alternative general criterion is offered in section 3, extending a former result in [16] to unbounded positions, within a nontopological framework. It will be used for strong robust representations of risk measures by probability measures in section 4. We shall succeed in giving a complete solution. In particular the aboved mentioned strict criterion by Föllmer and Schied will turn out to be necessary and sufficient. The investigations of the sections 1 - 4 will then be applied to recover in section 5 the already known representation results within a given market model. The proofs of the main results will be provided separately in the sections 7, 8 and 9 as well as in appendix B. They rely on some technical arguments gathered in section 6 and a measure theoretical tool presented in appendix A.

1 Some basic representations of convex risk measures

Let us fix a set Ω. Financial positions will be expressed by mappings $X \in \mathbb{R}^{\Omega}$. As a special case $\Omega = \widetilde{\Omega} \times \mathbb{T}$ with $\widetilde{\Omega}$ denoting a set of scenarios, equipped with a family $(\mathcal{F}_t)_{t\in\mathbb{T}}$ of $\sigma-$algebras, and $\mathbb{T}$ being a time set, we may consider financial positions $X \in \mathbb{R}^{\Omega\times\mathbb{T}}$ with $X(\cdot, t)$ being $\mathcal{F}_t-$measurable for every $t \in \mathbb{T}$. They may be viewed as discounted pay off streams, liquidated at the dates from the time set. The available financial positions are gathered by a nonvoid vector subspace $\mathfrak{X} \subseteq \mathbb{R}^{\Omega}$ containing the constants. Sometimes we shall in addition assume that $X \wedge Y := \min\{X, Y\},\ X \vee Y := \max\{X, Y\} \in \mathfrak{X}$ for $X, Y \in \mathfrak{X}$. In this case $\mathfrak{X}$ is a so-called Stonean vector lattice. For the space of bounded positions from $\mathfrak{X}$ the symbol $\mathfrak{X}_b$ will be used. Furthermore let us fix a vector subspace $\mathfrak{C} \subseteq \mathfrak{X}$ of financial positions for hedging, including the constants. In particular we may take into account liquid derivatives like put and call options as financial instruments. They are associated with a positive linear function $\pi : \mathfrak{C} \to \mathbb{R},\ \pi(1) = 1$, where $\pi(Y)$ stands for the initial costs to obtain Y. Prominent special cases of this setting are the following:

- $\mathbb{T} = \{1, ..., n\},\ (\mathcal{F}_t)_{t\in\mathbb{T}}$ family of $\sigma-$algebras on a set of scenarios $\widetilde{\Omega},\ \mathfrak{X}$ consisting of all $X \in \mathbb{R}^{\Omega\times\mathbb{T}}$ with $X(\cdot, t)\ \mathcal{F}_t -$ measurable for $t \in \mathbb{T},\ \mathfrak{C} = \mathbb{R}^n,\ \pi(y_1, ..., y_n) := \frac{1}{n}\sum\limits_{i=1}^{n} y_i$

  $n-$**period positions, one-period positions** if $n = 1$

- $\mathbb{T} = [0, T],\ (\mathcal{F}_t)_{t\in\mathbb{T}}$ be a filtration of $\sigma-$algebras on a set of scenarios $\widetilde{\Omega}$, $\mathfrak{X}$ set of financial positions X with $X(\omega, \cdot)$ being a cadlag function, $\mathfrak{C} = \mathbb{R},\ \pi$ identity on $\mathbb{R}$

  cadlag positions

Let us now introduce the concept of risk measures suggested by Frittelli and Scandolo in [9]. As for one-period positions we may choose the axiomatic viewpoint, defining a **risk measure w.r.t.** π to be a functional $\rho : \mathfrak{X} \to \mathbb{R}$ which satisfies the properties

- **monotonicity:** $\rho(X) \leq \rho(Y)$ for $X \geq Y$
- **translation invariance w.r.t.** π**:** $\rho(X + Y) = \rho(X) - \pi(Y)$ for $X \in \mathfrak{X}, Y \in \mathfrak{C}$

The meaning of these conditions may be transferred from the genuine concept of risk measures. Moreover, it can be shown that a risk measure ρ w.r.t. π satisfies $\rho(X) = \inf\{\pi(Y) \mid\ Y \in \mathfrak{C}, \rho(X + Y) \leq 0\}$ for any $X \in \mathfrak{X}$ ([9], Proposition 3.6). Regarding $\rho^{-1}(] - \infty, 0])$ as the acceptable positions, an outcome $\rho(X)$ expresses the infimal costs to hedge it. This retains the original meaning of risk measures as capital requirements.

In the following we shall focus on so-called **convex risk measures**, defined to mean risk measures which are convex mappings. Convexity is a reasonable condition for a risk measure due to its interpretation that diversification should not increase risk. From the technical point of view convexity is a necessary property for the desired dual representations of risk measures.

Let us now fix a convex risk measure $\rho : \mathfrak{X} \to \mathbb{R}$ w.r.t. π. It is associated with the so-called Fenchel-Legendre transform

$$\beta_\rho : \mathfrak{X}^* \to] - \infty, \infty],\ \Lambda \mapsto \sup_{X\in\mathfrak{X}} (-\Lambda(X) - \rho(X)),$$

where $\mathfrak{X}^*$ gathers all real linear forms on $\mathfrak{X}$. It is easy to verify that every Λ from the domain of β_ρ has to be a positive linear form extending π. The standard tools from convex analysis provide basic representation results for ρ with β_ρ as a penalty function.

Proposition 1.1 *Let $\mathfrak{X}^{*\pi}_{+}$ denote the space of all positive linear forms on $\mathfrak{X}$ which extend π, and let τ be any topology on $\mathfrak{X}$ such that $(\mathfrak{X}, \tau)$ is a locally convex topological vector space with topological dual $\mathfrak{X}'$. Then* $\rho(X) =$

$\max_{\Lambda \in \mathfrak{X}_+^{*\pi}} (-\Lambda(X) - \beta_\rho(\Lambda))$ *for every* $X \in \mathfrak{X}$. *Moreover,* $\rho(X) = \sup_{\Lambda \in \mathfrak{X}_+^{*\pi} \cap \mathfrak{X}'} (-\Lambda(X) - \beta_\rho(\Lambda))$ *holds for every* $X \in \mathfrak{X}$ *if and only if* ρ *is lower semicontinuous w.r.t.* τ.

The proof may be found in Appendix B.

The aim of the paper is to improve the representation results by allowing only representing linear forms which are in turn representable by probability measures. For notational purposes let us introduce the counterpart of β_ρ w.r.t. the probability measures on the $\sigma-$algebra $\sigma(\mathfrak{X})$ on Ω generated by $\mathfrak{X}$

$$\alpha_\rho : \mathcal{M}_1 \to]-\infty, \infty], \;\; \mathrm{P} \mapsto \sup_{X \in \mathfrak{X}} (-E_{\mathrm{P}}[X] - \rho(X)).$$

Here $\mathcal{M}_1$ is defined to consist of all probability measures P on $\sigma(\mathfrak{X})$ such that all positions from $\mathfrak{X}$ are P $-$integrable, and $E_{\mathrm{P}}[X]$ denotes the expected value of X w.r.t. P . We shall speak of a **robust representation by probability measures from** $\mathcal{M}$ or a $\sigma-$**additive robust representation of** ρ **w.r.t.** $\mathcal{M}$ if $\mathcal{M} \subseteq \mathcal{M}_1$ nonvoid, and $\rho(X) = \sup_{\mathrm{P} \in \mathcal{M}} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$ for every $X \in \mathfrak{X}$. As an immediate consequence of Proposition 1.1 we obtain a first characterization of such representations.

Proposition 1.2 *Let* F *be a vector space of bounded countably additive set functions on* $\sigma(\mathfrak{X})$ *which separates points in* $\mathfrak{X}$ *such that each* $X \in \mathfrak{X}$ *is integrable w.r.t. any* $\mu \in F$. *Then in the case that the set* $\mathcal{M}_1(F)$ *of all* $\mathrm{P} \in \mathcal{M}_1 \cap F$ *with* $E_{\mathrm{P}}|\mathfrak{C} = \pi$ *is nonvoid*

$$\rho(X) = \sup_{\mathrm{P} \in \mathcal{M}_1(F)} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P})) \textit{ for all } X \in \mathfrak{X}$$

if and only if ρ *is lower semicontinuous w.r.t. weak topology* $\sigma(\mathfrak{X}, F)$ *on* $\mathfrak{X}$ *induced by* F.

Remark 1.3 *Retaking assumptions and notations from Proposition 1.2,* ρ *admits a robust representation in terms of* $\mathcal{M}_1(F)$ *if* F *contains the Dirac measures, and if* $\liminf_i \rho(X_i) \geq \rho(X)$ *holds for every net* $(X_i)_{i \in I}$ *in* $\mathfrak{X}$ *which converges pointwise to some* $X \in \mathfrak{X}$.

In general the lower semicontinuity of ρ w.r.t. the topology from Proposition 1.2 is not easy to verify. Therefore we are looking for more accessible conditions. The considerations will be based on the idea to reduce the investigations to bounded financial positions. As shown in Lemma 6.5 below, in case of $\mathfrak{X}$ being a Stonean vector lattice, this may be achieved if the linear forms from the domain of β_ρ are representable by finitely additive set functions in the sense explained there. Fortunately, we might express this condition equivalently by the property that $\lim_{n \to \infty} \rho(-\lambda(X-n)^+) = \rho(0)$ ($(X-n)^+$ positive part of $X-n$) is satisfied for every $\lambda > 0$ and any nonnegative $X \in \mathfrak{X}$, which is obviously true if all positions in $\mathfrak{X}$ are bounded (cf. Proposition 6.6 below).

Before going into the development of criteria for $\sigma-$additive representations let us collect some necessary conditions. In the case that the positions from $\mathfrak{X}$ are essentially bounded mappings w.r.t. a reference probability measure of a given market model the so-called Fatou property plays a prominent role. Adapting this concept, we shall say that a risk measure ρ fulfill the **Fatou property** if $\liminf_{n\to\infty} \rho(X_n) \geq \rho(X)$ whenever $(X_n)_n$ denotes a uniformly bounded sequence in $\mathfrak{X}$ which converges pointwise to some bounded $X \in \mathfrak{X}$. The Fatou property implies obviously that $\rho|\mathfrak{X}_b$ is **continuous from above**, defined to mean $\rho(X_n) \nearrow \rho(X)$ for $X_n \searrow X$. Both conditions coincide if $\sup_n X_n \in \mathfrak{X}$ for any uniformly bounded sequence $(X_n)_n$ in $\mathfrak{X}$.

Proposition 1.4 *Let ρ admit a $\sigma-$additive robust representation w.r.t. some nonvoid $\mathcal{M} \subseteq \mathcal{M}_1$, then ρ satisfies the Fatou property and $\rho|\mathfrak{X}_b$ is continuous from above. Moreover, if $\mathfrak{X}$ is a Stonean vector lattice, and if $\mathfrak{L} \subseteq \mathfrak{X}$ denotes any Stonean vector lattice which contains $\mathfrak{C}$ as well as generates $\sigma(\mathfrak{X})$, then $\rho(X) = \sup_{X \leq Y \in E} \inf_{Y \geq Z \in \mathfrak{X}} \rho(Z)$ for every bounded nonegative $X \in \mathfrak{X}$, where $E := \{\sup_n Y_n \mid Y_n \in \mathfrak{L}, Y_n \geq 0, \sup_n Y_n$ bounded$\}$.*

The proof may be found in section 7.

As mentioned in the introduction, a robust representation of ρ by probability measures is not guaranteed in general by the Fatou property or continuity from above, even if $\mathfrak{X}$ contains bounded positions only. In the next section we shall investigate additional conditions to guarantee the sufficiency by the Fatou property and a nonsequential counterpart.

2 Representation of convex risk measures by probability measures and the Fatou properties

It will turn out by the investigations within this section that in the case of uncertainty about the market model the nonsequential counterpart of the Fatou property takes over partly the role that the Fatou property plays when a reference probability measure is given. We shall say that ρ satisfies the **nonsequential Fatou property** if $\liminf_i \rho(X_i) \geq \rho(X)$ holds whenever $(X_i)_{i\in I}$ is a uniformly bounded net in $\mathfrak{X}$ which converges pointwise to some bounded $X \in \mathfrak{X}$. The following condition provides an important special situation when the Fatou property and its nonsequential counterpart are equivalent.

(2.1) For any $r > 0$, every $Z \in \mathfrak{X}_b$ from the closure of $A_r := \{X \in \mathfrak{X}_b \mid \rho(X) \leq 0, \ \sup_{\omega\in\Omega} |X(\omega)| \leq r\}$ w.r.t. the topology of pointwise convergence on $\mathfrak{X}_b$ is the pointwise limit of a sequence in A_r.

Lemma 2.1 *Under (2.1) ρ satisfies the nonsequential Fatou property if and only if it fulfills the Fatou property.*

The proof is delegated to section 9.

Remark:

The sequential condition (2.1) is closely related with the concepts of double limit relations. For a comprehensive exposition the reader is referred to [15]. In general one may try to apply double limit relations to the sets A_r from (2.1) and suitable sets of bounded countably additive set functions on $\sigma(\mathfrak{X})$.

The main result of this section relies on the following assumption, denoting by $B(\Omega)$ the space of all bounded real-valued mappings on Ω.

(2.2) The sets $B_r := \{X \in \mathfrak{X}_b \mid \sup_{\omega\in\Omega} |X(\omega)| \leq r\}$ $(r > 0)$ are closed w.r.t. the topology of pointwise convergence on $B(\Omega)$.

Theorem 2.2 *Let either $\mathfrak{X} = \mathfrak{X}_b$ or $\mathfrak{X}$ be a Stonean vector lattice such that $\lim_{n\to\infty} \rho(-\lambda(X-n)^+) = \rho(0)$ holds for any nonnegative $X \in \mathfrak{X}$, $\lambda > 0$. Furthermore let F denote a vector space of bounded countably additive set functions on $\sigma(\mathfrak{X})$ which contains all Dirac measures as well as at least one probability measure P with $E_{\mathrm{P}}|\mathfrak{C} = \pi$ such that every $X \in \mathfrak{X}$ is integrable w.r.t. any $\mu \in F$. Additionally, F is supposed to be complete w.r.t. the seminorm $\|\cdot\|_F$, defined by $\|\mu\|_F := \sup\{|\int X \, d\mu| \mid X \in \mathfrak{X}_b, \sup_{\omega\in\Omega} |X(\omega)| \leq 1\}$. Consider the following statements:*

.1 ρ satisfies the nonsequential Fatou property.

.2 ρ has a $\sigma-$additive robust representation w.r.t. $\mathcal{M}_1 \cap F$.

.3 ρ fulfills the Fatou property.

If (2.2) is valid, then .1 $\Rightarrow$ .2 $\Rightarrow$ .3, and all statements are equivalent provided that condition (2.1) holds in addition. In the case that the sets A_r from (2.1) are even relatively compact w.r.t. the weak topology $\sigma(\mathfrak{X}, F)$ we have .1 $\Leftrightarrow$ .2 $\Rightarrow$ .3.

The proof will be performed in section 9.

Remark 2.3 *The nonsequential Fatou property is not necessary for a $\sigma-$additive representation of risk measures. Take for example $\mathfrak{X}$ the space of all boundend Borel-measurable mappings on $\mathbb{R}$, and define ρ by $\rho(X) = -E_{\mathrm{P}}[X]$, where P denotes any probability measure which is absolutely convex w.r.t. the Lebesgue-Borel measure on $\mathbb{R}$. Obviously, on one hand ρ is a convex risk measure w.r.t. the identity on $\mathbb{R}$, having a trivial $\sigma-$additive robust representation. On the other hand, consider the set I of the cofinite subsets of $\mathbb{R}$, directed by set inclusion, and the net $(X_i)_{i\in I}$ of all its indicator mappings. It converges pointwise to 0, but unfortunately $\liminf_i \rho(X_i) = -1 < 0 = \rho(0)$.*

Remark 2.4 *Let F be any vector space of bounded countably additive set functions on $\sigma(\mathfrak{X})$ such that each $X \in \mathfrak{X}$ is integrable w.r.t. every $\mu \in F$, and such that $\mathfrak{X}_b$ separates points in F. Additionally, F is supposed to be closed w.r.t. the norm of total variation. Then the sets A_r from (2.1) are relatively $\sigma(\mathfrak{X}, F)-$compact if and only if $\mathfrak{X}_b$ may be identified via evaluation mapping with the topological dual of F w.r.t. the norm of total variation (cf. proof in section 9).*

In the case of an at most countable Ω, we have a simplified situation which admits an application of the full Theorem 2.2. The reason is that then the topology of pointwise convergence on the space $B(\Omega)$ is metrizable.

Corollary 2.5 *Let Ω be at most countable, and let $\mathfrak{X} \subseteq B(\Omega)$ be sequentially closed w.r.t. the pointwise topology on $B(\Omega)$. Then ρ has a robust representation by probability measures from $\mathcal{M}_1$ if and only it satisfies the Fatou property, or equivalently, if and only if ρ is continuous from above.*

As another application of Theorem 2.2 we shall retain in the proof of Theorem 5.3 below the above mentioned result that in face of a market model the Fatou propery describes equivalently robust representations of convex risk measures for essentially bounded positions by probability measures. Unfortunately, it is unclear whether we may avoid in Theorem 2.2 condition (2.2) in order to guarantee a $\sigma-$additive robust representation of risk measures by the nonsequential Fatou property. Moreover, the nonsequential Fatou property is unsatisfactory in the way that it does not work for trivial representations like those indicated in Remark 2.3. However, we may only provide a sufficient substitution by the Fatou property under the quite restrictive condition (2.1). So it seems that in presence of model uncertainty the Fatou property and its nonsequential counterpart are appropriate conditions for $\sigma-$additive representations of convex risk measures in quite exceptional situations only. Therefore we shall look for alternatives in the following section.

3 Robust representation of convex risk measures by inner regular probability measures

Throughout this section let $\mathfrak{X}$ be a Stonean vector lattice, and let $\mathfrak{L} \subseteq \mathfrak{X}$ denote any Stonean vector lattice which contains $\mathfrak{C}$ as well as generates $\sigma(\mathfrak{X})$ and which induces the set system $\mathcal{S} := \{\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[) \mid X_n \in \mathfrak{L} \text{ nonnegative, bounded}, x_n > 0\}$. Additionally, let E consist of all bounded $\sup_n Y_n$, where $(Y_n)_n$ is a sequence of nonnegative bounded positions from $\mathfrak{L}$. In view of the inner Daniell-Stone theorem (cf. [14], Theorem 5.8, final remark after Addendum 5.9) every probability measure $\mathrm{P} \in \mathcal{M}_1$ has to be inner regular w.r.t. $\mathcal{S}$, i.e. $\mathrm{P}(A) = \sup_{A \supseteq B \in \mathcal{S}} \mathrm{P}(B)$ for every $A \in \sigma(\mathfrak{X})$. So within this setting we are dealing with robust representations of ρ by probability measures from $\mathcal{M}_1(\mathcal{S})$ defined to consist of all probability measures belonging to $\mathcal{M}_1$ which are inner regular w.r.t. $\mathcal{S}$ and which represent π on $\mathfrak{C}$. We are ready to formulate the general representation result w.r.t. inner regular probability measures.

Theorem 3.1 *Let Δ_c ($c \in]-\rho(0), \infty[$) gather all $\mathrm{P} \in \mathcal{M}_1(\mathcal{S})$ with $\alpha_\rho(\mathrm{P}) \leq c$, and let ρ satisfy the following properties.*

(1) $\rho(X) = \sup_{X \leq Y \in E} \inf_{Y \geq Z \in \mathfrak{X}} \rho(Z)$ *for all nonnegative bounded $X \in \mathfrak{X}$,*

(2) $\inf_{Y \geq Z \in \mathfrak{X}} \rho(Z) = \inf_{Y \geq Z \in \mathfrak{L}} \rho(Z)$ *for* $Y \in E$,

(3) $\rho(X_n) \searrow \rho(X)$ *for any isotone sequence* $(X_n)_n$ *of bounded positions* $X_n \in \mathfrak{L}$ *with* $X_n \nearrow X \in \mathfrak{L}$, X *bounded,*

(4) $\lim_{n \to \infty} \rho(-\lambda(X - n)^+) = \rho(0)$ *for every nonnegative* $X \in \mathfrak{X}$ *and* $\lambda > 0$.

Then we may state:

.1 The initial topology $\tau_{\mathfrak{L}}$ *on* $\mathcal{M}_1(\mathcal{S})$ *induced by the mappings* $\psi_X : \mathcal{M}_1(\mathcal{S}) \to \mathbb{R}$, $\mathrm{P} \mapsto E_{\mathrm{P}}[X]$, $(X \in \mathfrak{L})$ *is completely regular and Hausdorff.*

.2 Each Δ_c $(c \in]-\rho(0), \infty[)$ *is compact w.r.t.* $\tau_{\mathfrak{L}}$, *and furthermore for every* Λ *from the domain of* β_ρ *there is some* $\mathrm{P} \in \mathcal{M}_1(\mathcal{S})$ *with* $\Lambda|\mathfrak{L} = E_{\mathrm{P}}|\mathfrak{L}$ *and* $\alpha_\rho(\mathrm{P}) \leq \beta_\rho(\Lambda)$.

.3 $\rho(X) = \sup_{\mathrm{P} \in \mathcal{M}_1(\mathcal{S})} (E_{\mathrm{P}}[-X] - \alpha_\rho(\mathrm{P}))$ *for all* $X \in \mathfrak{X}$.

The proof of Theorem 3.1 is delegated to section 7.

Remarks 3.2 *In view of Proposition 1.4, assumption (1) in Theorem 3.1 is necessary for a robust representation of* ρ *by probability measures. Let us now point out some special situations where the assumptions on* ρ, *imposed in Theorem 3.1, may be simplified:*

.1 If $\mathfrak{X}$ *is restricted to bounded positions, then assumption (4) is redundant. Also (1), (2) hold in general in the case* $\mathfrak{X} = \mathfrak{L}$.

.2 By Lemma 6.4 below assumption (3) is fulfilled in general whenever $\mathfrak{L}_{+b}$, *consisting of all nonnegative bounded* $X \in \mathfrak{L}$, *is a so-called* **Dini cone**, *i.e.* $\inf_n \sup_{\omega \in \Omega} X_n(\omega) = \sup_{\omega \in \Omega} \inf_n X_n(\omega)$ *for any antitone sequence* $(X_n)_n$ *in* $\mathfrak{L}_{+b}$ *with pointwise limit in* $\mathfrak{L}_{+b}$. *The most prominent Dini cones are the cones of nonnegative upper semicontinuous and nonnegative continuous real-valued mappings on compact Hausdorff spaces due to the general Dini lemma (cf. [12], Theorem 3.7).*

.3 If $E \subseteq \mathfrak{X}$, *then assumptions (1), (2) read as follows:*

(1) $\rho(X) = \sup_{X \leq Y \in E} \rho(Y)$ *for all nonnegative bounded* $X \in \mathfrak{X}$,

(2) $\rho(Y) = \inf_{Y \geq Z \in \mathfrak{L}} \rho(Z)$ *for* $Y \in E$.

We may specialize to $\mathfrak{X} = \mathfrak{L}$, and a direct application of Theorem 3.1 in combination with Lemma 6.4 below leads to the following condition to ensure that every linear form Λ from the domain of β_ρ is representable by a probability measure. Note that here $\mathcal{M}_1(\mathcal{S}) = \mathcal{M}_1$.

Corollary 3.3 *Let* $\mathfrak{X}$ *be a Stonean vector lattice, and let* $\lim_{n \to \infty} \rho(-\lambda(X - n)^+) = \rho(0)$ *be valid for every nonnegative* $X \in \mathfrak{X}, \lambda > 0$. *Then every linear form from the domain of* β_ρ *is representable by some probability measure from* $\mathcal{M}_1$ *if and only if* $\rho(X_n) \searrow \rho(X)$ *whenever* $(X_n)_n$ *is an isotone sequence of bounded positions from* $\mathfrak{X}$ *which converges pointwise to some bounded* $X \in \mathfrak{X}$.

Remark:

Corollary 3.3 extends a respective result for bounded one-period positions ([16], Proposition 3).

Let us now consider some special situations where Theorem 3.1 might be used.

Remark 3.4 *Let $\Omega = \widetilde{\Omega} \times \mathbb{T}$ with $\widetilde{\Omega}$ denoting a set of scenarios, equipped with a metrizable topology $\tau_{\widetilde{\Omega}}$ as well as the induced $\sigma-$algebra $\mathcal{B}(\widetilde{\Omega})$, and $\mathbb{T}$ being a time set, endowed with a separably metrizable topology $\tau_{\mathbb{T}}$ as well as the generated $\sigma-$algebra $\mathcal{B}(\mathbb{T})$. Furthermore let $\mathfrak{X}$ consist of all bounded real-valued mappings on $\widetilde{\Omega} \times \mathbb{T}$ which are measurable w.r.t. the product $\sigma-$algebra $\mathcal{B}(\widetilde{\Omega}) \otimes \mathcal{B}(\mathbb{T})$, and let $\mathfrak{L}$ be the set of all bounded real-valued mappings on $\widetilde{\Omega} \times \mathbb{T}$ which are continuous w.r.t. the product topology $\tau_{\widetilde{\Omega}} \times \tau_{\mathbb{T}}$. Finally $\mathcal{S}$ is defined to gather the closed subsets of $\widetilde{\Omega} \times \mathbb{T}$ w.r.t. the metrizable topology $\tau_{\widetilde{\Omega}} \times \tau_{\mathbb{T}}$. Using the introduced notations, $\sigma(\mathfrak{X}) = \mathcal{B}(\Omega) \otimes \mathcal{B}(\mathbb{T})$ is generated by $\mathcal{S}$, $\mathfrak{L} \subseteq \mathfrak{X}$, and we may restate Theorem 3.1 with E being the space of all bounded nonnegative lower semicontinuous mappings on $\widetilde{\Omega} \times \mathbb{T}$. This version generalizes an analogous result for the one-period positions (cf. [16], Theorem 2), and will be proved in section 7.*

We may also utilize Theorem 3.1 for cadlag positions.

Remark 3.5 *Let $\mathbb{T} = [0, T], \mathfrak{C} = \mathbb{R}$, let $(\mathcal{F}_t)_{t \in \mathbb{T}}$ be a filtration of $\sigma-$algebras on some nonvoid set $\widetilde{\Omega}$, and let $\mathfrak{X}$ be the set of cadlag positions. Then $\sigma(\mathfrak{X})$ is the so-called optional $\sigma-$algebra. We may associate for stopping times $S_1, S_2,\ S_1 \leq S_2$, the stochastic interval $[S_1, S_2[$, defined by $[S_1, S_2[(\omega, t) := 1$ if $S_1(\omega) \leq t < S_2(\omega)$, and $[S_1, S_2[(\omega, t) := 0$ otherwise. $\mathfrak{I}$ stands for the set of all such stochastic intervals. It can be shown that $\sigma(\mathfrak{X})$ is generated by the stochastic intervals $[S, \infty[$ (cf. [5], IV, 64).*
For $\mathfrak{L}$ let us choose the vector space spanned by the stochastic intervals $[S, \infty[$, which is also spanned by the positions $\max\limits_{i \in \{1,\ldots,r\}} [S_i, \widetilde{S}_i[$ $(r \in \mathbb{N})$, where $[S_i, \widetilde{S}_i[\ \in \mathfrak{I}$ for $i \in \{1, ..., r\}$. Moreover, $\mathfrak{L}$ is indeed a Stonean vector lattice, and $\{X^{-1}([x, \infty[) \mid X \in \mathfrak{L}$ nonnegative, $x > 0\}$ is an algebra consisting of all the subsets $\bigcup\limits_{i=1}^{r} \left([S_i, \widetilde{S}_i[\right)^{-1}(\{1\})$ with $r \in \mathbb{N}$ and $[S_i, \widetilde{S}_i[\ \in \mathfrak{I}$ for $i \in \{1, ..., r\}$.
Using the introduced notations, we may restate Theorem 3.1.

Theorem 3.1 may be used as a basis to derive conditions for a strong robust representation of ρ, i.e. a $\sigma-$additive robust representation where solutions of the associated optimization problems exist. We shall succeed in finding a full characterization in the next section.

4 Strong robust representation of convex risk measures by probability measures

We want to look for conditions which induce a strong robust representation of ρ by probability measures in the sense that

$$\rho(X) = \max_{\mathrm{P}\in\mathcal{M}_1} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$$

holds for any $X \in \mathfrak{X}$. The considerations are reduced to a Stonean vector lattice $\mathfrak{X}$ being stable w.r.t. countable convex combinations of antitone sequences of financial positions. In this case the following result gives a complete answer to the problem of strong robust representations.

Theorem 4.1 *Let $\mathfrak{X}$ be a Stonean vector lattice and let us assume that for every antitone sequence $(X_n)_n$ in $\mathfrak{X}$ with $X_n \searrow 0$ and each sequence $(\lambda_n)_n$ in $[0,1]$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ there is some pointwise limit $\sum_{n=1}^{\infty} \lambda_n X_n$ of $(\sum_{n=1}^{m} \lambda_n X_n)_m$ belonging to $\mathfrak{X}$. Then the following statements are equivalent:*

.1 $\rho(X) = \max_{\mathrm{P}\in\mathcal{M}_1} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$ *holds for every* $X \in \mathfrak{X}$.

.2 $\rho(X_n) \searrow \rho(X)$ *for* $X_n \nearrow X$.

.3 $\lim_{n\to\infty} \rho(-\lambda(X-n)^+) = \rho(0)$ *hold for arbitrary nonnegative* $X \in \mathfrak{X}, \lambda > 0$, *and* $\rho(X_n) \searrow \rho(X)$ *for any isotone sequence* $(X_n)_n$ *of bounded positions from* $\mathfrak{X}$ *with* $X_n \nearrow X$, X *being bounded.*

In any of these cases every linear form from the domain of β_ρ is representable by a probability measure. Moreover, for any Stonean vector lattice $\mathfrak{L} \subseteq \mathfrak{X}$ which contains $\mathfrak{C}$ as well as generates the same $\sigma-$algebra as $\mathfrak{X}$ and induces the set system $\mathcal{S}$ consisting of all sets $\bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[)$ ($X_n \in \mathfrak{L} \cap \mathfrak{X}_b$ nonnegative, $x_n > 0$) any of the statements .1 - .3 is implied by

.4 $\lim_{n\to\infty} \rho(-\lambda(X-n)^+) = \rho(0)$ *hold for arbitrary nonnegative* $X \in \mathfrak{X}, \lambda > 0$, *and* $\inf_{1_{A_n} \geq Z \in \mathfrak{X}} \rho(\lambda Z) \searrow \rho(\lambda)$ *for* $\lambda > 0$ *whenever* $(1_{A_n})_n$ *is an isotone sequence of indicator mappings of subsets* $A_n \in \mathcal{S}$ *with* $\bigcup_{n=1}^{\infty} A_n = \Omega$.

We have even equivalence of the statements .1 - .4 if the indicator mappings 1_A ($A \in \mathcal{S}$) belong to $\mathfrak{X}$.

The proof may be found in section 8.
For bounded one-period positions, Theorem 4.1 enables us to give an equivalent characterization of convex risk measures that admit strong robust representations by probability measures.

Corollary 4.2 *Let $\mathcal{F}$ denote some $\sigma-$algebra on Ω, and let $\mathfrak{X}$ consist of all bounded $\mathcal{F}-$measurable real-valued mappings. Then the following statements are equivalent:*

.1 $\rho(X) = \max_{\mathrm{P}\in\mathcal{M}_1} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$ *holds for every* $X \in \mathfrak{X}$.

.2 $\rho(X_n) \searrow \rho(X)$ *for* $X_n \nearrow X$.

.3 $\rho(\lambda 1_{A_n}) \searrow \rho(\lambda)$ *for* $\lambda > 0$ *whenever* $(1_{A_n})_n$ *is an isotone sequence of indicator mappings of subsets* $A_n \in \mathcal{F}$ *with* $\bigcup_{n=1}^{\infty} A_n = \Omega$.

Let us adapt Theorem 4.1 to the special situations of Remarks 3.4, 3.5.

Corollary 4.3 *In the special context of Remark 3.4 with the notations introduced there all the statements .1 - .4 of Theorem 4.1 are equivalent.*

Remark:

Corollary 4.3 generalizes a result for one-period positions (cf. [16], Theorem 1).

Remark 4.4 *Let* $\mathbb{T} = [0, T], \mathfrak{C} = \mathbb{R}$, *let* $(\mathcal{F}_t)_{t\in\mathbb{T}}$ *be a filtration of* $\sigma-$*algebras on a set of scenarios* $\widetilde{\Omega}$, *and let* $\mathfrak{X}$ *be the set of cadlag positions. Then all statements .1 - .4 from Theorem 4.1 are equivalent, choosing* $\mathfrak{L}$ *to be the vector space spanned by the stochastic intervals* $[S, \infty[$ *(cf. Remark 3.5).*

5 Robust representations of convex risk measures in presence of given market models

Througout this section we assume that we have a market model with a reference probability measure P on a $\sigma-$algebra $\mathcal{F}$ on the set of scenarios Ω. In the following we shall retain, and partly generalize, already known results concerning the $\sigma-$additive robust representations of the convex risk measure ρ within the setting of a market model. The point is that they may be derived from the results presented in the sections 1, 2 and 4. We shall use the following notations. The spaces of $\mathrm{P}-$integrable mappings of order $p \in [1, \infty[$ and $\mathrm{P}-$essentially bounded mappings will be denoted by $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ and $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ respectively. For $p \in [1, \infty]$ the symbol $L_p(\Omega, \mathcal{F}, \mathrm{P})$ will be used for the space formed by identifying functions in $\mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ that agree $\mathrm{P}-$a.s.. The equivalence class of any $X \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ will be indicated by $< X >$.

The first result may be found in [19] for π being the identity on $\mathbb{R}$. Using Propostion 1.1, we obtain a slight generalization.

Proposition 5.1 *Let* $\mathfrak{X} = \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P})$ $(p \in [1, \infty[)$ *with conjugate space* $\mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$. *Furthermore let* $\rho(X) = \rho(Y)$ *for* $X = Y$ P *a.s.. If* $\mathcal{M}_1(q)$ *denotes the set of all* $\mathrm{Q} \in \mathcal{M}_1$ *having some* $\mathrm{P}-$*density from* $\mathcal{L}_q(\Omega, \mathcal{F}, \mathrm{P})$, *then*

$$\rho(X) = \max_{\mathrm{Q}\in\mathcal{M}_1(q)} (-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q})) \text{ for all } X \in \mathcal{L}_p(\Omega, \mathcal{F}, \mathrm{P}).$$

Proof:

Let $L_p(\Omega, \mathcal{F}, \mathrm{P})$ be equipped with the order $\succeq$, defined by $< X > \succeq < Y >$ if $X \geq Y$ $\mathrm{P}-$a.s., which induces the operations of minimum and maximum. It is known that $(L_p(\Omega, \mathcal{F}, \mathrm{P}), \|\cdot\|_p, \succeq)$ ($\|\cdot\|_p$ L_p- norm) is a Banach lattice, and therefore all positive linear forms w.r.t. $\succeq$ are continuous w.r.t. $\|\cdot\|_p$ (cf. [10], p. 151/152, Corollary 3). Next notice that $\beta_\rho(\Lambda) < \infty$ implies that $\Lambda(Z) = 0$ holds for $Z = 0$ $\mathrm{P}-$a.s., so that $\hat{\Lambda}(< X >) := \Lambda(X)$ describes a well defined positive linear form on $L_p(\Omega, \mathcal{F}, \mathrm{P})$ w.r.t. $\succeq$. Then the claimed representation of ρ follows immediately from Proposition 1.1 and the representation result for norm-continuous linear forms on $L_p(\Omega, \mathcal{F}, \mathrm{P})$.

■

In the case of $\mathfrak{X} = \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ we may generalize the equivalent characterization of strong robust representations for ρ shown in [11].

Theorem 5.2 *Let $\mathfrak{X} = \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$, and $\rho(X) = \rho(Y)$ for $X = Y$ P a.s.. Then $\rho(X) = \max_{\mathrm{Q}\in\mathcal{M}_1}(-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q}))$ for all $X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ if and only if $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$ $\mathrm{P}-$a.s..*

Proof:

The statement follows immediately from Theorem 4.1 since the condition $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$ $\mathrm{P}-$a.s. is equivalent with the property $\rho(X_n) \searrow \rho(X)$ for $X_n \nearrow X$. ■

The next result retains an equivalent characterization of the robust representations for ρ which may be found in [7] (Theorem 4.31).

Theorem 5.3 *Let $\mathfrak{X} := \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ and π be the identity on $\mathfrak{C} = \mathbb{R}$. Furthermore ρ is supposed to satisfy $\rho(X) = \rho(Y)$ for $X = Y$ $\mathrm{P}-$a.s.. If $\mathcal{M}_1(\mathrm{P})$ denotes the set of probability measures on $\mathcal{F}$ which are absolutely continuous w.r.t. P, then the following statements are equivalent.*

.1 $\rho(X) = \sup_{\mathrm{Q}\in\mathcal{M}_1(\mathrm{P})}(-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q}))$ *for all* $X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$.

.2 $\rho(X_n) \nearrow \rho(X)$ *for* $X_n \searrow X$ $\mathrm{P}-$*a.s.*.

.3 $\liminf_{n\to\infty} \rho(X_n) \geq \rho(X)$ *whenever* $(X_n)_n$ *is a uniformly* $\mathrm{P}-$*essentially bounded sequence in* $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ *with* $X_n \to X$ $\mathrm{P}-$*a.s.*.

Proof:

First of all, $\widehat{\rho} : L_\infty(\Omega, \mathcal{F}, \mathrm{P}) \to \mathbb{R}$, $\widehat{\rho}(< X >) := \rho(X)$ is well defined.

Next let $S_{L_{1+}} := \{< g > \in L_1(\Omega, \mathcal{F}, \mathrm{P}) \mid g \geq 0$ $\mathrm{P}-$a.s., $E_{\mathrm{P}}[g] = 1\}$ be endowed with the relative topology of the L_1-norm topology on $L_1(\Omega, \mathcal{F}, \mathrm{P})$. We may introduce, via $\Phi(< X >)(< g >) = E_{\mathrm{P}}[Xg]$, an injective vector space homomorphism Φ from $L_\infty(\Omega, \mathcal{F}, \mathrm{P})$ onto a vector subspace of $\mathcal{C}_b(S_{L_{1+}})$, defined to consist of the bounded, continuous real-valued mappings on $S_{L_{1+}}$,. The inverse mapping $\Phi^{-1} : \Phi(L_\infty(\Omega, \mathcal{F}, \mathrm{P})) \to L_\infty(\Omega, \mathcal{F}, \mathrm{P})$ may be used to define the convex risk measure $\widetilde{\rho} := \widehat{\rho} \circ \Phi^{-1}$ on $\widetilde{\mathfrak{X}} := \Phi(L_\infty(\Omega, \mathcal{F}, \mathrm{P}))$ w.r.t. to the identity $\tilde{\pi}$ on $\mathbb{R}$. Notice that $\Phi(\mathbb{R})$ consists of all constant real-valued mappings on $S_{L_{1+}}$.

Furthermore let $\tilde{F}$ be the linear span of the Dirac measures $\delta_{<g>}$ $(< g >\in S_{L_{1+}})$. For any $\nu \in \tilde{F}$ there is some $< g >\in L_1(\Omega, \mathcal{F}, \mathrm{P})$ such that $\int \Phi(< X >)\, d\nu = \int Xg\, d\mathrm{P}$ holds for every $< X >\in L_\infty(\Omega, \mathcal{F}, \mathrm{P})$, and $\|\nu\|_{\tilde{F}} := \sup\{| \int \Phi(< X >)\, d\nu| \mid \sup_{<g>\in S_{L_{1+}}} |\Phi(< X >)(< g >)| \leq 1\} = \| < g > \|_1$. Here $\|\cdot\|_1$ denotes the L_1-norm.

Conversely, for each $< g >$ from $L_1(\Omega, \mathcal{F}, \mathrm{P})$ with arbitrary representation $< g >= \sum_{i=1}^{r} \lambda_i < g_i >$ $(r \in \mathbb{N}, \lambda_i \in \mathbb{R}$, $< g_i >\in S_{L_{1+}}; i = 1, ..., r)$, we may define $\nu := \sum_{i=1}^{r} \lambda_i \delta_{<g_i>} \in \tilde{F}$ which satisfies $\int \Phi(< X >)\, d\nu = \int Xg\, d\mathrm{P}$ for every $< X >\in L_\infty(\Omega, \mathcal{F}, \mathrm{P})$. Therefore $\tilde{F}$ is complete w.r.t. the seminorm $\| \cdot \|_{\tilde{F}}$, and in order to apply Theorem 2.2 we have to show that the conditions (2.1), (2.2) are fulfilled for the sets $A_r := \tilde{\rho}^{-1}(]-\infty, 0]) \cap B_r$ and $B_r := \{\Phi(< X >) \in \widetilde{\mathfrak{X}} \mid \sup_{<g>\in S_{L_{1+}}} |\Phi(< X >)(< g >)| \leq r\}$ $(r > 0)$.

For this purpose fix $r > 0$. Since $L_\infty(\Omega, \mathcal{F}, \mathrm{P})$ represents the norm dual of $L_1(\Omega, \mathcal{F}, \mathrm{P})$, the application of the Banach-Alaoglu theorem yields that $\Phi^{-1}(B_r)$ is $\sigma(L_\infty(\Omega, \mathcal{F}, \mathrm{P}), L_1(\Omega, \mathcal{F}, \mathrm{P}))-$compact. This in turn implies by construction that B_r is compact w.r.t. the topology $\sigma(\widetilde{\mathfrak{X}}, \tilde{F})$ of pointwise convergence.

Moreover, by definition of Φ, the mapping $\varphi : \widetilde{\mathfrak{X}} \to L_1(\Omega, \mathcal{F}, \mathrm{P}), \Phi(< X >) \mapsto < X >$, is injective, and continuous w.r.t. $\sigma(\widetilde{\mathfrak{X}}, \tilde{F})$ and the weak topology on $L_1(\Omega, \mathcal{F}, \mathrm{P})$. Since the closure $cl(A_r)$ w.r.t. $\sigma(\widetilde{\mathfrak{X}}, \tilde{F})$ is even compact, the restriction $\varphi|cl(A_r) : cl(A_r) \to \varphi(cl(A_r))$ is a homeomorphism w.r.t. the associated relative topologies. In particular $\varphi(cl(A_r))$ is the weak closure of $\varphi(A_r)$, and hence, by Eberlein-Smulian theorem, every element is the limit point of a sequence in $\varphi(A_r)$ w.r.t. the weak topology. Therefore each point from $cl(A_r)$ is the pointwise limit of a sequence in A_r.

Now in view of Proposition 1.4 the relationships .1 $\Rightarrow$ .2 $\Leftrightarrow$ .3 are clear. The implication .3 $\Rightarrow$ .1 follows from Theorem 2.2 by the following argument. Let $(X_n)_n$ be a sequence in $\mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ and let $X \in \mathcal{L}_\infty(\Omega, \mathcal{F}, \mathrm{P})$ such that $(\Phi(< X_n >)_n$ is uniformly bounded and converges pointwise to $\Phi(< X >)$. We may find a subsequence $(X_{i(n)})_n$ with $\liminf_{n\to\infty} \rho(X_n) = \lim_{n\to\infty} \rho(X_{i(n)})$. Since the $\mathrm{P}-$essential sup norm on $L_\infty(\Omega, \mathcal{F}, \mathrm{P})$ coincides with the operator norm w.r.t. $\| \cdot \|_1$, the sequence $(X_n)_n$ is $\mathrm{P}-$essentially bounded. Therefore Komlos' subsequence theorem (cf. [7], Lemma 1.69) guarantees a sequence $(Y_n)_n$ of convex combinations Y_n from $\{X_{i(m)} \mid m \geq n\}$ which converges $\mathrm{P}-$a.s. pointwise to X and satisfies $\liminf_{n\to\infty} \rho(X_n) \geq \liminf_{n\to\infty} \rho(Y_n)$. ■

6 Some auxiliary results

Throughout this section we want to gather some technical arguments which will be often used when proving the main results of the paper. In the following ρ denotes a convex risk measure w.r.t. π associated with the

Fenchel-Legendre transform β_ρ and its counterpart α_ρ for the probability measures.

Lemma 6.1 *Let $X_1, X_2 \in \mathfrak{X}$ with $X_1 \leq X_2$. Then there exists some $c^* \in]-\rho(0), \infty[$ such that the representation $\rho(Y) = \max_{\Lambda \in \{\beta_\rho \leq c^*\}} (-\Lambda(Y) - \beta_\rho(\Lambda))$ holds for every $Y \in \mathfrak{X}$ with $X_1 \leq Y \leq X_2$. Moreover, for every $Y \in \mathfrak{X}$ with $X_1 \leq Y \leq X_2$ we have $\beta_\rho(\Lambda) \leq c^*$ if $\rho(Y) = -\Lambda(Y) - \beta_\rho(\Lambda)$.*

Proof:

Let $Y \in \mathfrak{X}$ with $X_1 \leq Y \leq X$. By Proposition 1.1 there is some $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$ with $\rho(Y) = -\Lambda(Y) - \beta_\rho(\Lambda)$. Then

$$\beta_\rho(\Lambda) = -\Lambda(2Y) - \rho(Y) + \Lambda(Y) \leq \rho(2Y) + \beta_\rho(\Lambda) - \rho(Y) + \Lambda(Y) = \rho(2Y) - 2\rho(Y) \leq \rho(2X_1) - 2\rho(X_2).$$

Therefore any $c > \max\{\rho(2X_1) - 2\rho(X_2), -\rho(0)\}$ is as required. ■

Lemma 6.2 *Let $X \in \mathfrak{X}$ with $X \leq \inf_{Z \in E} Z$, where $E \subseteq \mathfrak{X}$ is assumed to be downward directed, i.e. for $Z_1, Z_2 \in E$ there is some $Z \in E$ with $Z \leq \min\{Z_1, Z_2\}$. Furthermore let $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$.*
Then $\inf_{Z \in E} \Lambda(Z) = \Lambda(X)$ if $\inf_{Z \in E} \rho(-\lambda(Z - X)) = \rho(0)$ for arbitrary $\lambda > 0$.

Proof:

For arbitrary $\lambda > 0$ and every $Z \in E$ we have $\beta_\rho(\Lambda) \geq -\Lambda(-\lambda(Z - X)) - \rho(-\lambda(Z - X))$, and therefore by assumption

$$0 \leq \inf_{Z \in E} \Lambda(Z - X) \leq \frac{\beta_\rho(\Lambda) + \rho(0)}{\lambda} .$$

Finally, by taking $\lambda \uparrow \infty$, we obtain $\inf_{Z \in E} \Lambda(Z - X) = 0$ because $0 \leq \beta_\rho(\Lambda) + \rho(0) < \infty$. The proof is now complete. ■

We may divide the domain of β_ρ into the classes $\{\beta_\rho \leq c\}$ $(-\rho(0) = \inf \beta\rho < c < \infty)$. The following topological property of these classes is crucial.

Lemma 6.3 *$\{\beta_\rho \leq c\}$ is compact w.r.t. the product topology on $\mathbb{R}^{\mathfrak{X}}$ for every $c \in]-\rho(0), \infty[$.*

Proof:

Let $c \in]-\rho(0), \infty[$, and let $(\Lambda_i)_{i \in I}$ be a net in $\{\beta_\rho \leq c\}$ which converges to some $\Lambda \in \mathbb{R}^{\mathfrak{X}}$ w.r.t. the product topology. Obviously, Λ is a positive linear form on $\mathfrak{X}$ which extends π. Furthermore

$$-\Lambda(X) - \rho(X) = \lim_i (-\Lambda_i(X) - \rho(X)) \leq \limsup_i \beta_\rho(\Lambda_i) \leq c \text{ for } X \in \mathfrak{X}.$$

Therefore $\{\beta_\rho \leq c\}$ is closed w.r.t. the product topology, and the proof may be completed by the application of Tychonoff's theorem because $\{\beta_\rho \leq c\} \subseteq \mathop{\times}\limits_{X \in \mathfrak{X}} [-c - \rho(X), c + \rho(-X)]$. ■

As an application of Lemma 6.3 we may show the following useful technical argument.

Lemma 6.4 *Let $E \subseteq \mathfrak{X}$ consist of nonnegative positions, and let E be upward directed, i.e. for $Z_1, Z_2 \in E$ there is some $Z \in E$ with $Z \geq \max\{Z_1, Z_2\}$. Furthermore let $X := \sup_{Z\in E} Z \in \mathfrak{X}$, and let $Y \in \mathfrak{X}$ be nonnegative and bounded. Then $\inf_{Z\in E} \rho(Z - Y) = \rho(X - Y)$ holds if $\inf_{Z\in E}(\Lambda(X) - \Lambda(Z)) = 0$ for every $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$.*

Proof:
Due to Lemma 6.1 there exists some real $c^* > -\rho(0)$ with $\rho(\widetilde{X} - Y) = \sup_{\Lambda\in\{\beta_\rho\leq c^*\}} (-\Lambda(\widetilde{X} - Y) - \beta_\rho(\Lambda))$ for all $\widetilde{X} \in E \cup \{X\}$. Then we may conclude

$$0 \leq \inf_{Z\in E} \rho(Z - Y) - \rho(X - Y) \leq \inf_{Z\in E} \sup_{\Lambda\in\{\beta_\rho\leq c^*\}} F_Z(\Lambda),$$

where $F_Z : \{\beta_\rho \leq c^*\} \to \mathbb{R},\ \Lambda \mapsto \Lambda(X) - \Lambda(Z)$, for $Z \in E$.
In the view of Lemma 6.3 $(\{\beta_\rho \leq c^*\}, \tau)$ is a compact Hausdorff space, where τ denotes the relative topology of the product topology on $\mathbb{R}^{\mathfrak{X}}$ to $\{\beta_\rho \leq c^*\}$. Since E is upward directed, $M := \{F_Z \mid Z \in E\}$ is a downward directed family of real-valued mappings, i.e. for $Z_1, Z_2 \in E$ there exists some $Z \in E$ with $F_Z \leq \min\{F_{Z_1}, F_{Z_1}\}$. Furthermore all functions from M are continuous w.r.t. τ, and $\inf_{Z\in E} F_Z(\Lambda) = 0$ for $\Lambda \in \{\beta_\rho \leq c^*\}$ by assumption. Therefore the application of the general Dini lemma (cf. [12], Theorem 3.7) leads to $\inf_{Z\in E} \sup_{\Lambda\in\{\beta_\rho\leq c^*\}} F_Z(\Lambda) = 0$, which completes the proof. ∎

In the next step we want to look for conditions which allow to reduce investigations to bounded positions. For this purpose we have to recall some concepts from integration theory, adapted to our setting. If Q denotes a probability content on the $\sigma-$algebra $\sigma(\mathfrak{X})$, i.e. a finitely additive nonnegative set function with $\mathrm{Q}(\Omega) = 1$, then we shall call a $\sigma(\mathfrak{X})-$measurable mapping X with positive and negative part X^+ and X^- **integrable w.r.t.** Q if

$$\int_0^\infty \mathrm{Q}(\{X^+ \geq x\})\ dx,\ \int_0^\infty \mathrm{Q}(\{X^- \geq x\})\ dx < \infty.$$

The terminology stems from the fact that Q may be extended via the so-called **asymmetric Choquet integral** E_{Q} defined by ([6], Chapter 5, p. 87)

$$E_{\mathrm{Q}}[X] := \int_0^\infty \mathrm{Q}(\{X^+ \geq x\})\ dx\ -\ \int_0^\infty \mathrm{Q}(\{X^- \geq x\})\ dx.$$

It is a positive linear form on the space of all Q$-$integrable mappings (cf. [6], Proposition 5.1, Theorem 6.3, Corollary 6.4), and hence the restriction to the bounded ones is even continuous w.r.t. to the sup norm. Therefore the restriction of E_{Q} to the bounded $\sigma(\mathfrak{X})-$measurable mappings is just the respective integral defined in functional analysis (e.g. [7], Appendix A.5). Using the introduced notions, a real linear form Λ on $\mathfrak{X}$ is defined to be

representable by a probability content if there is some probability content Q such that every $X \in \mathfrak{X}$ is integrable w.r.t. Q and $\Lambda(X) = E_{\mathrm{Q}}[X]$.

If $\mathfrak{X}$ is a Stonean vector lattice, then $X \wedge Y = \min\{X, Y\}$, $X \vee Y = \max\{X, Y\} \in \mathfrak{X}$ for $X, Y \in \mathfrak{X}$ in particular $X^+ := X \vee 0, X^- := (-X) \vee 0 \in \mathfrak{X}$ for any $X \in \mathfrak{X}$. In this case, if all linear forms from the domain of β_ρ are representable by probability contents, then ρ is concentrated on the bounded positions, and as a consequence ρ admits a robust representation by probability measures if its restriction to the bounded positions does so.

Lemma 6.5 *Let $\mathfrak{X}$ be a Stonean vector lattice, and let every linear form $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$ be representable by some probability content Q on $\sigma(\mathfrak{X})$. Then we can state:*

.1 The sequence $\big(\rho(X^+ - (X^- \wedge n))\big)_n$ converges to $\rho(X)$ for every $X \in \mathfrak{X}$.

.2 The sequence $\big(\rho((X \wedge m) - Y)\big)_m$ converges to $\rho(X - Y)$ for nonnegative $X, Y \in \mathfrak{X}$, Y being bounded.

.3 $\inf\{|\rho((X^+ \wedge m) - (X^- \wedge n)) - \rho(X)| \mid m, n \in \mathbb{N}\} = 0$ for $X \in \mathfrak{X}$, and in addition

$$\sup_{X \in \mathfrak{X}} (-E_{\mathrm{Q}}[X] - \rho(X)) = \sup_{X \in \mathfrak{X}_b} (-E_{\mathrm{Q}}[X] - \rho(X))$$

for every probability content Q on $\sigma(\mathfrak{X})$ such that each $X \in \mathfrak{X}$ is integrable w.r.t. Q.

.4 If $\mathcal{M} \subseteq \alpha_\rho^{-1}(\mathbb{R})$ with $\rho(X) = \sup_{\mathrm{Q} \in \mathcal{M}} (-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q}))$ for all bounded $X \in \mathfrak{X}$, then

$$\rho(X) = \sup_{\mathrm{Q} \in \mathcal{M}} (-E_{\mathrm{Q}}[X] - \alpha_\rho(\mathrm{Q})) \text{ for all } X \in \mathfrak{X}.$$

Proof:

The most important tool of the proof is Greco's representation theorem. The reader is kindly referred to [14] (Theorem 2.10 with Remark 2.3).

Since any $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$ is representable by a probability content, statement .2 follows immediately from Greco's representation theorem and Lemma 6.4.

<u>proof of .1</u>:

Let $X \in \mathfrak{X}$, and let $\varepsilon > 0$. Then there exists some probability content Q with $\beta_\rho(E_{\mathrm{Q}}|\mathfrak{X}) < \infty$ such that the inequality $\rho(X) - \varepsilon < -E_{\mathrm{Q}}[X^+] - \beta_\rho(E_{\mathrm{Q}}|\mathfrak{X}) + E_{\mathrm{Q}}[X^-]$ holds. Application of Greco's representation theorem leads then to

$$\rho(X) - \varepsilon < -E_{\mathrm{Q}}[X^+] - \beta_\rho(E_{\mathrm{Q}}|\mathfrak{X}) + \lim_n E_{\mathrm{Q}}[X^- \wedge n] \leq \lim_n \rho(X^+ - (X^- \wedge n)) \leq \rho(X)$$

<u>proof of .3</u>:

Let Q be a probability content on $\sigma(\mathfrak{X})$ such that every $X \in \mathfrak{X}$ is integrable w.r.t. Q, and let $X \in \mathfrak{X}$. Then for $\varepsilon > 0$ we may choose by statement .1 and Greco's representation theorem some $n \in \mathbb{N}$ with

$$|-E_{\mathrm{Q}}[X] - \rho(X) - \big(-E_{\mathrm{Q}}[X^+ - (X^- \wedge n)] - \rho(X^+ - (X^- \wedge n))\big)| < \frac{\varepsilon}{3}$$

Moreover, due to statement .2 and Greco's representation theorem again, there exists some $m \in \mathbb{N}$ such that

$$|E_{\mathrm{Q}}[X^+ - (X^- \wedge n)] - E_{\mathrm{Q}}[(X^+ \wedge m) - (X^- \wedge n)]|, |\rho(X^+ - (X^- \wedge n)) - \rho((X^+ \wedge m) - (X^- \wedge n))| < \frac{\varepsilon}{3}$$

We may conclude by

$$-E_{\mathrm{Q}}[X] - \rho(X) < -E_{\mathrm{Q}}[(X^+ \wedge m) - (X^- \wedge n)] - \rho((X^+ \wedge m) - (X^- \wedge n)) + \varepsilon \leq \sup_{Y \in \mathfrak{X}_b} (-E_{\mathrm{Q}}[Y] - \rho(Y)) + \varepsilon$$

and then $\sup\limits_{X \in \mathfrak{X}} (-E_{\mathrm{Q}}[X] - \rho(X)) = \sup\limits_{X \in \mathfrak{X}_b} (-E_{\mathrm{Q}}[X] - \rho(X))$. The rest of statement .3 follows easily from statements .1, .2.

proof of .4:

By assumption $\hat{\rho} : \mathfrak{X} \to \mathbb{R},\ X \mapsto \sup\limits_{\mathrm{P} \in \mathcal{M}} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P}))$, is a well defined convex risk measure w.r.t. π with $\hat{\rho} \leq \rho$ and $\hat{\rho}(X) = \rho(X)$ for bounded $X \in \mathfrak{X}$. In particular $\beta_{\hat{\rho}}^{-1}(\mathbb{R}) \subseteq \beta_\rho^{-1}(\mathbb{R})$, which implies that every $\Lambda \in \beta_{\hat{\rho}}^{-1}(\mathbb{R})$ is representable by a probability content due to the assumptions on β_ρ. Therefore the statements .1, .2 are also valid for $\hat{\rho}$, following the same line of reasoning used in the proof of them. Then firstly, fixing $\varepsilon > 0$, we may find for $X \in \mathfrak{X}$ an integer $n \in \mathbb{N}$ with

$$|\hat{\rho}(X) - \hat{\rho}(X^+ - (X^- \wedge n))|, |\rho(X) - \rho(X^+ - (X^- \wedge n))| < \frac{\varepsilon}{4}\ .$$

Furthermore there is some $m \in \mathbb{N}$ such that

$$|\hat{\rho}((X^+ \wedge m) - (X^- \wedge n)) - \hat{\rho}(X^+ - (X^- \wedge n))|, |\rho((X^+ \wedge m) - (X^- \wedge n)) - \rho(X^+ - (X^- \wedge n))| < \frac{\varepsilon}{4}\ .$$

Thus $|\hat{\rho}(X) - \rho(X)| < \varepsilon$, and hence $\hat{\rho}(X) = \rho(X)$, which completes the proof. ■

In order to apply Lemma 6.5 we are now interested in conditions on ρ that ensure that linear forms from the domain of β_ρ are representable by probability contents. We shall succeed in providing a full characterization.

Proposition 6.6 *Let $\mathfrak{X}$ be a Stonean vector lattice. Then every linear form from $\beta_\rho^{-1}(\mathbb{R})$ is representable by a probability content if and only if $\lim\limits_{n\to\infty} \rho(-\lambda(X-n)^+) = \rho(0)$ for every $\lambda > 0$ and nonnegative $X \in \mathfrak{X}$.*

Proof:

For the if part let $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$. Then by assumption and Lemma 6.2 the sequence $\big(\Lambda((X-n)^+)\big)_n$ converges to 0 for nonnegative $X \in \mathfrak{X}$. Hence, due to Greco's representation theorem (cf. [14], Theorem 2.10 with Remark 2.3) Λ is representable by a probability content.

Conversely, let every linear form from the domain of β_ρ be representable by a probability content, and let $\lambda > 0$ as well as $X \in \mathfrak{X}$ be nonnegative. In view of Lemma 6.1 there is some $c^* \in]-\rho(0), \infty[$ with $\rho(Y) = \sup\limits_{\Lambda \in \{\beta_\rho \leq c^*\}} (-\Lambda(Y) - \beta_\rho(\Lambda))$ for any $Y \in \mathfrak{X}$ with $-\lambda X \leq Y \leq 0$. In particular

$$|\rho(-\lambda(X-n)^+) - \rho(0)| \leq \sup_{\Lambda \in \{\beta_\rho \leq c^*\}} (\Lambda(\lambda(X-n)^+)) \text{ for every } n.$$

In view of Greco's representation theorem we have $\inf_n \Lambda(\lambda(X-n)^+) = 0$ for any $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$. So by Lemma 6.3 we may apply the general Dini lemma as in the proof of Lemma 6.4 to conclude $\lim_{n\to\infty} |\rho(-\lambda(X-n)^+) - \rho(0)| = 0$. This completes the proof. ∎

Remark: If $\mathfrak{X}$ is a Stonean vector lattice which consists of bounded positions only, then Proposition 6.6 is trivial.

7 Proof of Proposition 1.4 and Theorem 3.1

Throughout this section let $\mathfrak{L}, E$ and $\mathcal{S}$ as in the context of Proposition 1.4 and Theorem 3.1. Furthermore $\mathfrak{X}$ is assumed to be a Stonean vector lattice. The next two results are for preparation.

Lemma 7.1 *Let $\mathfrak{X}_{+b}$ consist of all nonnegative bounded $X \in \mathfrak{X}$, and let P be a probability measure on $\sigma(\mathfrak{X})$. Then $E_{\mathrm{P}}[X] = \inf_{X \leq Y \in E} E_{\mathrm{P}}[Y]$ for every $X \in \mathfrak{X}_{+b}$, and $\sup_{X \in \mathfrak{X}_{+b}} (-E_{\mathrm{P}}[X] - \rho(X)) = \sup_{X \in E} (-E_{\mathrm{P}}[X] - \inf_{X \geq Z \in \mathfrak{X}} \rho(Z))$.*

Proof:

Let us use the abbreviations $c := \sup_{X \in \mathfrak{X}_{+b}} (-E_{\mathrm{P}}[X] - \rho(X))$ and $d := \sup_{X \in E} (-E_{\mathrm{P}}[X] - \inf_{X \geq Z \in \mathfrak{X}} \rho(Z))$. Setting $\mathcal{T} := \{\Omega \setminus A \mid A \in \mathcal{S}\}$, we have $\big\{\sum_{i=1}^r \lambda_i 1_{G_i} \mid r \in \mathbb{N}, \lambda_1, ..., \lambda_r \in]0, \infty[, G_1, ..., G_r \in \mathcal{T}\big\} \subseteq E$, where 1_A denotes the indicator mapping of the subset A (cf. [14], Proposition 3.2). Since $\mathfrak{L}$ generates $\sigma(\mathfrak{X})$, the inner Daniell-Stone theorem tells us that P satisfies

$$\mathrm{P}(A) = \sup\{\mathrm{P}(B) \mid A \supseteq B \in \mathcal{S}\} = \inf\{\mathrm{P}(B) \mid A \subseteq B \in \mathcal{T}\}$$

for every $A \in \sigma(\mathfrak{X})$ (cf. [14], Theorem 5.8, final remark after Addendum 5.9).

Every nonnegative bounded function from $\mathfrak{X}$ may be described as a lower(!) envelope of a sequence of simple $\sigma(\mathfrak{X})-$measurable mappings. This implies $E_{\mathrm{P}}[X] = \inf\{E_{\mathrm{P}}[Y] \mid X \leq Y \in E\}$ for all bounded nonnegative $X \in \mathfrak{X}$. In particular $c \leq d$. Moreover for any $X \in E$ and $\varepsilon > 0$ there is some $Y \in \mathfrak{X}_{+b}$ with $Y \leq X$ and $\inf_{X \geq Z \in \mathfrak{X}} \rho(Z) + \varepsilon > \rho(Y)$. This implies the inequalities $-E_{\mathrm{P}}[X] - \inf_{X \geq Z \in \mathfrak{X}} \rho(Z) \leq -E_{\mathrm{P}}[Y] - \rho(Y) + \varepsilon \leq c + \varepsilon$. Hence $d \leq c$, which completes the proof. ∎

Lemma 7.2 *Let P be a probability measure on $\sigma(\mathfrak{X})$ with $\sup_{X \in \mathfrak{L}_{+b}} (-E_{\mathrm{P}}[X] - \rho(X)) < \infty$, where $\mathfrak{L}_{+b} = \mathfrak{L} \cap \mathfrak{X}_{+b}$ with $\mathfrak{X}_{+b}$ consisting of all nonnegative positions from $\mathfrak{X}_b$. If condition (2) of Theorem 3.1 is satisfied, then every $X \in \mathfrak{X}$ is $\mathrm{P}-$integrable, and $\sup_{X \in \mathfrak{X}_b} (-E_{\mathrm{P}}[X] - \rho(X)) = \sup_{X \in \mathfrak{L}_{+b}} (-E_{\mathrm{P}}[X] - \rho(X))$.*

Proof:

We have $\sup_{X \in E} (-E_{\mathrm{P}}[X] - \inf_{X \geq Z \in \mathfrak{X}} \rho(Z)) \leq \sup_{X \in \mathfrak{L}_{+b}} (-E_{\mathrm{P}}[X] - \rho(X))$ by definition of E and condition (2) of Theorem 3.1. Moreover, $\mathfrak{L}_{+b} \subseteq E$, and therefore the application of Lemma 7.1 with translation invariance of ρ leads to

(*) $\sup_{X\in\mathfrak{X}_b}(-E_{\mathrm{P}}[X]-\rho(X)) = \sup_{X\in\mathfrak{X}_{+b}}(-E_{\mathrm{P}}[X]-\rho(X)) = \sup_{X\in E}(-E_{\mathrm{P}}[X]-\inf_{X\geq Z\in\mathfrak{X}}\rho(Z)) = \sup_{X\in\mathfrak{L}_{+b}}(-E_{\mathrm{P}}[X]-\rho(X)) =: c < \infty$.

Let $X \in \mathfrak{X}$ be nonnegative. It is an upper envelope of an isotone sequence $(X_n)_n$ of nonnegative simple $\sigma(\mathfrak{X})-$measurable mappings. Hence, in view of the monotone convergence theorem it remains to show that the sequence $(E_{\mathrm{P}}[X_n])_n$ is bounded from above. Indeed, bearing (*) in mind, $E_{\mathrm{P}}[X_n]-\rho(-X_n)\leq c$ for each n, which implies $\sup_n E_{\mathrm{P}}[X_n]\leq c+\rho(-X)$. The proof is now complete. ■

Proof of Proposition 1.4:

In view of Proposition 1.2 ρ is lower semicontinuous w.r.t. the weak topology $\sigma(\mathfrak{X},F)$ where F is the space of all bounded countably additive set functions on $\sigma(\mathfrak{X})$ such that every $X\in\mathfrak{X}$ is integrable w.r.t. any $\mu\in F$. This implies that ρ satisfies the Fatou property, and thus $\rho|\mathfrak{X}_b$ is continuous from above. The remaining part of Proposition 1.4 follows immediately from Lemma 7.1. ■

Proof of Theorem 3.1:

Let $\mathfrak{X}_{+b},\mathfrak{L}_{+b}$ consist of all nonnegative bounded positions from $\mathfrak{X}$ and $\mathfrak{L}$ respectively. For any Λ from the domain of β_ρ assumption (4) ensures it may be represented by a probability content Q in the sense explained just before Lemma 6.5 (cf. Proposition 6.6). Then condition (3), Lemma 6.2 and the inner Daniell-Stone theorem (cf. [14], Theorem 5.8, final remark after Addendum 5.9) provide a probability measure P on $\sigma(\mathfrak{X})$ with $\mathrm{P}(A)=\sup_{A\supseteq B\in\mathcal{S}}\mathrm{P}(B)$ for every $A\in\sigma(\mathfrak{X})$ such that $E_{\mathrm{P}}[Z]=\Lambda(Z)$ holds for every $Z\in\mathfrak{L}\cap\mathfrak{X}_b$ (note that $\mathfrak{L}\cap\mathfrak{X}_b$ generates $\sigma(\mathfrak{X})$ by assumption on $\mathfrak{L}$ since $\mathfrak{L}$ is a Stonean vector lattice). Then any $X\in\mathfrak{X}$ is $\mathrm{P}-$integrable by condition (2) with Lemma 7.2. In particular we may define for every $Z\in\mathfrak{L}$ with positive and negative part Z^+ and Z^- respectively, via $Y_n := Z^+\wedge n - Z^-\wedge n$ a sequence $(Y_n)_n$ in $\mathfrak{L}\cap\mathfrak{X}_b$ which converges pointwise to Z and satisfies by dominated convergence as well as the Greco theorem (cf. [14], Theorem 2.10) the idenities $\Lambda(Z)=\lim_{n\to\infty}\Lambda(Y_n)=\lim_{n\to\infty}E_{\mathrm{P}}[Y_n]=E_{\mathrm{P}}[Z]$. This also means $\mathrm{P}\in\mathcal{M}_1(\mathcal{S})$ because $\mathfrak{C}\subseteq\mathfrak{L}$. Applying Lemma 7.2 again, and bearing Lemma 6.5 with Proposition 6.6 in mind, we obtain $\alpha_\rho(\mathrm{P})=\sup_{X\in\mathfrak{L}_{+b}}(-E_{\mathrm{P}}[X]-\rho(X))\leq\beta_\rho(\Lambda)$. Summarizing the discussion we have shown

(*) For every Λ from the domain of β_ρ there is some $\mathrm{P}\in\mathcal{M}_1(\mathcal{S})$ such that $E_{\mathrm{P}}|\mathfrak{L}=\Lambda|\mathfrak{L}$ and $\alpha_\rho(\mathrm{P})\leq\beta_\rho(\Lambda)$.

After these preliminary considerations we are ready to prove Theorem 3.1.

Statement .1 is borrowed from [17] (p.12 there).

proof of statement .2:

In order to verify statement .2 we may use (*), and it remains to show that the sets Δ_c ($c\in]\rho(0),\infty[$) are compact w.r.t. the topology $\tau_{\mathfrak{L}}$ introduced in statement .1. For this purpose let $(\mathrm{P}_i)_{i\in I}$ be a net in Δ_c with arbitrary

$c\in]\rho(0),\infty[$. So $(E_{\mathrm{P}}|\mathfrak{X})_{i\in I}$ is a net in $\{\beta_\rho\leq c\}$, which in turn is compact w.r.t. to product topology on $\mathbb{R}^{\mathfrak{X}}$ by Lemma 6.3. Therefore there exist a subnet $(\mathrm{P}_{i(k)})_{k\in K}$ and some $\Lambda\in\{\beta_\rho\leq c\}$ with $\lim_k E_{\mathrm{P}_{i(k)}}[X]=\Lambda(X)$ for every $X\in\mathfrak{X}$. Then $(\mathrm{P}_{i(k)})_{k\in K}$ converges to some $\mathrm{P}\in\Delta_c$ due to (*). This finishes the prove of statement .2.

proof of .3:

Drawing on Lemma 6.5 with Proposition 6.6 and the translation invariance of ρ it remains to show

$$\rho(X)=\sup_{\mathrm{P}\in\mathcal{M}_1(\mathcal{S})}(-E_{\mathrm{P}}[X]-\alpha_\rho(\mathrm{P}))\text{ for all }X\in\mathfrak{X}_{+b}.$$

For this purpose let $X\in\mathfrak{X}_{+b}$, and let $\varepsilon>0$. Then by (1), (2) and definition of E we may find an isotone sequence $(Y_n)_n$ in $\mathfrak{L}_{+b}$ with $X\leq\sup_n Y_n\in E$ and $\rho(X)<\inf_n\rho(Y_n)+\varepsilon$. In view of Lemma 6.1 and (*) there is some $c^*\in]-\rho(0),\infty[$ with $\rho(Y_n)=\sup_{\mathrm{P}\in\Delta_{c^*}}(-E_{\mathrm{P}}[Y_n]-\alpha_\rho(\mathrm{P}))$ for any n. Furthermore $F_n(\mathrm{P}):=-E_{\mathrm{P}}[Y_n]-\alpha_\rho(\mathrm{P})$ defines an antitone sequence of mappings $F_n:=\Delta_{c^*}\to\mathbb{R}$ which are upper semicontinuous w.r.t. the relative topology of $\tau_{\mathfrak{L}}$ (see Lemmata 7.2, 6.5 again) such that $F_n\searrow F$, defined by $F(\mathrm{P}):=-E_{\mathrm{P}}[Y]-\alpha_\rho(\mathrm{P})$. Due to .2 we may apply the generalized Dini lemma (cf. [12], Theorem 3.7) and obtain

$$\rho(X)-\varepsilon<\inf_n\rho(Y_n)=\sup_{\mathrm{P}\in\Delta_{c^*}}(-E_{\mathrm{P}}[Y]-\alpha_\rho(\mathrm{P}))\leq\sup_{\mathrm{P}\in\Delta_{c^*}}(-E_{\mathrm{P}}[X]-\alpha_\rho(\mathrm{P}))\leq\rho(X),$$

which completes the proof. ■

Proof of Remark 3.4:

Obviously, $\sigma(\mathfrak{X})=\mathcal{B}(\widetilde{\Omega})\otimes\mathcal{B}(\mathbb{T})$, and any closed subset A of $\widetilde{\Omega}\times\mathbb{T}$ w.r.t. the metrizable topology $\tau_{\widetilde{\Omega}}\times\tau_{\mathbb{T}}$ may be described by $A=\bigcap_{n=1}^{\infty}X_n^{-1}([x_n,\infty[)$ for some sequence $(X_n)_n$ of nonnegative continuous mappings and a sequence $(x_n)_n$ of positive real numbers.

Now let $X\in\mathfrak{L}$. Then $X(\cdot,t)$ is $\mathcal{B}(\widetilde{\Omega})-$measurable for $t\in\mathbb{T}$ and $X(\omega,\cdot)$ is continuous w.r.t. $\tau_{\mathbb{T}}$ for $\omega\in\Omega$, which implies $X\in\mathfrak{X}$ because $\tau_{\mathbb{T}}$ is separably metrizable (cf. e.g. [3], Lemma III-14). Therefore $\mathfrak{L}\subseteq\mathfrak{X}$, and $\sigma(\mathfrak{X})$ is generated by $\mathcal{S}$. Then the statement of Remark 3.4 follows immediately from Theorem 3.1. ■

8 Proof of Theorem 4.1

Obviously, $.2\Rightarrow.3$, and statement .2 implies statement .4 if the indicator mappings 1_A $(A\in\mathcal{S})$ belong to $\mathfrak{X}$. So in view of Corollary 3.3 and Proposition 1.1 it remains to prove the implications $.1\Rightarrow.2$ and $.4\Rightarrow.3$.

proof of $.1\Rightarrow.2$:

Let $\mathcal{Q}$ consist of all sequences $(\lambda_n)_n$ in $[0,1]$ with $\sum_{n=1}^{\infty}\lambda_n=1$. Aditionally, let $(X_n)_n$ be any isotone sequence in $\mathfrak{X}$ which converges pointwise to some $X\in\mathfrak{X}$. Then by assumption $\sum_{n=1}^{\infty}\lambda_nX_n$ is a well defined member of $\mathfrak{X}$ for

$(\lambda_n)_n \in \mathcal{Q}$. Then for arbitrary $(\lambda_n)_n \in \mathcal{Q}$, $m \in \mathbb{N}$ and every $\omega \in \Omega$

$$X_1(\omega)(1 - \sum_{n=1}^{m} \lambda_n) \leq \sum_{n=1}^{\infty} \lambda_n X_n(\omega) - \sum_{n=1}^{m} \lambda_n X_n(\omega) \leq X(\omega)(1 - \sum_{n=1}^{m} \lambda_n).$$

This implies

(*) $\lim_{m\to\infty} \big(\Lambda(\sum_{n=1}^{\infty} \lambda_n X_n) - \Lambda(\sum_{n=1}^{m} \lambda_n X_n)\big) = 0$ for any $\Lambda \in \beta_\rho^{-1}(\mathbb{R})$.

Let us define $\Delta_c := \{\alpha_\rho \leq c\}$. Then in view of Lemma 6.1 and statement .1 there exists some real $c^* > -\rho(0)$ with

$$\rho(Y) = \max_{\Lambda \in \{\beta_\rho \leq c^*\}} (-\Lambda(Y) - \beta_\rho(\Lambda)) = \max_{\mathrm{P} \in \Delta_{c^*}} (-E_{\mathrm{P}}[Y] - \alpha_\rho(\mathrm{P}))$$

for $Y \in \{X, \sum_{n=1}^{\infty} \lambda_n X_n \mid (\lambda_n)_n \in \mathcal{Q}\}$.
The sequence $(f_n)_n$ in $\mathbb{R}^{\{\beta_\rho \leq c^*\}}$, defined by $f_n(\Lambda) = -\Lambda(X_n) - \beta_\rho(\Lambda)$, is uniformly bounded because

$$-2c^* - \rho(-X) \leq -\beta_\rho(\Lambda) - \rho(-X) - \beta_\rho(\Lambda) \leq f_n(\Lambda) \leq \rho(X_n) \leq \rho(X_1).$$

Furthermore for $(\lambda_n)_n \in \mathcal{Q}$ the mapping $\sum_{n=1}^{\infty} \lambda_n f_n$ is well defined with

$$\sum_{n=1}^{\infty} \lambda_n f_n(\Lambda) = \lim_{m\to\infty} (-\Lambda(\sum_{n=1}^{m} \lambda_n X_n) - \beta_\rho(\Lambda) \sum_{n=1}^{m} \lambda_n) = -\Lambda(\sum_{n=1}^{\infty} \lambda_n X_n) - \beta_\rho(\Lambda)$$

for $\beta_\rho(\Lambda) \leq c^*$ due to (*). Hence

$$\sup_{\Lambda \in \{\beta_\rho \leq c^*\}} \sum_{n=1}^{\infty} \lambda_n f_n(\Lambda) = \rho(\sum_{n=1}^{\infty} \lambda_n X_n) = \max_{\mathrm{P} \in \Delta_{c^*}} (-E_{\mathrm{P}}\big[\sum_{n=1}^{\infty} \lambda_n X_n\big] - \alpha_\rho(\mathrm{P})) = \max_{\mathrm{P} \in \Delta_{c^*}} \sum_{n=1}^{\infty} \lambda_n f_n(E_{\mathrm{P}}|\mathfrak{X}).$$

The application of Simons' lemma (cf. [20], Lemma 2) and the monotone convergence theorem leads then to

$$\rho(X) = \sup_{\mathrm{P} \in \Delta_{c^*}} (-E_{\mathrm{P}}[X] - \alpha_\rho(\mathrm{P})) = \sup_{\mathrm{P} \in \Delta_{c^*}} \limsup_{n} f_n(E_{\mathrm{P}}|\mathfrak{X}) \geq \inf\{\sup_{\Lambda \in \{\beta_\rho \leq c^*\}} f(\Lambda) \mid f \in co(\{f_n \mid n \in \mathbb{N}\})\},$$

where $co(\{f_n \mid n \in \mathbb{N}\})$ denotes the convex hull of $\{f_n \mid n \in \mathbb{N}\}$ in $\mathbb{R}^{\{\beta_\rho \leq c^*\}}$. Thus for $\varepsilon > 0$ there is some convex combination $f = \sum_{i=1}^{r} \lambda_i f_{n_i}$ with $n_1 < ... < n_r$ and $\rho(X) + \varepsilon > \sup_{\Lambda \in \{\beta_\rho \leq c^*\}} f(\Lambda) = \rho(\sum_{i=1}^{r} \lambda_i f_{n_i})$. In particular the inequalities $\rho(X) + \varepsilon > \rho(X_n) \geq \rho(X)$ hold for $n \geq n_r$, which implies $\lim_{n\to\infty} \rho(X_n) = \rho(X)$.

<u>proof of .4 $\Rightarrow$ .3</u>:

Let $\mathfrak{X}_{+b}$ gather all nonnegative $X \in \mathfrak{X}_b$. Drawing on Corollary 3.3 it suffices to prove that every linear form Λ from the domain of β_ρ is representable by a probability measure. So let Λ belong to $\beta_\rho^{-1}(\mathbb{R})$. Then by part of statement .4 as well as Proposition 6.6, the linear form Λ is representable by some probability content Q on $\sigma(\mathfrak{X})$ in the sense as introduced just before Lemma 6.5. We want to apply Proposition A.1 (cf. appendix A) to verify that Q is a probability measure. Since $\Omega \setminus \bigcap_{n=1}^{\infty} X_n^{-1}([x_n, \infty[) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \big(x_n - (X_n \wedge x_n)\big)^{-1}([\frac{1}{m}, \infty[)$ holds for every pair $(X_n)_n, (x_n)_n$ of sequences in $\mathfrak{L} \cap \mathfrak{X}_{+b}$ and $]0, \infty[$ respectively, it remains to show by assumption that $\lim_{n\to\infty} \mathrm{Q}(A_n) = 1$ whenever $(A_n)_n$ is an isotone sequence in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} A_n = \Omega$. Fortunately, this follows immediately from the second part of statement .4 and

$$|\,\mathrm{Q}(\Omega) - \mathrm{Q}(A_n)| \;\leq\; \frac{\beta_\rho(\Lambda) + \rho(0) + \inf\limits_{1_{A_n} \geq Z \in \mathfrak{X}} \rho(Z) - \rho(\lambda)}{\lambda} \quad \text{for all } \lambda > 0;$$

Note that $\beta_\rho(\Lambda) \geq E_{\mathrm{Q}}[-\lambda 1_{A_n}] - \inf\limits_{1_{A_n} \geq Z \in \mathfrak{X}} \rho(\lambda Z) + (\lambda\,\mathrm{Q}(\Omega) + \rho(\lambda) - \rho(0))$. ■

9 Proofs of results from section 2

Proof of Lemma 2.1:

It remains to show the if part. For this purpose let $(X_i)_{i\in I}$ denote a uniformly bounded net in $\mathfrak{X}_b$ with pointwise limit $X \in \mathfrak{X}_b$. Setting $c := \liminf_i \rho(X_i)$ and fixing $\varepsilon > 0$, we may find a subnet $(X_{i(k)})_{k\in K}$ with $\rho(X_{i(k)}) < c + \varepsilon$ for every $k \in K$. Hence $(X_{i(k)} + c + \varepsilon)_{k\in K}$ is a net in some A_r which converges pointwise to $X + c + \varepsilon$. So in view of condition (2.1) $X + c + \varepsilon$ is also the pointwise limit of some sequence $(Y_n)_n$ from A_r. If ρ satisfies the Fatou property we may conclude $\rho(X + c + \varepsilon) \leq \liminf_{n\to\infty} \rho(Y_n) \leq 0$, and hence $\rho(X) \leq c + \varepsilon$. This completes the proof.

■

Proof of Theorem 2.2:

Let us retake assumptions and notations from Theorem 2.2

The implication .2 $\Rightarrow$ .3 is always valid as indicated in Proposition 1.4.

Let us now introduce the space $\hat{F}$ consisting of all real linear forms on $\mathfrak{X}_b$ which are representable by some $\mu \in F$. The operator norm $\|\cdot\|_{\hat{F}}$ on $\hat{F}$ w.r.t. the sup norm $\|\cdot\|_\infty$ satisfies $\|\int \cdot\, d\mu\|_{\hat{F}} = \|\mu\|_F$ for every $\mu \in F$. Since F is supposed to be complete w.r.t $\|\cdot\|_F$, $(\hat{F}, \|\cdot\|_{\hat{F}})$ is a Banach space. The topological dual $\hat{F}'$ of $\hat{F}$ will be endowed with the respective operator norm $\|\cdot\|$, and $B_{\hat{F}'}$ denotes the unit ball in $\hat{F}'$.

Since F contains all Dirac measures, $\mathfrak{X}_b$ may be embedded isometrically into $\hat{F}'$ w.r.t. $\|\cdot\|_\infty$ and $\|\cdot\|$ by the evaluation mapping $\hat{e} : \mathfrak{X}_b \to \hat{F}'$. Next let us fix an arbitrary $J \in \hat{F}'$ outside the closure $cl(\hat{e}(\mathfrak{X}_b) \cap B_{\hat{F}'})$ of $\hat{e}(\mathfrak{X}_b) \cap B_{\hat{F}'}$ w.r.t. the weak * topology $\sigma(\hat{F}', \hat{F})$. By Hahn-Banach theorem we may find some $\sigma(\hat{F}', \hat{F})-$continuous real linear form Λ on $\hat{F}'$ with

$$\sup\{\Lambda(\tilde{J}) \mid \tilde{J} \in cl(\hat{e}(\mathfrak{X}_b) \cap B_{\hat{F}'})\} < \Lambda(J).$$

In addition there is some $\mu \in F$ with $\Lambda(\tilde{J}) = \tilde{J}(\int \cdot\, d\mu)$ for any $\tilde{J} \in \hat{F}'$. Without loss of generality we may assume $\|\mu\|_F = 1$. Since $\hat{e}$ is isometric, we obtain then

$$\|J\| > \sup\{\int X\, d\mu \mid X \in \mathfrak{X}_b, \sup_{\omega\in\Omega} |X(\omega)| \leq 1\} = \|\mu\|_F = 1.$$

Hence $\hat{e}(\mathfrak{X}_b) \cap B_{\hat{F}'}$ is $\sigma(\hat{F}', \hat{F})-$dense in $B_{\hat{F}'}$.

Now let condition (2.2) be valid, and let us assume that ρ satisfies the nonsequential Fatou property. Using Dirac measures δ_ω ($\omega \in \Omega$), we may define for any $J \in \hat{F}'$ a mapping $X_J \in \mathbb{R}^\Omega$ via $X_J(\omega) := J(\int \cdot \, \delta_\omega)$. Each X_J is bounded because $|X_J(\omega)| \leq \|J\|$ holds for every $\omega \in \Omega$. Furthermore for any $J \in \hat{F}'$ there exists a uniformly bounded net $(X_i)_{i\in I}$ in $\mathfrak{X}_b$ such that $(\hat{e}(X_i))_{i\in I}$ converges to J w.r.t. $\sigma(\hat{F}', \hat{F})$. In particular X_J is the pointwise limit of $(X_i)_{i\in I}$, which means that it belongs to $\mathfrak{X}_b$ due to (2.2). Hence the mapping $\hat{\rho} : \hat{F}' \to \mathbb{R},\ J \mapsto \rho(X_J)$ is well defined with $\hat{\rho}(\hat{e}(X)) = \rho(X)$ for $X \in \mathfrak{X}_b$.

For every $r > 0$ and any net $(J_i)_{i\in I}$ in $\hat{\rho}^{-1}(]-\infty, 0]) \cap rB_{\hat{F}'}$ we may select by Banach-Alaoglu theorem a subnet $(J_{i(k)})_{k\in K}$ and some $J \in rB_{F'}$ such that $(J_{i(k)})_{k\in K}$ converges to J w.r.t. $\sigma(\hat{F}', \hat{F})$. Then $(X_{J_{i(k)}})_{k\in K}$ is a uniformly bounded net in $\mathfrak{X}_b$ which converges pointwise to X_J. Since ρ fulfills the nonsequential Fatou property, we obtain

$$\hat{\rho}(J) = \rho(X_J) \leq \liminf_k \rho(X_{J_{i(k)}}) = \liminf_k \hat{\rho}(J_{i(k)}) \leq 0.$$

Thus the sets $\hat{\rho}^{-1}(]-\infty, 0]) \cap rB_{\hat{F}'}$ ($r > 0$) are $\sigma(\hat{F}', \hat{F})-$compact, which means that $\hat{\rho}^{-1}(]-\infty, 0])$ is closed w.r.t. $\sigma(\hat{F}', \hat{F})$ by Krein-Smulian theorem. Now it is easy to check that $\rho^{-1}(]-\infty, 0]) \cap \mathfrak{X}_b$ is closed w.r.t. $\sigma(\mathfrak{X}_b, F)$, which implies that all level sets $\rho^{-1}(]-\infty, c]) \cap \mathfrak{X}_b$ ($c \in \mathbb{R}$) are $\sigma(\mathfrak{X}_b, F)-$closed due to the translation invariance of ρ. This shows statement .2, drawing on Propositions 1.2, 6.6 and Lemma 6.5. As a further consequence we have equivalence of the statements .1 - .3 under (2.1), (2.2) in view of Lemma 2.1.

If we strengthen condition (2.2) by the assumption that the sets A_r from (2.1) are compact w.r.t. $\sigma(\mathfrak{X}, F)$, it remains to show the implication .2 $\Rightarrow$ .1. Indeed statement .2 implies that ρ is lower semicontinuous w.r.t. the weak topology $\sigma(\mathfrak{X}, F)$ by Propositions 1.2. Furthermore for any uniformly bounded net $(X_i)_{i\in I}$ in $\mathfrak{X}_b$ with pointwise limit $X \in \mathfrak{X}_b$ we may suppose without loss of generality that $(X_i)_{i\in I}$ is a net in some A_r due to translation invariance of ρ. Then, drawing on relative $\sigma(\mathfrak{X}, F)-$compactness of A_r, the mapping X is the $\sigma(\mathfrak{X}, F)-$limit of $(X_i)_{i\in I}$. This implies $\liminf_i \rho(X_i) \geq \rho(X)$, and completes the proof. ∎

Proof of Remark 2.4:

Let $\hat{e}$ denote the evaluation mapping from $\mathfrak{X}_b$ into the topological dual F' of F w.r.t. the norm of total variation. It is isometric w.r.t. the sup norm $\|\cdot\|_\infty$ and the operator norm $\|\cdot\|$. Then the if part is obvious in view of Banach-Alaoglu theorem. Conversely, translation invariance and relative $\sigma(\mathfrak{X}, F)-$compactness of the sets A_r imply the $\sigma(F', F)-$compactness of the sets $\hat{e}(\mathfrak{X}_b) \cap rB_{F'}$ ($r > 0$), where $B_{F'}$ denotes the unit ball w.r.t. $\|\cdot\|$. This means that $\hat{e}(\mathfrak{X}_b)$ is closed w.r.t. $\sigma(F', F)$ due to the Krein-Smulian theorem, and then $\hat{e}(\mathfrak{X}) = F'$ because $\mathfrak{X}_b$ separates points in F, and thus $\hat{e}(\mathfrak{X}_b)$ is dense w.r.t. $\sigma(F', F)$. The proof is finished. ∎

A Appendix

Proposition A.1 *Let $(\widetilde{\Omega}, \widetilde{\mathcal{F}})$ be a measurable space, and let $\{\emptyset, \widetilde{\Omega}\} \subseteq \mathcal{S} \subseteq \widetilde{\mathcal{F}}$ be stable under finite union and countable intersection, generating $\widetilde{\mathcal{F}}$. Furthermore, for every $A \in \mathcal{S}$ there exists a sequence $(A_n)_n$ in $\mathcal{S}$ such that $\widetilde{\Omega} \setminus A = \bigcup_{n=1}^{\infty} A_n$. Then every probability content Q on $\widetilde{\mathcal{F}}$ is a probability measure if and only if $\lim_{n\to\infty} \mathrm{Q}(A_n) = 1$ holds for any isotone sequence $(A_n)_n$ in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} A_n = \widetilde{\Omega}$.*

Proof:

Let Q be a probability content on $\widetilde{\mathcal{F}}$, and let us denote $\varphi := \mathrm{Q}\,|\mathcal{S}$. The only if part of the statement is obvious. For the if part we want to show

(*) $\lim_{n\to\infty} \varphi(A_n) = \varphi(A)$ for every isotone sequence $(A_n)_n$ in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{S}$

Since for $A, B \in \mathcal{S}$ with $A \subseteq B$ there is by assumption an isotone sequence $(A_n)_n$ in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} A_n = B \setminus A$, we may apply a version of the general extension theorem by König (cf. [13], Theorem 7.12 with Proposition 4.5). Hence condition $(*)$ together with the assumptions on $\mathcal{S}$ guarantee a probability measure P on the $\sigma-$algebra $\widetilde{\mathcal{F}}$ with $\mathrm{P}\,|\mathcal{S} = \varphi$, and $\mathrm{P}(A) = \sup_{A \supseteq B \in \mathcal{S}} \mathrm{P}(B)$ for every $A \in \widetilde{\mathcal{F}}$. In particular $\mathrm{P} \leq \mathrm{Q}$ which implies $\mathrm{P} = \mathrm{Q}$ due to additivity of Q and P. Therefore it remains to prove the condition $(*)$.

<u>proof of (*)</u>:

Let $(A_n)_n$ be an isotone sequence in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{S}$. By assumption there exists some isotone sequence $(B_n)_n$ in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} B_n = \widetilde{\Omega} \setminus A$. Then $(B_n \cup A)_n$ and $(B_n \cup A_n)_n$ are isotone sequences in $\mathcal{S}$ with $\bigcup_{n=1}^{\infty} (B_n \cup A_n) = \widetilde{\Omega} = \bigcup_{n=1}^{\infty} (B_n \cup A)$, and therefore $\lim_{n\to\infty} (\varphi(B_n) + \varphi(A_n)) = 1 = \lim_{n\to\infty} \varphi(B_n) + \varphi(A)$. Hence $\lim_{n\to\infty} \varphi(A_n) = \varphi(A)$, which shows (*), and completes the proof.

■

B Appendix

Proof of Proposition 1.1:

Let $X \in \mathfrak{X}$, and let $\rho'_+(X, \cdot) : \mathfrak{X} \to \mathbb{R}$ denote the respective rightsided derivative of ρ at X defined by $\rho'_+(X, Y) := \lim_{h\to 0_+} \frac{\rho(X+hY)-\rho(X)}{h}$. It it well known from convex analysis that $\rho'_+(X, \cdot)$ is well defined and sublinear satisfying $\rho'_+(X, Y - X) \leq \rho(Y) - \rho(X)$ for all $Y \in \mathfrak{X}$. Then we may choose by Hahn-Banach theorem some linear form $\tilde{\Lambda}$ on $\mathfrak{X}$ with $\tilde{\Lambda} \leq \rho'_+(X, \cdot)$. Moreover, we obtain for $Z \geq 0$

$$\tilde{\Lambda}(Z) \leq \rho'_+(X, (X + Z) - X) \leq \rho(X + Z) - \rho(X) \leq 0.$$

Therefore $\Lambda := -\tilde{\Lambda}$ is a positive linear form fulfilling $\Lambda(X-Y) \le \rho'_+(X, Y-X) \le \rho(Y) - \rho(X)$ for $Y \in \mathfrak{X}$, and $\Lambda(Y) \le \pi(Y)$ for $Y \in \mathfrak{C}$. This implies $\Lambda|\mathfrak{C} = \pi$ due to linearity of $\Lambda|\mathfrak{C}$ and π. Furthermore we have shown $\beta_\rho(\Lambda) = -\Lambda(X) - \rho(X)$.

For the proof of the equivalence stated in Proposition 1.1 note that the only if part is obvious, the if part follows immediately from the Fenchel-Moreau theorem (cf. [2], Theorem 4.2.2), observing that $\beta_\rho(\Lambda) < \infty$ only if Λ is positive linear and extends π (see also Proposition 3.9 in [9]). This completes the proof. ■

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References

[1] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, Coherent measures of risk, Math. Finance 9 (1999), 203-228.

[2] J.-P. Aubin, I. Ekeland, "Applied Nonlinear Analysis", John Wiley and Sons, New York, 1984.

[3] C. Castaing, M. Valadier, "Convex Analysis and Measurable Multifunctions", Lecture Notes in Mathematics Vol. 580, Springer, Berlin, Heidelberg, New York, 1977.

[4] F. Delbaen, Coherent Measures of Risk on General Probability Spaces, in: K. Sandmann, P.J. Schönbucher (Eds.), "Advances in Finance and Stochastics", Springer, Berlin, 2002, pp. 1-37.

[5] C. Dellacherie, P.-A. Meyer, "Probabilities and Potential", North-Holland, Amsterdam, 1978.

[6] D. Denneberg, "Non-Additive Measure and Integral", Kluwer, Dordrecht, 1994.

[7] H. Föllmer, A. Schied, "Stochastic Finance", de Gruyter, Berlin, New York, 2004 (2nd ed.).

[8] M. Fritelli, E. Rosazza Gianin, Dynamic Convex Risk Measures, in: G. Szego (Ed), "Risk Measures for the 21st Century", Wiley, New York, 2004, pp. 227-248.

[9] M. Frittelli, G. Scandolo, Risk measures and capital requirements for processes, Math. Finance 16 (2006), 589-612.

[10] B. Fuchssteiner, W. Lusky, "Convex Cones", North-Holland, Amsterdam, New York, 1981.

[11] E. Jouini, W. Schachermayer and N. Touzi, Law invariant risk measures have the Fatou property, Advances in Mathematical Economics, 9 (2006), 49-71.

[12] H. König, On some basic theorems in convex analysis, in: B. Korte (Ed.), "Modern Applied Mathematics - Optimization and Operations Research", North-Holland, Amsterdam,1982, pp. 107-144.

[13] H. König, "Measure and Integration", Springer, Berlin et al., 1997.

[14] H. König, Measure and Integration: Integral representations of isotone functionals, Annales Universitatis Saraviensis 9 (1998), 123-153.

[15] H. König, N. Kuhn, Angelic spaces and the double limit relation, J. London Math. Soc. 35 (1987), 454-470.

[16] V. Krätschmer, Robust representation of convex risk measures by probability measure, Finance and Stochastics 9 (2005), 597-608.

[17] V. Krätschmer, Compactness in spaces of inner regular measures and a general Portmanteau lemma, SFB 649 discussion paper 2006-081, downloadable at http://sfb649.wiwi.hu-berlin.de.

[18] F. Riedel, Dynamic Coherent Risk Measures, Stochastic Processes and Applications 112 (2004), 185-200.

[19] A. Ruszczynski, A. Shapiro, Optimization of convex risk functions, Math. Oper. Res. 31 (2006), 433-451.

[20] S. Simons, A convergence theorem with boundary, Pacific J. Math. 40 (1972), 703-708.

[21] S. Weber, Distribution-invariant dynamic risk measures, information, and dynamic consistency, Math. Finance 16 (2006), 419-442.

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