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Pricing Bermudan options using regression: optimal rates of convergence for lower estimates

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Pricing Bermudan options using regression: optimal rates of convergence for lower estimates

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Abstract

The problem of pricing Bermudan options using Monte Carlo and a nonparametric regression is considered. We derive optimal non-asymptotic bounds for a lower biased estimate based on the suboptimal stopping rule constructed using some estimates of continuation values. These estimates may be of different nature, they may be local or global, with the only requirement being that the deviations of these estimates from the true continuation values can be uniformly bounded in probability.

Keywords: Bermudan options; Regression; Boundary condition.

1 Introduction

An American option grants the holder the right to select the time at which to exercise the option, and in this differs from a European option which may be exercised only at a fixed date. A general class of American option pricing problems can be formulated through an $\mathbb{R}^d$ Markov process $\{X(t),\, 0 \le t \le T\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathrm{P})$. It is assumed that $X(t)$ is adapted to $(\mathcal{F}_t)_{0\le t\le T}$ in the sense that each X_t is $\mathcal{F}_t$ measurable. Recall that each $\mathcal{F}_t$ is a σ-algebra of subsets of Ω such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \le t$. We interpret $\mathcal{F}_t$ as all relevant financial information available up to time t. We restrict attention to options admitting a finite set of exercise opportunities $0 = t_0 < t_1 < t_2 < \ldots < t_L = T$, sometimes called Bermudan options. If exercised at time t_l, $l = 1, \ldots, L$, the option pays $f_l(X(t_l))$, for some known functions $f_0, f_1, \ldots, f_L$ mapping $\mathbb{R}^d$ into $[0, \infty)$. Let $\mathcal{T}_n$ denote the set of stopping times taking values in $\{n, n+1, \ldots, L\}$. A standard result in the theory of contingent claims states that the equilibrium price

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$V_n(x)$ of the American option at time t_n in state x given that the option was not exercised prior to t_n is its value under an optimal exercise policy:

$$V_n(x) = \sup_{\tau \in \mathcal{T}_n} \mathrm{E}[f_\tau(X(t_\tau))|X(t_n) = x], \quad x \in \mathbb{R}^d.$$

Pricing an American option thus reduces to solving an optimal stopping problem. Solving this optimal stopping problem and pricing an American option are straightforward in low dimensions. However, many problems arising in practice (see e.g. Glasserman (2004)) have high dimensions, and these applications have motivated the development of Monte Carlo methods for pricing American option. Pricing American style derivatives with Monte Carlo is a challenging task because the determination of optimal exercise strategies requires a backwards dynamic programming algorithm that appears to be incompatible with the forward nature of Monte Carlo simulation. Much research was focused on the development of fast methods to compute approximations to the optimal exercise policy. Notable examples include the functional optimization approach in Andersen (2000), mesh method of Broadie and Glasserman (1997), the regression-based approaches of Carriere (1996), Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999) and Egloff (2005). A common feature of all above mentioned algorithms is that they deliver estimates $\widehat{C}_0(x), \ldots, \widehat{C}_{L-1}(x)$ for the so called continuation values:

$$(1.1) \quad C_k(x) := \mathrm{E}[V_{k+1}(X(t_{k+1}))|X(t_k) = x], \quad k = 0, \ldots, L-1.$$

An estimate for V_0, the price of the option at time t_0 can then be defined as

$$\widetilde{V}_0(x) := \max\{f_0(x), \widehat{C}_0(x)\}, \quad x \in \mathbb{R}^d.$$

This estimate basically inherits all properties of $\widehat{C}_0(x)$. In particular, it is usually impossible to determine the sign of the bias of $\widetilde{V}_0$ since the bias of $\widehat{C}_0$ may change its sign. One way to get a lower bound (low biased estimate) for V_0 is to construct a (generally suboptimal) stopping rule

$$\widehat{\tau} = \min\{0 \le k \le L : \widehat{C}_k(X(t_k)) \le f_k(X(t_k))\}$$

with $\widehat{C}_L \equiv 0$ by definition. Simulating a new independent set of trajectories and averaging the pay-offs stopped according to $\widehat{\tau}$ on these trajectories gives us a lower bound $\widehat{V}_0$ for V_0. As was observed by practitioners, the so constructed estimate $\widehat{V}_0$ has rather stable behavior with respect to the estimates of continuation values $\widehat{C}_0(x), \ldots, \widehat{C}_{L-1}(x)$, that is even rather poor estimates of continuation values may lead to a good estimate $\widehat{V}_0$. The aim of this paper is to find a theoretical explanation of this observation and to investigate the properties of $\widehat{V}_0$. In particular, we derive optimal non-asymptotic bounds for the bias $V_0 - \mathrm{E}\,\widehat{V}_0$ assuming some uniform probabilistic bounds for $C_r - \widehat{C}_r$.

It is shown that the bounds for $V_0 - \mathrm{E}\,\widehat{V}_0$ are usually much tighter than ones for $V_0 - \mathrm{E}\,\widetilde{V}_0$ implying a better quality of $\widehat{V}_0$ as compared to the quality of $\widetilde{V}_0$ constructed using one and the same set of estimates for continuation values.

The issues of convergence for regression algorithms have been already studied in several papers. Clément, Lamberton and Protter (2002) were first who proved the convergence of the Longstaff-Schwartz algorithm. Glasserman and Yu (2005) have shown that the number of Monte Carlo paths has to be in general exponential in the number of basis functions used for regression in order to ensure convergence. Recently, Egloff, Kohler and Todorovic (2007) (see also Kohler (2008)) have derived the rates of convergence for continuation values estimates obtained by the so called dynamic look-ahead algorithm (see Egloff (2004)) that "interpolates" between Longstaff-Schwartz and Tsitsiklis-Roy algorithms. They presented the convergence rates for $\widetilde{V}_0$ which coincide with the rates of $\widehat{C}_0$ and are determined by the smoothness properties of the true continuation values $C_0, \ldots, C_{L-1}$. It turns out that the convergence rates for $\widehat{V}_0$ depend not only on the smoothness of continuation values (as opposite to $\widetilde{V}_0$), but also on the behavior of the underlying process near the exercise boundary. Interestingly enough, there are cases where these rates become almost independent either of the smoothness properties of $\{C_k\}$ or of the dimension of X and the bias of $\widehat{V}_0$ decreases exponentially in the number of Monte Carlo paths used to construct $\{\widehat{C}_k\}$.

The paper is organized as follows. In Section 2.1 we introduce and discuss the so called boundary assumption which describes the behavior of the underlying process X near the exercise boundary and heavily influences the properties of $\widehat{V}_0$. In Section 2.2 we derive non-asymptotic bounds for the bias $V_0 - \mathrm{E}\,\widehat{V}_0$ and prove that these bounds are optimal in the minimax sense. Finally, we illustrate our results by a numerical example.

2 Main results

2.1 Boundary assumption

For the considered Bermudan option let us introduce a continuation region $\mathcal{C}$ and an exercise (stopping) region $\mathcal{E}$:

$$
\begin{aligned}
\mathcal{C} &:= \{(i,x) : f_i(x) < C_i(x)\}\,, \\
\mathcal{E} &:= \{(i,x) : f_i(x) \geq C_i(x)\}\,.
\end{aligned}
\tag{2.2}
$$

Furthermore, let us assume that there exist constants $B_{0,k} > 0$, $k = 0, \ldots, L-1$ and $\alpha > 0$ such that the inequality

$$
\mathrm{P}_{t_k|t_0}(0 < |C_k(X(t_k)) - f_k(X(t_k))| \leq \delta) \leq B_{0,k}\delta^{\alpha}, \quad \delta > 0, \tag{2.3}
$$

holds for all $k = 0, \ldots, L-1$, where $\mathrm{P}_{t_k|t_0}$ is the conditional distribution of $X(t_k)$ given $X(t_0)$. Assumption (2.3) provides a useful characterization of the behavior of the continuation values $\{C_k\}$ and payoffs $\{f_k\}$ near the exercise boundary $\partial\mathcal{E}$. Although this assumption seems quite natural to look at, we make in this paper, to the best of our knowledge, a first attempt to investigate its influence on the convergence rates of lower bounds based on suboptimal stopping rules.

In the situation when all functions $C_k - f_k$, $k = 0, \dots, L-1$ are smooth and have non-vanishing derivatives in the vicinity of the exercise boundary, we have $\alpha = 1$. Other values of α are possible as well. We illustrate this by two simple examples.

Example 1 Fix some $\alpha > 0$ and consider a two period ($L = 1$) Bermudan power put option with the payoffs

$$f_0(x) = f_1(x) = (K^{1/\alpha} - x^{1/\alpha})^+, \quad x \in \mathbb{R}_+, \quad K > 0. \tag{2.4}$$

Denote by Δ the length of the exercise period, i.e. $\Delta = t_1 - t_0$. If the process X follows the Black-Scholes model with volatility σ and zero interest rate, then one can show that

$$\begin{aligned} C_0(x) := \mathrm{E}[f_1(X(t_1))|X(t_0) = x] = K^{1/\alpha}\Phi(-d_2) \\ - x^{1/\alpha} e^{\Delta(\alpha^{-1}-1)(\sigma^2/2\alpha)}\Phi(-d_1) \end{aligned}$$

with Φ being the cumulative distribution function of the standard normal distribution,

$$d_1 = \frac{\log(x/K) + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\sigma^2\Delta}{\sigma\sqrt{\Delta}}$$

and $d_2 = d_1 - \sigma\sqrt{\Delta}/\alpha$. As can be easily seen, the function $C_0(x) - f_0(x)$ satisfies $|C_0(x) - f_0(x)| \asymp x^{1/\alpha}$ for $x \to +0$ and $C_0(x) > f_0(x)$ for all $x > 0$ if $\alpha \geq 1$. Hence

$$\mathrm{P}(0 < |C_0(X(t_0)) - f_0(X(t_0))| \leq \delta) \lesssim \delta^\alpha, \quad \delta \to 0, \quad \alpha \geq 1.$$

Taking different α in the definition of the payoffs (2.4), we get (2.3) satisfied for α ranging from 1 to ∞.

In fact, even the extreme case "$\alpha = \infty$" may take place as shown in the next example.

Example 2 Let us consider again a two period Bermudan option such that the corresponding continuation value $C_0(x) = \mathrm{E}[f_1(X(t_1))|X(t_0) = x]$ is positive and monotone increasing function of x on any compact set in $\mathbb{R}$. Fix some $x_0 \in \mathbb{R}$ and choose δ_0 satisfying $\delta_0 < C_0(x_0)$. Define the payoff function $f_0(x)$ in the following way

$$f_0(x) = \begin{cases} C_0(x_0) + \delta_0, & x < x_0, \\ C_0(x_0) - \delta_0, & x \geq x_0. \end{cases}$$

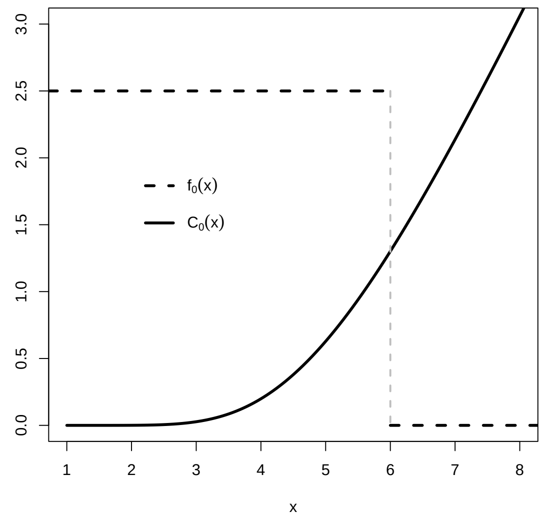

Figure 1: Illustration to Example 2.

So, $f_0(x)$ has a "digital" structure. Figure 1 shows the plots of C_0 and f_0 in the case where X follows the Black-Scholes model and $f_1(x) = (x - K)^+$. It is easy to see that

$$\mathrm{P}_{t_0}(0 < |C_0(X(t_0)) - f_0(X(t_0))| \leq \delta_0) = 0.$$

On the other hand

$$\begin{aligned} \mathcal{C} &= \{x \in \mathbb{R} : C_0(x) \geq f_0(x)\} = \{x \in \mathbb{R} : x \geq x_0\}, \\ \mathcal{E} &= \{x \in \mathbb{R} : C_0(x) < f_0(x)\} = \{x \in \mathbb{R} : x < x_0\}. \end{aligned}$$

So, both continuation and exercise regions are not trivial in this case.

The last example is of particular interest because as will be shown in the next sections the bias of $\widehat{V}_0$ decreases in this case exponentially in the number of Monte Carlo paths used to estimate the continuation values, a lower bound $\widehat{V}_0$ was constructed from.

2.2 Non-asymptotic bounds for $V_0 - \mathrm{E}\,\widehat{V}_0$

Let $\widehat{C}_{k,M}$, $k = 1, \ldots, L-1$, be some estimates of continuation values obtained using M paths of the underlying process X starting from x_0 at time t_0. We may think of $(X^{(1)}(t), \ldots, X^{(M)}(t))$ as being a vector process on the product probability space with σ-algebra $\mathcal{F}^{\otimes M}$ and the product measure $\mathrm{P}_{x_0}^{\otimes M}$ defined on $\mathcal{F}^{\otimes M}$ via

$$\mathrm{P}_{x_0}^{\otimes M}(A_1 \times \ldots \times A_M) = \mathrm{P}_{x_0}(A_1) \cdot \ldots \cdot \mathrm{P}_{x_0}(A_M),$$

with $A_m \in \mathcal{F}$, $m = 1, \ldots, M$. Thus, each $\widehat{C}_{k,M}$, $k = 0, \ldots, L-1$, is measurable with respect to $\mathcal{F}^{\otimes M}$. The following proposition provides non-asymptotic bounds for the bias $V_0 - \mathrm{E}\,\widehat{V}_{0,M}$ of a lower bound $\widehat{V}_{0,M}$ given uniform probabilistic bounds for $\{\widehat{C}_{k,M}\}$.

Proposition 2.1. *Suppose that there exist constants B_1, B_2 and a positive sequence γ_M such that for any $\delta > \delta_0 > 0$ it holds*

$$\mathrm{P}_{x_0}^{\otimes M}\left(|\widehat{C}_{k,M}(x) - C_k(x)| \geq \delta \gamma_M^{-1/2}\right) \leq B_1 \exp(-B_2 \delta) \tag{2.5}$$

for almost all x with respect to $\mathrm{P}_{t_k|t_0}$, the conditional distribution of $X(t_k)$ given $X(t_0)$, $k = 0, \ldots, L-1$. Define

$$V_{0,M} := \mathrm{E}\left[f_{\widehat{\tau}_M}(X(t_{\widehat{\tau}_M}))|X(t_0) = x_0\right] \tag{2.6}$$

with

$$\widehat{\tau}_M := \min\left\{0 \leq k \leq L : \widehat{C}_{k,M}(X(t_k)) \leq f_k(X(t_k))\right\}. \tag{2.7}$$

If the boundary condition (2.3) *is fulfilled, then*

$$0 \leq V_0 - \mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}[V_{0,M}] \leq B\left[\sum_{l=0}^{L-1} B_{0,l}\right]\gamma_M^{-(1+\alpha)/2}$$

with some constant B depending only on α, B_1 and B_2.

The above convergence rates are, in fact, optimal in the following sense.

Proposition 2.2. *Fix a set of non-zero payoff functions $f_0, \ldots, f_L$ and let $\mathcal{P}_\alpha$ be a class of pricing measures such that the boundary condition* (2.3) *is fulfilled with some $\alpha > 0$. For any positive sequence γ_M satisfying*

$$\gamma_M^{-1} = o(1), \quad \gamma_M = O(M), \quad M \to \infty,$$

there exist a subset $\mathcal{P}_{\alpha,\gamma}$ of $\mathcal{P}_\alpha$ and a constant $B > 0$ such that for any $M \geq 1$, any stopping rule $\widehat{\tau}_M$ and any set of estimates $\{\widehat{C}_{k,M}\}$ measurable w.r.t. $\mathcal{F}^{\otimes M}$, we have for some $\delta > 0$ and $k = 0, \ldots, L-1$,

$$\sup_{\mathrm{P} \in \mathcal{P}_{\alpha,\gamma}} \mathrm{P}^{\otimes M}\left(|\widehat{C}_{k,M}(x) - C_k(x)| \geq \delta\gamma_M^{-1/2}\right) > 0$$

for almost all x w.r.t. any $\mathrm{P} \in \mathcal{P}_{\alpha,\gamma}$ and

$$\sup_{\mathrm{P} \in \mathcal{P}_{\alpha,\gamma}}\left\{\sup_{\tau \in \mathcal{T}_0} \mathrm{E}_{\mathrm{P}}^{\mathcal{F}_{t_0}}[f_\tau(X(t_\tau))] - \mathrm{E}_{\mathrm{P}^{\otimes M}}[\mathrm{E}_{\mathrm{P}}^{\mathcal{F}_{t_0}}\, f_{\widehat{\tau}_M}(X(t_{\widehat{\tau}_M}))]\right\} \geq B\gamma_M^{-(1+\alpha)/2}.$$

Finally, we discuss the case when "$\alpha = \infty$", meaning that there exists $\delta_0 > 0$ such that

$$\mathrm{P}_{t_k|t_0}(0 < |C_k(X(t_k)) - f_k(X(t_k))| \le \delta_0) = 0 \tag{2.8}$$

for $k = 0, \ldots, L-1$. This is very favorable situation for pricing. It turns out that if the continuation values estimates $\{\widehat{C}_{k,M}\}$ satisfy a kind of exponential inequality and (2.8) holds, then the bias of $\widehat{V}_{0,M}$ converges to zero exponentially fast in γ_M.

Proposition 2.3. *Suppose that for any $\delta > 0$ there exist constants B_1, B_2 possibly depending on δ and a sequence of positive numbers γ_M not depending on δ such that*

$$\mathrm{P}_{x_0}^{\otimes M}\left(|\widehat{C}_{k,M}(x) - C_k(x)| \ge \delta\right) \le B_1 \exp(-B_2 \gamma_M) \tag{2.9}$$

for almost all x with respect to $\mathrm{P}_{t_k|t_0}$, $k = 0, \ldots, L-1$. Assume also that there exist a constant $B_f > 0$ such that

$$\mathrm{E}\left[\max_{k=0,\ldots,L} f_k^2(X(t_k))\right] \le B_f, \quad k = 0, \ldots, L. \tag{2.10}$$

If the condition (2.8) is fulfilled with some $\delta_0 > 0$, then

$$0 \le V_0 - \mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}[V_{0,M}] \le B_3 \exp(-B_4 \gamma_M)$$

with some constant B_3 and B_4 depending only on B_1, B_2 and B_f.

Discussion Let us make a few remarks on the results of this section. First, Proposition 2.1 implies that the convergence rates of $\widehat{V}_{0,M}$ are always faster than the convergence rates of $\{\widehat{C}_{k,M}\}$ provided that $\alpha > 0$. Indeed, while the convergence rates of $\{\widehat{C}_{k,M}\}$ are of order $\gamma_M^{-1/2}$, the bias of $\widehat{V}_{0,M}$ converges to zero as fast as $\gamma_M^{-(1+\alpha)/2}$. As to the variance of $\widehat{V}_{0,M}$, it can be made arbitrary small by averaging $\widehat{V}_{0,M}$ over a large number of sets, each consisting of M trajectories, and by taking a large number of new Monte Carlo paths used to average the payoffs stopped according to $\widehat{\tau}_M$.

Second, if the condition (2.8) holds true, then the bias of $\widehat{V}_{0,M}$ decreases exponentially in γ_M, indicating that even very unprecise estimates of the continuation values would lead to the estimate $\widehat{V}_{0,M}$ of acceptable quality.

Finally, let us stress that the results obtained in this section are quite general and do not depend on the particular form of the estimates $\{\widehat{C}_{k,M}\}$, only the inequality (2.5) being crucial for the result to hold. This inequality holds for various types of estimators. These may be global least squares estimators or local polynomial estimators. In particular, it can be shown that if all continuation values $\{C_k\}$ belong to the Hölder class $\Sigma(\beta, H, \mathbb{R}^d)$ and the conditional law of X satisfies some regularity assumptions, then the local polynomial estimates of continuation values satisfy inequality (2.5) with $\gamma_M = M^{2\beta/(2\beta+d)} \log^{-1}(M)$.

3 Numerical example: Bermudan max call

This is a benchmark example studied in Broadie and Glasserman (1997) and Glasserman (2004) among others. Specifically, the model with d identically distributed assets is considered, where each underlying has dividend yield δ. The risk-neutral dynamic of assets is given by

$$\frac{dX_k(t)}{X_k(t)} = (r - \delta)dt + \sigma dW_k(t), \quad k = 1, ..., d,$$

where $W_k(t)$, $k = 1, ..., d$, are independent one-dimensional Brownian motions and r, δ, σ are constants. At any time $t \in \{t_0, ..., t_L\}$ the holder of the option may exercise it and receive the payoff

$$f(X(t)) = (\max(X_1(t), ..., X_d(t)) - K)^+.$$

We take $d = 2$, $r = 5\%$, $\delta = 10\%$, $\sigma = 0.2$ and $t_i = iT/L$, $i = 0, ..., L$, with $T = 3$, $L = 9$ as in Glasserman (2004, Chapter 8). First, we estimate all continuation values via the dynamic programming algorithm using the so called Nadaraya-Watson regression estimator

$$\widehat{C}_{r,M}(x) = \frac{\sum_{m=1}^{M} K((x - X^{(m)}(t_r))/h) Y_{r+1}^{(m)}}{\sum_{m=1}^{M} K((x - X^{(m)}(t_r))/h)} \tag{3.11}$$

with $Y_{r+1}^{(m)} = \max(f_{r+1}(X^{(m)}(t_{r+1})), \widehat{C}_{r+1,M}(X^{(m)}(t_{r+1})))$, $r = 0, \ldots, L-1$. Here K is a kernel, $h > 0$ is a bandwidth and $(X^{(m)}(t_1), \ldots, X^{(m)}(t_L))$, $m = 1, \ldots, M$, is the set of paths of the process X, all starting from the point $x_0 = (90, 90)$ at $t_0 = 0$. As can be easily seen the estimator (3.11) is a local polynomial estimator of degree 0. Upon estimating $\widehat{C}_{1,M}$, we define an estimate for the price of the option at time $t_0 = 0$ as

$$\widetilde{V}_0 := \frac{1}{M} \sum_{m=1}^{M} Y_1^{(m)}.$$

Next, using the so constructed estimates of continuation values we construct a stopping policy $\widehat{\tau}$ which is defined pathwise as

$$\widehat{\tau}^{(n)} := \min \left\{ 1 \leq k \leq L : \widehat{C}_{k,M}(\widetilde{X}^{(n)}(t_k)) \leq f_k(\widetilde{X}^{(n)}(t_k)) \right\}, \quad n = 1, \ldots, N,$$

where $(\widetilde{X}^{(n)}(t_1), \ldots, \widetilde{X}^{(n)}(t_L))$, $n = 1, \ldots, N$, is a new independent set of trajectories of the process X, all starting from $x_0 = (90, 90)$ at $t_0 = 0$. The stopping policy $\widehat{\tau}$ yields a lower bound

$$\widehat{V}_0 = \frac{1}{N} \sum_{n=1}^{N} f_{\widehat{\tau}^{(n)}}(\widetilde{X}^{(n)}(t_{\widehat{\tau}^{(n)}})).$$

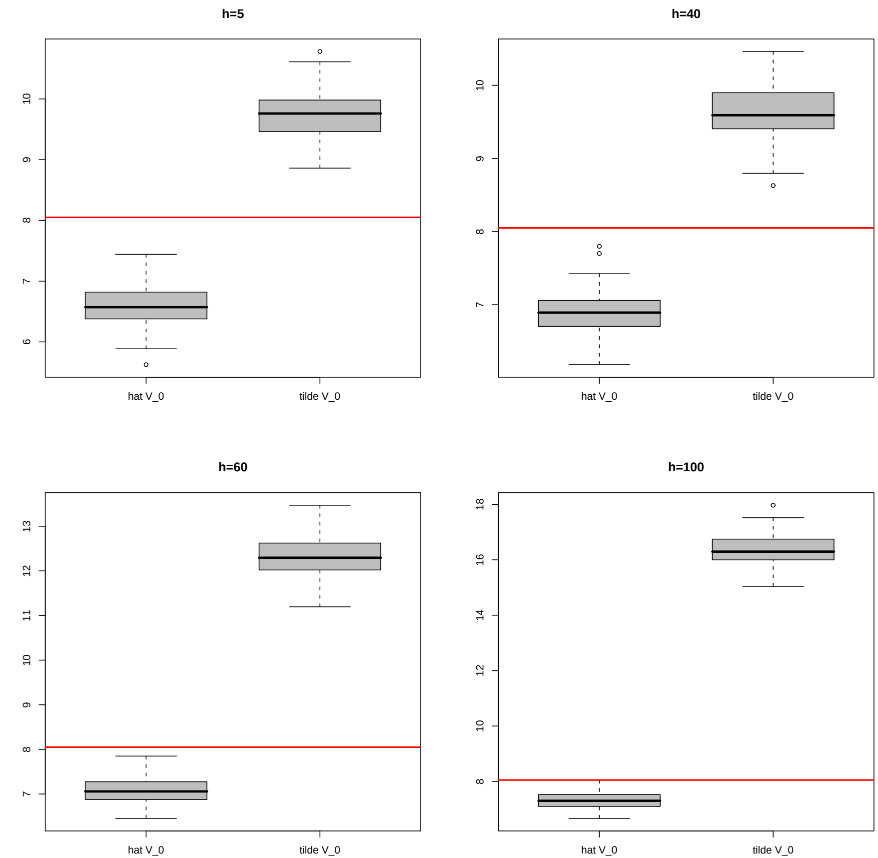

Figure 2: Boxplots of the estimates $\widehat{V}_0$ and $\widetilde{V}_0$ for different values of the bandwidth h. The true option price is shown is a red base line.

In Figure 2 we show the boxplots of $\widetilde{V}_0$ and $\widehat{V}_0$ based on 100 sets of trajectories each of the size $M = 1000$ for different values of the bandwidth h, where the triangle kernel $K(x) = (1 - \|x\|^2)^+$ is used to construct (3.11). Also the true value V_0 of the option (8.08 in this case), computed using a two-dimensional binomial lattice, is shown as a red base line. Several observations can be made by an examination of Figure 2. First, while the bias of $\widehat{V}_0$ is always smaller then the bias of $\widetilde{V}_0$, the largest difference takes place for large h. This can be explained by the fact that for large h more observations $Y_{r+1}^{(m)}$ with $X^{(m)}(t_r)$ lying far away from the given point x become involved in the construction of $\widehat{C}_{r,M}(x)$. This has a consequence of increasing the bias of the estimate (3.11). The most interesting phenomenon is, however, the behavior of $\widehat{V}_0$ which turns out to be quite stable with respect to h. So even in the case of rather poor estimates of continuation values (when h is large) $\widehat{V}_0$ still looks reasonable.

We stress that the aim of this example is not to show the strength of the local polynomial estimation algorithms (for this we would take larger M and higher order kernels) but rather to illustrate the main claim of this paper, namely the claim about the efficiency of $\widehat{V}_0$ as compared to the estimates based on the direct use of continuation values estimates.

4 Conclusion

In this paper we have derived optimal rates of convergence for lower biased estimates of the price of a Bermudan option based on suboptimal exercise policies obtained from some estimates of the optimal continuation values. We have shown that these rates are usually much faster than the convergence rates of the corresponding continuation values estimates. This may explain the efficiency of these lower bounds observed in practice. Moreover, it turns out that there are some cases where the expected values of the lower bounds based on suboptimal stopping rules achieve very fast convergence rates which are exponential in the number of paths used to estimate the corresponding continuation values. This suggests that the algorithms based on suboptimal stopping rules (e.g. Longstaff-Schwartz algorithm) rather than on the direct use of the continuation values estimates might be preferable.

5 Proofs

5.1 Proof of Proposition 2.1

Define

$$\begin{aligned} \tau_j &:= \min\{j \le k < L : C_k(X(t_k)) \le f_k(X(t_k))\}, \quad j = 0, \ldots, L, \\ \widehat{\tau}_{j,M} &:= \min\{j \le k < L : \widehat{C}_k(X(t_k)) \le f_k(X(t_k))\}, \quad j = 0, \ldots, L \end{aligned}$$

and

$$V_{k,M}(x) := \mathrm{E}[f_{\widehat{\tau}_{k,M}}(X(t_{\widehat{\tau}_{k,M}}))|X(t_k) = x], \quad x \in \mathbb{R}^d.$$

The so called *Snell envelope* process V_k is related to τ_k via

$$V_k(x) = \mathrm{E}[f_{\tau_k}(X(t_{\tau_k}))|X(t_k) = x], \quad x \in \mathbb{R}^d.$$

The following lemma provides a useful inequality which will be repeatedly used in our analysis.

Lemma 5.1. *For any $k = 0, \ldots, L-1$, it holds with probability one*

$$\begin{aligned}(5.12)\quad 0 \le V_k(X(t_k)) - V_{k,M}(X(t_k)) \\ \le \mathrm{E}^{\mathcal{F}_{t_k}} \Bigg[\sum_{l=k}^{L-1} |f_l(X(t_l)) - C_l(X(t_l))| \\ \times \left(\mathbf{1}_{\{\widehat{\tau}_{l,M} > l,\, \tau_l = l\}} + \mathbf{1}_{\{\widehat{\tau}_{l,M} = l,\, \tau_l > l\}} \right) \Bigg].\end{aligned}$$

Proof. We shall use induction to prove (5.12). For $k = L-1$ we have

$$\begin{aligned}V_{L-1}(X(t_{L-1})) - V_{L-1,M}(X(t_{L-1})) = \\ = \mathrm{E}^{\mathcal{F}_{t_{L-1}}} \left[(f_{L-1}(X(t_{L-1})) - f_L(X(t_L)))\mathbf{1}_{\{\tau_{L-1} = L-1,\, \widehat{\tau}_{L-1,M} = L\}} \right] \\ + \mathrm{E}^{\mathcal{F}_{t_{L-1}}} \left[(f_L(X(t_L)) - f_{L-1}(X(t_{L-1})))\mathbf{1}_{\{\tau_{L-1} = L,\, \widehat{\tau}_{L-1,M} = L-1\}} \right] \\ = |f_{L-1}(X(t_{L-1})) - C_{L-1}(X(t_{L-1}))| \mathbf{1}_{\{\widehat{\tau}_{L-1,M} \ne \tau_{L-1}\}}\end{aligned}$$

since events $\{\tau_{L-1} = L\}$ and $\{\widehat{\tau}_{L-1,M} = L\}$ are measurable w.r.t. $\mathcal{F}_{t_{L-1}}$. Thus, (5.12) holds with $k = L-1$. Suppose that (5.12) holds with $k = L'+1$. Let us prove it for $k = L'$. Consider a decomposition

$$f_{\tau_{L'}}(X(t_{\tau_{L'}})) - f_{\widehat{\tau}_{L',M}}(X(t_{\widehat{\tau}_{L',M}})) \quad = \quad S_1 + S_2 + S_3$$

with

$$\begin{aligned}S_1 &:= \left(f_{\tau_{L'}}(X(t_{\tau_{L'}})) - f_{\widehat{\tau}_{L',M}}(X(t_{\widehat{\tau}_{L',M}})) \right) \mathbf{1}_{\{\tau_{L'} > L',\, \widehat{\tau}_{L',M} > L'\}} \\ S_2 &:= \left(f_{\tau_{L'}}(X(t_{\tau_{L'}})) - f_{\widehat{\tau}_{L',M}}(X(t_{\widehat{\tau}_{L',M}})) \right) \mathbf{1}_{\{\tau_{L'} > L',\, \widehat{\tau}_{L',M} = L'\}} \\ S_3 &:= \left(f_{\tau_{L'}}(X(t_{\tau_{L'}})) - f_{\widehat{\tau}_{L',M}}(X(t_{\widehat{\tau}_{L',M}})) \right) \mathbf{1}_{\{\tau_{L'} = L',\, \widehat{\tau}_{L',M} > L'\}}.\end{aligned}$$

Since

$$\begin{aligned}\mathrm{E}^{\mathcal{F}_{t_{L'}}}[S_1] &= \mathrm{E}^{\mathcal{F}_{t_{L'}}} \left[\left(V_{L'+1}(X(t_{L'+1})) - V_{L'+1,M}(X(t_{L'+1})) \right) \right] \mathbf{1}_{\{\tau_{L'} > L',\, \widehat{\tau}_{L',M} > L'\}}, \\ \mathrm{E}^{\mathcal{F}_{t_{L'}}}[S_2] &= \left(\mathrm{E}^{\mathcal{F}_{t_{L'}}} \left[f_{\tau_{L'+1}}(X(t_{\tau_{L'+1}})) \right] - f_{L'}(X(t_{L'})) \right) \mathbf{1}_{\{\tau_{L'} > L',\, \widehat{\tau}_{L',M} = L'\}} \\ &= \left(C_{L'}(X(t_{L'})) - f_{L'}(X(t_{L'})) \right) \mathbf{1}_{\{\tau_{L'} > L',\, \widehat{\tau}_{L',M} = L'\}}\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}^{\mathcal{F}_{t_{L'}}}[S_3] &= \left(f_{L'}(X(t_{L'})) - \mathrm{E}^{\mathcal{F}_{t_{L'}}}\left[f_{\widehat{\tau}_{L'+1,M}}(X(t_{\widehat{\tau}_{L'+1,M}}))\right]\right)\mathbf{1}_{\{\tau_{L'}=L',\,\widehat{\tau}_{L',M}>L'\}} \\
&= \left(f_{L'}(X(t_{L'})) - C_{L'}(X(t_{L'}))\right)\mathbf{1}_{\{\tau_{L'}=L',\,\widehat{\tau}_{L',M}>L'\}} \\
&\quad + \mathrm{E}^{\mathcal{F}_{t_{L'}}}\left[\left(V_{L'+1}(X(t_{L'+1})) - V_{L'+1,M}(X(t_{L'+1}))\right)\mathbf{1}_{\{\tau_{L'}=L',\,\widehat{\tau}_{L',M}>L'\}}\right],
\end{aligned}$$

we get with probability one

$$\begin{aligned}
V_{L'}(X(t_{L'})) - V_{L',M}(X(t_{L'}) &\leq |f_{L'}(X(t_{L'})) - C_{L'}(X(t_{L'}))| \\
&\quad \times\left(\mathbf{1}_{\{\widehat{\tau}_{L',M}>L',\,\tau_{L'}=L'\}} + \mathbf{1}_{\{\widehat{\tau}_{L',M}=L',\,\tau_{L'}>L'\}}\right) \\
&\quad + \mathrm{E}^{\mathcal{F}_{t_{L'}}}\left[V_{L'+1}(X(t_{L'+1})) - V_{L'+1,M}(X(t_{L'+1}))\right].
\end{aligned}$$

Our induction assumption implies now that

$$\begin{aligned}
&V_{L'}(X(t_{L'})) - V_{L',M}(X(t_{L'})) \leq \\
&\qquad \mathrm{E}^{\mathcal{F}_{t_{L'}}}\left[\sum_{l=L'}^{L-1} |f_l(X_l) - C_l(X_l)|\left(\mathbf{1}_{\{\widehat{\tau}_{l,M}>l,\,\tau_l=l\}} + \mathbf{1}_{\{\widehat{\tau}_{l,M}=l,\,\tau_l>l\}}\right)\right]
\end{aligned}$$

and hence (5.12) holds for $k = L'$. □

Let us continue with the proof of Proposition 2.1. Consider the sets $\mathcal{E}_l$, $\mathcal{A}_{l,j} \subset \mathbb{R}^d$, $l = 0, \ldots, L-1$, $j = 1, 2, \ldots$, defined as

$$\begin{aligned}
\mathcal{E}_l &:= \left\{x \in \mathbb{R}^d : \widehat{C}_{l,M}(x) \leq f_l(x),\, C_l(x) > f_l(x)\right\} \\
&\quad \cup \left\{x \in \mathbb{R}^d : \widehat{C}_{l,M}(x) > f_l(x),\, C_l(x) \leq f_l(x)\right\}, \\
\mathcal{A}_{l,0} &:= \left\{x \in \mathbb{R}^d : 0 < |C_l(x) - f_l(x)| \leq \gamma_M^{-1/2}\right\}, \\
\mathcal{A}_{l,j} &:= \left\{x \in \mathbb{R}^d : 2^{j-1}\gamma_M^{-1/2} < |C_l(x) - f_l(x)| \leq 2^j\gamma_M^{-1/2}\right\}, \quad j > 0.
\end{aligned}$$

We may write

$$\begin{aligned}
V_0(X(t_0)) - V_{0,M}(X(t_0)) &\leq \mathrm{E}^{\mathcal{F}_{t_0}}\left[\sum_{l=0}^{L-1} |f_l(X(t_l)) - C_l(X(t_l))|\mathbf{1}_{\{X(t_l)\in\mathcal{E}_l\}}\right] \\
&= \sum_{j=0}^{\infty} \mathrm{E}^{\mathcal{F}_{t_0}}\left[\sum_{l=0}^{L-1} |f_l(X(t_l)) - C_l(X(t_l))|\mathbf{1}_{\{X(t_l)\in\mathcal{A}_{l,j}\cap\mathcal{E}_l\}}\right] \\
&\leq \gamma_M^{-1/2}\sum_{l=0}^{L-1} \mathrm{P}_{t_l|t_0}\left(0 < |C_l(X(t_l)) - f_l(X(t_l))| \leq \gamma_M^{-1/2}\right) \\
&\quad + \sum_{j=1}^{\infty} \mathrm{E}^{\mathcal{F}_{t_0}}\left[\sum_{l=0}^{L-1} |f_l(X(t_l)) - C_l(X(t_l))|\mathbf{1}_{\{X(t_l)\in\mathcal{A}_{l,j}\cap\mathcal{E}_l\}}\right].
\end{aligned}$$

Using the fact that

$$|f_l(X(t_l)) - C_l(X(t_l))| \le |\widehat{C}_{l,M}(X(t_l) - C_l(X(t_l))|, \quad l = 0, \ldots, L-1,$$

on $\mathcal{E}_l$, we get for any $j \ge 1$ and $l \ge 0$

$$\begin{aligned}
&\mathrm{E}^{\mathcal{F}_{t_0}}\,\mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}\left[|f_l(X(t_l)) - C_l(X(t_l))|\mathbf{1}_{\{X(t_l)\in\mathcal{A}_{l,j}\cap\mathcal{E}_l\}}\right] \\
&\qquad \le 2^j\gamma_M^{-1/2}\,\mathrm{E}^{\mathcal{F}_{t_0}}\,\mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}\Big[\mathbf{1}_{\{|\widehat{C}_{l,M}(X(t_l) - C_l(X(t_l))| \ge 2^{j-1}\gamma_M^{-1/2}\}} \\
&\qquad\qquad\qquad \times \mathbf{1}_{\{0<|f_l(X(t_l)) - C_l(X(t_l))| \le 2^j\gamma_M^{-1/2}\}}\Big] \\
&\qquad \le 2^j\gamma_M^{-1/2}\,\mathrm{E}^{\mathcal{F}_{t_0}}\Big[\mathrm{P}_{x_0}^{\otimes M}(|\widehat{C}_{l,M}(X(t_l)) - C_l(X(t_l))| \ge 2^{j-1}\gamma_M^{-1/2}) \\
&\qquad\qquad\qquad \times \mathbf{1}_{\{0<|f_l(X(t_l)) - C_l(X(t_l))| \le 2^j\gamma_M^{-1/2}\}}\Big] \\
&\le B_1 2^j\gamma_M^{-1/2}\exp\left(-B_2 2^{j-1}\right)\mathrm{P}_{t_l|t_0}(0 < |f_l(X(t_l)) - C_l(X(t_l))| \le 2^j\gamma_M^{-1/2}) \\
&\qquad\qquad\qquad\qquad \le B_1 B_{0,l} 2^{j(1+\alpha)}\gamma_M^{-(1+\alpha)/2}\exp\left(-B_2 2^{j-1}\right),
\end{aligned}$$

where Assumption 2.3 is used to get the last inequality. Finally, we get

$$\begin{aligned}
&V_0(X(t_0)) - \mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}\left[V_{0,M}(X(t_0))\right] \\
&\le \left[\sum_{l=0}^{L-1} B_{0,l}\right]\gamma_M^{-(1+\alpha)/2} + B'\left[\sum_{l=0}^{L-1} B_{0,l}\right]\gamma_M^{-(1+\alpha)/2}\sum_{j\ge 1} 2^{j(1+\alpha)}\exp(-B_2 2^{j-1}) \\
&\qquad\qquad\qquad\qquad \le B\left[\sum_{l=0}^{L-1} B_{0,l}\right]\gamma_M^{-(1+\alpha)/2}
\end{aligned}$$

with some constant B depending on B_1, B_2 and α.

5.2 Proof of Proposition 2.2

For the sake of simplicity we consider the case of a three period Bermudan option with two possible exercise dates t_1 and t_2 (exercise at t_0 is not possible). We also assume that the payoff function f_2 has a "digital" structure, i.e. it takes two values 0 and 1. The extension to a general case is straightforward but somewhat cumbersome.

We have

$$\begin{aligned}
(5.13)\quad & V_0(X(t_0)) - \widehat{V}_{0,M}(X(t_0)) = \\
&\qquad = \mathrm{E}^{\mathcal{F}_{t_0}}\left[(f_1(X(t_1)) - f_2(X(t_2)))1(\tau_1 = 1, \widehat{\tau}_{1,M} = 2)\right] \\
&\qquad + \mathrm{E}^{\mathcal{F}_{t_0}}\left[(f_2(X(t_2)) - f_1(X(t_1)))1(\tau_1 = 2, \widehat{\tau}_{1,M} = 1)\right] \\
&\qquad\qquad\qquad = \mathrm{E}^{\mathcal{F}_{t_0}}\left[|f_1(X(t_1)) - C_1(X(t_1))|\mathbf{1}_{\{\widehat{\tau}_{1,M}\ne\tau_1\}}\right].
\end{aligned}$$

For an integer $q \geq 1$ consider a regular grid on $[0,1]^d$ defined as

$$G_q = \left\{ \left(\frac{2k_1+1}{2q}, \ldots, \frac{2k_d+1}{2q} \right) : \, k_i \in \{0, \ldots, q-1\}, \, i = 1, \ldots, d \right\}.$$

Let $n_q(x) \in G_q$ be the closest point to $x \in \mathbb{R}^d$ among points in G_q. Consider the partition $\mathcal{X}'_1, \ldots, \mathcal{X}'_{q^d}$ of $[0,1]^d$ canonically defined using the grid G_q (x and y belong to the same subset if and only if $n_q(x) = n_q(y)$). Fix an integer $m \leq q^d$. For any $i \in \{1, \ldots, m\}$, define $\mathcal{X}_i = \mathcal{X}'_i$ and $\mathcal{X}_0 = \mathbb{R}^d \setminus \bigcup_{i=1}^m \mathcal{X}_i$, so that $\mathcal{X}_0, \ldots, \mathcal{X}_m$ form a partition of $\mathbb{R}^d$. Denote by $\mathcal{B}_{q,j}$ the ball with the center in $n_q(\mathcal{X}_j)$ and radius $1/2q$.

Define a hypercube $\mathcal{H} = \{P_{\bar\sigma} : \bar\sigma = (\sigma_1, \ldots, \sigma_m) \in \{-1,1\}^m\}$ of probability distributions $\mathrm{P}_{\bar\sigma}$ of the r.v. $(X(t_1), f_2(X(t_2)))$ valued in $\mathbb{R}^d \times \{0,1\}$ as follows. For any $\mathrm{P}_{\bar\sigma} \in \mathcal{H}$ the marginal distribution of $X(t_1)$ (given $X(t_0) = x_0$) does not depend on $\bar\sigma$ and has a bounded density μ w.r.t. the Lebesgue measure on $\mathbb{R}^d$ such that $\mathrm{P}_\mu(\mathcal{X}_0) = 0$ and

$$\mathrm{P}_\mu(\mathcal{X}_j) = \mathrm{P}_\mu(\mathcal{B}_{q,j}) = \int_{\mathcal{B}_{q,j}} \mu(x)\, dx = \omega, \quad j = 1, \ldots, m$$

for some $\omega > 0$. In order to ensure that the density μ remains bounded we assume that $q^d \omega = O(1)$.

The distribution of $f_2(X(t_2))$ given $X(t_1)$ is determined by the probability $\mathrm{P}_{\bar\sigma}(f_2(X(t_2)) = 1 | X(t_1) = x)$ which is equal to $C_{1,\bar\sigma}(x)$. Define

$$C_{1,\bar\sigma}(x) = f_1(x) + \sigma_j \phi(x), \quad x \in \mathcal{X}_j, \quad j = 1, \ldots, m,$$

and $C_{1,\bar\sigma}(x) = f_1(x)$ on $\mathcal{X}_0$, where $\phi(x) = \gamma_M^{-1/2} \varphi(q[x - n_q(x)])$, $\varphi(x) = A_\varphi \theta(\|x\|)$ with some constant $A_\varphi > 0$ and with $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ being a non-increasing infinitely differentiable function such that $\theta(x) \equiv 1$ on $[0, 1/2]$ and $\theta(x) \equiv 0$ on $[1, \infty)$. Without loss of generality we may assume that $f_1(x)$ is strictly positive on $[0,1]^d$, i.e. there exist two real numbers $0 < f_- < f_+ < 1$ such that $f_- \leq f_1(x) \leq f_+$. Taking A_φ small enough, we can then ensure that $0 \leq C_{1,\bar\sigma}(x) \leq 1$ on $\mathbb{R}^d$. Obviously, it holds $\phi(x) = A_\varphi \gamma_M^{-1/2}$ for $x \in \mathcal{B}_{q,j}$. As to the boundary assumption (2.3), we have

$$\begin{aligned}
&\mathrm{P}_\mu(0 < |f_1(X(t_1)) - C_{1,\bar\sigma}(X(t_1))| \leq \delta) = \\
&\qquad \sum_{j=1}^m \mathrm{P}_\mu(0 < |f_1(X(t_1)) - C_{1,\bar\sigma}(X(t_1))| \leq \delta, X(t_1) \in \mathcal{B}_{q,j}) \\
&\qquad\qquad = \sum_{j=1}^m \int_{\mathcal{B}_{q,j}} \mathbf{1}_{\{0 < \phi(x) \leq \delta\}} \mu(x)\, dx = m\omega \mathbf{1}_{\{A_\varphi \gamma_M^{-1/2} \leq \delta\}}
\end{aligned}$$

and (2.3) holds provided that $m\omega = O(\gamma_M^{-\alpha/2})$. Let $\widehat{\tau}_M$ be a stopping time measurable w.r.t. $\mathcal{F}^{\otimes M}$, then the identity (5.13) leads to

$$
\begin{aligned}
&\mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_{\bar\sigma}}[f_\tau(X(\tau))] - \mathrm{E}_{\mathrm{P}^{\otimes M}_{\bar\sigma}}[\mathrm{E}^{\mathcal{F}_{t_0}}\, f_{\widehat{\tau}_M}(X(\widehat{\tau}_M))] \\
&\qquad\qquad\qquad = \mathrm{E}_{\mathrm{P}^{\otimes M}_{\bar\sigma}}\, \mathrm{E}^{\mathcal{F}_{t_0}}_{P_\mu}\left[|\Delta_{\bar\sigma}(X(t_1))| \mathbf{1}_{\{\widehat{\tau}_{1,M}\neq\tau_1\}}\right],
\end{aligned}
$$

with $\Delta_{\bar\sigma}(X(t_1)) = f_1(X(t_1)) - C_{1,\bar\sigma}(X(t_1))$. By conditioning on $X(t_1)$, we get

$$
\begin{aligned}
&\mathrm{E}_{\mathrm{P}^{\otimes M}_{\bar\sigma}}\, \mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_\mu}\left[|\Delta_{\bar\sigma}(X(t_1))| \mathbf{1}_{\{\widehat{\tau}_{1,M}\neq\tau_1\}}\right] \\
&\qquad = \omega \sum_{j=1}^{m} \mathrm{E}_{\mathrm{P}^{\otimes M}_{\bar\sigma}}\, \mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_\mu}\left[\phi(X(t_1)) \mathbf{1}_{\{\widehat{\tau}_{1,M}\neq\tau_1\}} | X(t_1) \in \mathcal{B}_{q,j}\right] \\
&\qquad\qquad\qquad = A_\varphi m\omega\gamma_M^{-1/2}\, \mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_\mu}\, \mathrm{P}^{\otimes M}_{\bar\sigma}(\widehat{\tau}_{1,M} \neq \tau_1).
\end{aligned}
$$

Using now a well known Birgé's or Huber's lemma (see, e.g. Devroye, Györfi and Lugosi, 1996, p. 243), we get

$$
\sup_{\bar\sigma\in\{-1;+1\}^m} \mathrm{P}^{\otimes M}_{\bar\sigma}(\widehat{\tau}_{1,M} \neq \tau_1) \geq \left[0.36 \wedge \left(1 - \frac{MK_{\mathcal{H}}}{\log(|\mathcal{H}|)}\right)\right],
$$

where $K_{\mathcal{H}} := \sup_{P,Q\in\mathcal{H}} K(P,Q)$ and $K(P,Q)$ is a Kullback-Leibler distance between two measures P and Q. Since for any two measures P and Q from $\mathcal{H}$ with $Q \neq P$ it holds

$$
\begin{aligned}
K(P,Q) \;\leq\; & \sup_{\substack{\bar\sigma_1,\bar\sigma_2\in\{-1;+1\}^m \\ \bar\sigma_1\neq\bar\sigma_2}} \mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_\mu}\left[C_{1,\bar\sigma_2}(X(t_1)) \log\left\{\frac{C_{1,\bar\sigma_1}(X(t_1))}{C_{1,\bar\sigma_2}(X(t_1))}\right\}\right. \\
& \left. +(1 - C_{1,\bar\sigma_2}(X(t_1))) \log\left\{\frac{1 - C_{1,\bar\sigma_1}(X(t_1))}{1 - C_{1,\bar\sigma_2}(X(t_1))}\right\}\right] \\
\leq\; & (1 - f_+ - A_\varphi)^{-1}(f_- - A_\varphi)^{-1}\, \mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_\mu}\left[\phi^2(X(t_1)) \mathbf{1}_{\{X(t_1)\notin\mathcal{X}_0\}}\right]
\end{aligned}
$$

for small enough A_φ, and $\log(|\mathcal{H}|) = m\log(2)$, we get

$$
\begin{aligned}
&\sup_{\bar\sigma\in\{-1;+1\}^m} \left\{\mathrm{E}^{\mathcal{F}_{t_0}}_{\mathrm{P}_{\bar\sigma}}[f_{\tau,\bar\sigma}(X(\tau))] - \mathrm{E}_{\mathrm{P}^{\otimes M}_{\bar\sigma}}[\mathrm{E}^{\mathcal{F}_{t_0}}\, f_{\widehat{\tau}_M,\bar\sigma}(X(\widehat{\tau}_M))]\right\} \geq \\
&\qquad\qquad\qquad A_\varphi m\omega\gamma_M^{-1/2}(1 - AM\gamma_M^{-1}\omega) \gtrsim \gamma_M^{-(1+\alpha)/2},
\end{aligned}
$$

provided that $m\omega > B\gamma_M^{-\alpha/2}$ for some $B > 0$ and $AM\omega < \gamma_M$, where A is a positive constant depending on f_-, f_+ and A_φ. Using similar arguments, we derive

$$
\sup_{\bar\sigma\in\{-1;+1\}^m} \mathrm{P}^{\otimes M}_{\bar\sigma}(|C_{1,\bar\sigma}(x) - \widehat{C}_{1,M}(x)| > \delta\gamma_M^{-1/2}) > 0
$$

for almost x w.r.t. P_μ, some $\delta > 0$ and any estimator $\widehat{C}_{1,M}$ measurable w.r.t. $\mathcal{F}^{\otimes M}$.

5.3 Proof of Proposition 2.3

Using the arguments similar to ones in the proof of Proposition 2.1, we get

$$
\begin{aligned}
(5.14)\quad V_0(X(t_0)) - \mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}\left[V_{0,M}(X(t_0))\right] \le \\
&\delta_0 \sum_{l=0}^{L-1} \mathrm{P}_{t_l|t_0}(0 < |C_l(X(t_l)) - f_l(X(t_l))| \le \delta_0) \\
&+ \sum_{l=0}^{L-1} \mathrm{E}^{\mathcal{F}_{t_0}}\, \mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}\big[|C_l(X(t_l)) - f_l(X(t_l))| \\
&\qquad\qquad \times \mathbf{1}_{\{X(t_l)\in\mathcal{E}_l\}} \mathbf{1}_{\{|C_l(X(t_l)) - f_l(X(t_l))| > \delta_0\}}\big]
\end{aligned}
$$

with $\mathcal{E}_l$ defined as in the proof of Proposition 2.1. The first summand on the right-hand side of (5.14) is equal to zero due to (2.8). Hence, Cauchy-Schwarz and Minkowski inequalities imply

$$
\begin{aligned}
V_0(X(t_0)) - \mathrm{E}_{\mathrm{P}_{x_0}^{\otimes M}}\left[V_{0,M}(X(t_0))\right] \;&\le\; \sum_{l=0}^{L-1} \left[\mathrm{E}^{\mathcal{F}_{t_0}}\left| \mathrm{E}^{\mathcal{F}_{t_l}}\left[f_{\tau_{l+1}}(X(t_{\tau_{l+1}}))\right] - f_l(X(t_l))\right|^2\right]^{1/2} \\
&\qquad \times \left[\mathrm{E}^{\mathcal{F}_{t_0}}\, \mathrm{P}_{x_0}^{\otimes M}(|C_l(X(t_l)) - \widehat{C}_{l,M}(X(t_l))| > \delta_0)\right]^{1/2} \\
&\le\; 2B_f^{1/2} \sum_{l=0}^{L-1} \left[\mathrm{E}^{\mathcal{F}_{t_0}}\, \mathrm{P}_{x_0}^{\otimes M}(|C_l(X(t_l)) - \widehat{C}_{l,M}(X(t_l))| > \delta_0)\right]^{1/2}.
\end{aligned}
$$

Now the application of (2.9) finishes the proof.

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