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CRRA Utility Maximization under Risk Constraints

Santiago Moreno-Bromberg*
Traian A. Pirvu**
Anthony Réveillac*

* Humboldt-Universität zu Berlin, Germany
** McMaster University Hamilton, Canada

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Santiago Moreno–Bromberg
Institut für Mathematik
Humboldt-Universität zu Berlin &
Insitut für Banking und Finance
Universität Zürich
santiago.moreno@bf.uzh.ch

Traian A. Pirvu
Mathematics and Statistics Department
McMaster University
Hamilton, ON
Canada
tpirvu@math.mcmaster.ca

Anthony Réveillac
Institut für Mathematik
Humboldt-Universität zu Berlin
Berlin, Germany
areveill@math.hu-berlin.de

Abstract

This paper studies the problem of optimal investment with CRRA (constant, relative risk aversion) preferences, subject to dynamic risk constraints on trading strategies. The market model considered is continuous in time and incomplete; furthermore, financial assets are modeled by Itô processes. The dynamic risk constraints (time, state dependent) are generated by risk measures. The optimal trading strategy is characterized by a quadratic BSDE. Special risk measures (*Value–at–Risk*, *Tail Value–at–Risk* and *Limited Expected Loss*) are considered and a three–fund separation result is established in these cases. Numerical results emphasize the effect of imposing risk constraints on trading.

Preliminary - Comments Welcome

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1. Introduction

In this paper we consider the problem of a utility–maximizing agent, whose preferences are of of constant relative risk aversion (CRRA) type and whose trading strategies are subject to risk constraints. We work on a continuous–time, stochastic model with randomness being driven by Brownian noise. The market is incomplete and consists of several traded assets whose prices follow Itô processes. In practice managers set risk limits on the strategies executed by their traders. In fact,

the mechanisms used to control risk are more complex: financial institution have specialized internal departments in charge of risk assessments; on top of that there are external regulatory institutions to whom financial institutions must periodically report their risk exposure. It is natural, therefore, to study the portfolio problem with risk constraints, which has received a great deal of scrutiny lately. A very well known paper in this direction is [CK92]. The authors employ convex duality to characterize the optimal constrained portfolio. A more recent paper in the same direction is [HIM05]. Here the optimal constrained portfolio is characterized by a quadratic BSDE, which renders the method more amenable to numerical treatment. In these two (by now classical) papers the risk constraints are imposed via abstract convex (closed) sets. Lately, a line of research has been developed where the risk–constraint sets are specified employing a specific risk measure, e.g. VaR (Value at Risk). In the following we provide a brief overview of the related literature.

Existing Research: A risk measure that is commonly used by both practitioners and academics is VaR. Despite its success, VaR has as drawbacks not being subadditive and not recognizing the accumulation of risk. This encouraged researchers to develop other risk measures, e.g. TVaR (Tail Value at Risk). The works on optimal investment with risk constraints generated by VaR, TVaR (or other risk measures) split into two categories, which depend on whether or not the risk assessment is performed in a static or a dynamic fashion. Let us briefly touch on the first category. The seminal paper is [BS01], where the optimal dynamic portfolio and wealth-consumption policies of utility maximizing investors who use VaR to control their risk exposure is analyzed. In a complete–market, Itô-processes framework, VaR is computed in a static manner (the authors compute the VaR of the final wealth only). An interesting finding is that VaR limits, when applied only at maturity, may actually increase risk. One way to overcome this problem is to consider a risk measure that is based on the risk–neutral expectation of loss - the Limited Expected Loss (LEL). In [ESR01] a model with Capital–at–Risk (a version of VaR) limits, in the Black–Scholes–Samuelson framework is presented. The authors assume that portfolio proportions are held constant during the whole investment period, which makes the problem static. [DVLLLW10] extends [ESR01] from constant to deterministic parameters. In a market model with constant parameters, [GW09] extends [BS01] to cover the case of bounded expected loss. In a general, continuous–time Financial market model, [GW06] considers the portfolio problem under a downside risk constraint measured by an abstract convex risk measure. [Kup09] extends [ESR01] by imposing a uniform (in time) risk constraint.

In the category of dynamic risk measurements we recall the seminal paper [CHI08]. Following the financial industry practice, the VaR (or some other risk measure) is computed (and dynamically re–evaluated) using a time window (2 weeks in practice) over which the trading strategies are assumed to be held constant for the purpose of risk measurement. The finding of the authors is that dynamic VaR and TVaR constraints reduce the investment (proportion wise) in the risky asset. [LVT06] studies the impact of VaR constraint on equilibrium prices and the relationship with the leverage effect. [BCK05] shows that, in equilibrium, VaR reduces market volatility. [Pri10] finds that risk constraints may give rise to equilibrium asset pricing bubbles. Among others, [AP05], [Pir07], and [Yiu04] analyze the problem of investment and consumption subject to dynamic VaR constraints. [PirZit09] considers maximizing the growth rate of the portfolio in the context of dynamic VaR, TVaR and LEL constraints. In a complete market model, [Sas10] uses a martingale method to study the optimal investment under dynamic risk constraints and partial information.

Our Contribution: This paper extends the risk measurements introduced by [CHI08] by considering a relatively general class of risk measures (we only require them to be Carathéodory maps, and this class is rich enough to include many convex and coherent risk measures). The corresponding risk–constraint sets arising from such risk measures, and applied to the trading strategies, are time and state dependent. Moreover, they satisfy some important measurability properties.

We employ the method developed in [HIM05] in order to find the optimal trading strategy subject to the risk constraints. The main difference is that, unlike [HIM05], our constraint sets are time dependent, which renders the methodology developed in [HIM05] not directly applicable within our context. The difficulty stems from establishing the measurability of the BSDE's driver (the BSDE which characterizes the optimal trading strategy). This is done by means of the Measurable Maximum Theorem and the Kuratowski–Ryll–Nardzewski Selection Theorem. After this step is achieved we apply results from [BriandHu08] to get existence for the BSDE which in turn yields the optimal trading strategy.

We then restrict our risk measures to *Value–at–Risk*, *Tail Value–at–Risk* and *Limited Expected Loss*. By doing so we observe that the risk constraints have a particular structure: they are convex sets (for a fixed time and state) and depend on two statistics (portfolio return and variance). This leads to a three–fund separation result. More precisely, an investor subject to regulatory constraints will invest her wealth into three–funds: a savings account and two index funds. One index fund is a mix of the stocks with weights given by the Merton proportion. This index fund is related to market risk and most of the portfolio separation results refer to it. The second index is related to volatility risk. In a market with non–random drift and volatility the second index is absent. Thus, the second index can be explained by the demand of hedging volatility risk.

Numerical results we develop shed light into the structure of the optimal trading strategy. More precisely, using recent results concerning numerical methods for quadratic growth BSDEs, we present in Section 5 some numerical examples for the three risk measures Var, TVar and LEL. Our simulations clearly exhibit the effect of the risk constraint on the optimal strategy and on the associated value function.

The paper is organized as follows: In Section 2 we introduce the basic model, the risk measures and the corresponding risk constraints. Section 3 presents measurability properties of the candidate optimal trading strategy and its characterization via a quadratic BSDE. In Section 4 *Value–at–Risk*, *Tail Value–at–Risk* and *Limited Expected Loss* risk measures are considered and a three-fund separation result is obtained within this context. Numerical results are presented in Section 5. The paper ends with an appendix that contains some technical results.

2. Model Description and Problem Formulation

2.1. **The Financial Market.** Our model of a financial market, based on a a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ that satisfies the usual conditions, consists of $n+1$ assets. The first one, $\{S_0(t)\}_{t\in[0,T]}$, is a *riskless bond* with a strictly positive, constant interest rate $r > 0$. The remaining n assets are *stocks*, and they are modeled by an n–dimensional Itô–process $\{\boldsymbol{S}(t)\}_{t\in[0,T]} = \{(S_i(t))_{i=1,\dots,n}\}_{t\in[0,T]}$. Their dynamics are given by the following stochastic differential equations, in which $\{\boldsymbol{W}(t)\}_{t\in[0,T]} = \{(W_i(t))_{i=1,\dots,m}\}_{t\in[0,T]}$ is a m–dimensional standard Brownian motion:

$$\left.\begin{aligned} dS_0(t) &= S_0(t) r\, dt \\ dS_i(t) &= S_i(t)\Big(\alpha_i(t)\, dt + \sum_{j=1}^{m} \sigma_{ij}(t)\, dW_j(t)\Big),\ i = 1,\dots,n, \end{aligned}\right\} , t \in [0,T], \tag{2.1}$$

where the $\mathbb{R}^n$–valued process $\{\boldsymbol{\alpha}(t)\}_{t\in[0,T]} = \{(\alpha_i(t))_{i=1,\dots,n}\}_{t\in[0,T]}$ is the *mean rate of return*, and $\{\boldsymbol{\sigma}(t)\}_{t\in[0,T]} = \{(\sigma_{ij}(t))_{i=1,\dots,n}^{j=1,\dots,m}\}_{t\in[0,T]} \in \mathbb{R}^{n\times m}$ is the *variance–covariance* process. In order for the equations (2.1) to admit unique strong solutions, we impose the following regularity conditions on the coefficient processes $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\sigma}(t)$:

Assumption 2.1. *All the components of the processes $\{\boldsymbol{\alpha}(t)\}_{t\in[0,T]}$ and $\{\boldsymbol{\sigma}(t)\}_{t\in[0,T]}$ are predictable, and*

$$\sum_{i=1}^{n} \int_0^t |\alpha_i(u)|\, du + \sum_{i=1}^{n}\sum_{j=1}^{m} \int_0^t \sigma_{ij}(u)^2\, du < \infty, \ \textit{for all } t \in [0,\infty),\ \textit{a.s.}$$

To ease the exposition, we introduce the following notation: for an integrable $\mathbb{R}^m$-valued process $\boldsymbol{\gamma}(t) = (\gamma_i(t))_{i=1,\dots,n}$, and a sufficiently regular $\mathbb{R}^m$–valued process $\boldsymbol{\pi}(t) = (\pi_j(t))_{j=1,\dots,m}$ we write

$$\int_0^t \boldsymbol{\gamma}(u)\, du \triangleq \sum_{i=1}^{n} \int_0^t \gamma_i(u)\, dt, \quad \int_0^t \boldsymbol{\pi}(t)\, d\boldsymbol{W}(t) \triangleq \sum_{j=1}^{m} \int_0^t \pi_j(t)\, dW_j(t).$$

Further, we impose the following condition on the variance–covariance process $\boldsymbol{\sigma}(t)$:

Assumption 2.2. *The matrix $\boldsymbol{\sigma}(t)$ has independent rows for all $t \in [0,\infty)$ almost–surely.*

This assumption makes it impossible for different stocks to have the same diffusion structure. Otherwise, the market would either allow for arbitrage opportunities or redundant assets would exist. As a consequence of Assumption 2.2 we have that $n \le m$ - the number of risky assets does not exceed the number of “sources of uncertainty”. Also, the inverse $(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)')^{-1}$ is easily seen to exist, thus the equation

$$\boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)'\boldsymbol{\zeta}_M(t) = \boldsymbol{\mu}(t),$$

uniquely defines a predictable stochastic process $\{\boldsymbol{\zeta}_M(t)\}_{t\in[0,T]}$, named the *Merton–proportion process*, where $\{\boldsymbol{\mu}(t)\}_{t\in[0,T]} = \{(\mu_i(t))_{i=1,\dots,n}\}_{t\in[0,T]}$, with $\mu_i(t) = \alpha_i(t) - r$ for $i = 1,\dots,n$. At this point we make another assumption on the market coefficients:

Assumption 2.3. *We assume that*

$$\mathbb{E}\left[\exp\Big(\int_0^T ||\boldsymbol{\zeta}_M(t)\boldsymbol{\sigma}(u)||^2\,du\Big)\right] < \infty,$$

and the stochastic process $\boldsymbol{\sigma}'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}$ is uniformly bounded.

2.2. **Trading strategies and wealth.** Let $\mathcal{P}$ denote the predictable σ–algebra on $[0,T]\times\Omega$. The control variables are the proportions of current wealth the investor invests in the assets. More precisely, we have the following formal definition:

Definition 2.4. *An $\mathbb{R}^n$–valued stochastic process $\{\boldsymbol{\zeta}(t)\}_{t\in[0,T]} = \{(\zeta_i(t))_{i=1,\dots,n}\}_{t\in[0,T]}$ is called an* admissible portfolio–proportion process *if it is predictable (i.e. $\mathcal{P}$-measurable) and it satisfies*

$$\mathbb{E}\Big[\int_0^t \big|\boldsymbol{\zeta}'(u)(\boldsymbol{\alpha}(u) - r\mathbf{1})\big|\,du + \int_0^t ||\boldsymbol{\zeta}'(t)\boldsymbol{\sigma}(u)||^2\,du\Big] < \infty, \quad \textit{for all} \quad t\in[0,\infty). \tag{2.2}$$

Here $\boldsymbol{\zeta}'(t)$ denotes the transpose of $\boldsymbol{\zeta}(t)$, $\mathbf{1} = (1,\dots,1)'$ is a n–dimensional column vector all of whose coordinates are equal to 1, and $||\boldsymbol{x}||$ is the standard Euclidean norm. The set of admissible strategies will be denoted by $\mathcal{A}$.

Given a portfolio–proportion process $\boldsymbol{\zeta}(t)$, we interpret its n coordinates as the proportions of the current wealth $X^{\boldsymbol{\zeta}}(t)$ invested in each of the n stocks. In order for the portfolio to be self–financing, the remaining wealth $X^{\boldsymbol{\zeta}}(t)(1-\sum_{i=1}^n \zeta_i(t))$ is assumed to be invested in the riskless bond $S_0(t)$. If this quantity is negative, we are effectively borrowing at the rate $r > 0$. No short–selling restrictions are imposed, hence the proportions $\boldsymbol{\zeta}_i(t)$ are allowed to be negative, and they are not a priori bounded. The equation governing the evolution of the total wealth $\{X^{\boldsymbol{\zeta}}(t)\}_{t\in[0,T]}$ of the investor using the portfolio–proportion process $\{\boldsymbol{\zeta}(t)\}_{t\in[0,T]}$ is given by

$$\begin{aligned} dX^{\boldsymbol{\zeta}}(t) &= X^{\boldsymbol{\zeta}}(t)\Big(\boldsymbol{\zeta}'(t)\boldsymbol{\alpha}(t)\,dt + \boldsymbol{\zeta}'(t)\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\Big) + \Big(1-\boldsymbol{\zeta}'(t)\mathbf{1}\Big)X^{\boldsymbol{\zeta}}(t)r\,dt \\ &= X^{\boldsymbol{\zeta}}(t)\Big((r+\boldsymbol{\zeta}'(t)\boldsymbol{\mu}(t))\,dt + \boldsymbol{\zeta}'(t)\boldsymbol{\sigma}(t)\,d\boldsymbol{W}(t)\Big), \end{aligned} \tag{2.3}$$

where we recall that $\{\boldsymbol{\mu}(t)\}_{t\in[0,T]} = \{(\mu_i(t))_{i=1,\dots,n}\}_{t\in[0,T]}$, with $\mu_i(t) = \alpha_i(t) - r$ for $i = 1,\dots,n$, is the vector of *excess rates of return*. Under the regularity conditions (2.2) imposed on $\boldsymbol{\zeta}(t)$, Equation (2.3) admits a unique strong solution given by

$$X^{\boldsymbol{\zeta}}(t) = X(0)\exp\Big\{\int_0^t \Big(r+\boldsymbol{\zeta}'(u)\boldsymbol{\mu}(u) - \tfrac{1}{2}||\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)||^2\Big)\,du + \int_0^t \boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)\,d\boldsymbol{W}(u)\Big\}. \tag{2.4}$$

The initial wealth $X^{\boldsymbol{\zeta}}(0) = X(0)\in(0,\infty)$ is considered to be exogenously given. As a consequence of Assumption 2.3, and using (2.2), a strategy $\boldsymbol{\zeta}$ is admissible if and only if it is a predictable process such that

$$\mathbb{E}\Big[\int_0^T ||\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)||^2\,du\Big] < \infty. \tag{2.5}$$

Indeed we have

$$\boldsymbol{\zeta}'(u)\boldsymbol{\mu}(u) = (\boldsymbol{\sigma}^T(u)\boldsymbol{\zeta}(u))^T(\boldsymbol{\sigma}^T(u)\boldsymbol{\zeta}_M(u)) \leq ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(u)||\,||\boldsymbol{\zeta}_M^T(u)\boldsymbol{\sigma}(u)||,$$

by the Cauchy–Buniakowski–Schwarz inequality. Thus, inequality (2.5) follows from Assumption 2.3, Expression (2.2) and the Cauchy–Buniakowski–Schwarz inequality.

The expression appearing inside the first integral in (2.4) above will be given its own notation; the quadratic function $\tilde{Q} : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\tilde{Q}(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \triangleq r + \zeta_{\boldsymbol{\mu}} - \tfrac{1}{2}\zeta_{\boldsymbol{\sigma}}^2,$$

Another useful notation is for the the random field $Q : \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ by

$$Q(t, \boldsymbol{\zeta}) = \tilde{Q}(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||).$$

It is clear from Expression (2.4) that the evolution of wealth process $X^{\boldsymbol{\zeta}}(t)$ depends on the $\mathbb{R}^n$-dimensional process $\boldsymbol{\zeta}(t)$ only through two "sufficient statistics", namely

$$\zeta_{\boldsymbol{\mu}}(t) \triangleq \boldsymbol{\zeta}'(t)\boldsymbol{\mu}(t), \text{ and } \zeta_{\boldsymbol{\sigma}}(t) \triangleq ||\boldsymbol{\zeta}'(t)\boldsymbol{\sigma}(t)||. \tag{2.6}$$

These will be referred to in the sequel as *portfolio rate of return* and *portfolio volatility*, respectively.

2.3. **Projected distribution of wealth.** Let us recall that for the purposes of risk measurement, it is common practice to use an approximation of the distribution of the investor's wealth at a future date. Given the current time $t \geq 0$, and a length $\tau > 0$ of the measurement horizon $[t, t + \tau)$, the *projected distribution* of the wealth from trading will be calculated under the simplifying assumptions that

(1) the proportions of the wealth $\{\boldsymbol{\zeta}(s)\}_{s\in[t,t+\tau)}$ invested in various securities, as well as
(2) the market coefficients $\{\boldsymbol{\alpha}(s)\}_{s\in[t,t+\tau)}$ and $\{\boldsymbol{\sigma}(s)\}_{s\in[t,t+\tau)}$

will stay constant and equal to their present values throughout the time interval $[t, t+\tau)$. The wealth equations (2.3) and (2.4) yield that the *projected wealth loss* is - conditionally on $\mathcal{F}_t$ - distributed as $L = L(X(t), \zeta_{\boldsymbol{\mu}}(t), \zeta_{\boldsymbol{\sigma}}(t))$, where the law of $L(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ is the one of

$$x\Big(1 - \exp(Y(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}))\Big), \tag{2.7}$$

in which $Y(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ is a normal random variable with mean $\tilde{Q}(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})\tau$ and standard deviation $\sqrt{\tau}\zeta_{\boldsymbol{\sigma}}$. The quantities $\zeta_{\boldsymbol{\mu}}(t)$ and $\zeta_{\boldsymbol{\sigma}}(t)$ are the portfolio rate of return and volatility, defined in Equation (2.6). In the upcoming sections we turn our focus to risk measurements associated to the *relative projected wealth gain*, which will be defined as the distribution of the quantity

$$\frac{X^{\boldsymbol{\zeta}}(t+\tau-) - X^{\boldsymbol{\zeta}}(t)}{X^{\boldsymbol{\zeta}}(t)}.$$

This is not a technical requirement, and the method developed in Sections 2.4 to 3.1 still holds for risk measurements in absolute terms. The economic implications, however, may be stark, and the definition of the risk constraints below would require a certain recursive structure. The latter in the sense that admissibility (risk–wise) at time t will depend on the choice of the strategy at all previous times. We elaborate further on this in Remark 2.6. The measurement horizon τ and the market coefficients will play the role of "global variables".

2.4. **The risk constraints.** In this section we introduce the risk constraints that will be imposed on the trading strategies. We keep the presentation as general as possible, and so we make several sufficient assumptions on the risk measures. These allow us to show existence (and in some cases uniqueness) of optimal, constrained trading strategies. We begin by making precise how the risk of a given strategy is measured.

Let us define the gain over time interval $[t, t+\tau]$ by $\Delta_\tau X_t^{\boldsymbol{\zeta}} \triangleq X_{t+\tau-}^{\boldsymbol{\zeta}} - X_t^{\boldsymbol{\zeta}}$, and let $(\rho_t)_{t\in[0,T]}$ be a family of maps ρ_t with

$$\rho_t : \mathcal{C}_t \subset L^2(\mathcal{F}_T, P) \to L^2(\mathcal{F}_t, P),$$

where

$$\mathcal{C}_t \triangleq \left\{ \Delta_\tau X_t^{\boldsymbol{\zeta}} / X_t^{\boldsymbol{\zeta}} \,\middle|\, \boldsymbol{\zeta} \ \text{ is an admissible strategy} \right\}.$$

Notice that for all $t \in (0, T]$, we have that $\mathcal{C}_t \subset L^2(\mathcal{F}_T, P)$. We also define $\mathcal{C}_0 \triangleq L^2(\mathcal{F}_T, P)$. For a given admissible $(\tilde{\boldsymbol{\zeta}}(s))_{s\in[0,t)}$ and $\boldsymbol{\zeta} \in \mathbb{R}^n$ we define the strategy $\overline{\boldsymbol{\zeta}} : \Omega \times [0, t+\tau) \to \mathbb{R}^n$ as $\overline{\boldsymbol{\zeta}}(s) = \tilde{\boldsymbol{\zeta}}(s)$ for $s < t$ and $\overline{\boldsymbol{\zeta}}(s) = \boldsymbol{\zeta}$ for $t \le s < t+\tau$. By definition of the wealth process we obtain that $X_t^{\overline{\boldsymbol{\zeta}}} = X_{t-}^{\tilde{\boldsymbol{\zeta}}}$, moreover (under the assumptions made in Section 2.3) the quantity $\Delta_\tau X_t^{\overline{\boldsymbol{\zeta}}} / X_t^{\overline{\boldsymbol{\zeta}}}$ depends exclusively on $\boldsymbol{\zeta}$, and not on $\tilde{\boldsymbol{\zeta}}$. In order to establish the risk constraints, we define the acceptance sets

$$\mathcal{A}_t^{\rho,\tilde{\boldsymbol{\zeta}}}(\omega) \triangleq \left\{ \boldsymbol{\zeta} \in \mathbb{R}^n \,\middle|\, \rho_t\Big(\frac{\Delta_\tau X_t^{\overline{\boldsymbol{\zeta}}}}{X_t^{\overline{\boldsymbol{\zeta}}}(\omega)}\Big)(\omega) \le K_t(\omega) \right\} \quad t \in [0, T], \tag{2.8}$$

where K_t is a real–valued, exogenous, predictable process that satisfies $K_t \ge \rho_t(0)$ for all t in $[0, T]$, P–a.s.. Notice that $\boldsymbol{\zeta} = 0$ is in the constraint set. We observe that by construction, the sets $\mathcal{A}_t^{\rho,\tilde{\boldsymbol{\zeta}}}$ are independent of $\tilde{\boldsymbol{\zeta}}$, and we shall simply write $\mathcal{A}_t^{\rho}$. In analogous fashion we will slightly abuse notation and write $\Delta_\tau X_t^{\boldsymbol{\zeta}} / X_t^{\boldsymbol{\zeta}}$ for $\Delta_\tau X_t^{\overline{\boldsymbol{\zeta}}} / X_t^{\overline{\boldsymbol{\zeta}}}$. It follows from Equation (2.3) that in fact

$$\frac{\Delta_\tau X_t^{\boldsymbol{\zeta}}}{X_t^{\boldsymbol{\zeta}}} = \mathfrak{E}(\boldsymbol{\zeta}, t) - 1,$$

where

$$\mathfrak{E}(\boldsymbol{\zeta}, t) \triangleq \exp\left\{ \int_t^{t+\tau} \big(r + \zeta_{\boldsymbol{\mu}}(u) - \frac{1}{2}\zeta_{\boldsymbol{\sigma}}(u)^2\big) du + \int_t^{t+\tau} \boldsymbol{\zeta}(u)\boldsymbol{\sigma}(u) dW(u) \right\}.$$

Hence, the expressions for the sets of constraints $\mathcal{A}_t^{\rho}$ may be rewritten as

$$\mathcal{A}_t^{\rho}(\omega) = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^n \,\middle|\, \rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega) \le K_t(\omega) \right\}.$$

Moreover, under the assumption that $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\zeta}$ remain (for the purpose of risk assessment) constant over $[t, t+\tau)$, we may write

$$\mathfrak{E}(\boldsymbol{\zeta}, t) \triangleq \exp\Big\{ r\tau \Big\} \cdot \exp\Big\{ \tau\big(\zeta_{\boldsymbol{\mu}} - \frac{1}{2}\zeta_{\boldsymbol{\sigma}}^2\big) \Big\} \cdot \exp\Big\{ \boldsymbol{\zeta}\boldsymbol{\sigma}\Delta_\tau W_t \Big\},$$

and we shall denote by $\mathfrak{E}_1(\boldsymbol{\zeta}, t)$ and $\mathfrak{E}_2(\boldsymbol{\zeta}, t)$ the second and third factors of $\mathfrak{E}(\boldsymbol{\zeta}, t)$, respectively. We make the following assumption on the family $(\rho_t)_{t\in[0,T]}$:

Assumption 2.5. *The family of maps*

$$\rho_t : \mathcal{C}_t \subset L^2(\mathcal{F}_T, P) \to L^2(\mathcal{F}_t, P)$$

satisfies that the mapping

$$(\boldsymbol{\zeta}, (\omega, t)) \mapsto \rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega)$$

is a Carathéodory function; that is, for every (ω, t) in $\Omega \times [0, T]$, the map $\boldsymbol{\zeta} \mapsto \rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega)$ is continuous and for every $\boldsymbol{\zeta}$ in R^n the map $(\omega, t) \mapsto \rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega)$ is $\mathcal{P}$–measurable.

Before continuing our analysis, we give two simple examples of families $(\rho_t)_{t\in[0,T]}$ which satisfy Assumption 2.5.

Ex. 1: Let ρ_0 be a coherent, continuous risk measure on $L^2(\mathcal{F}_T, P)$, and for every admissible $\boldsymbol{\zeta}$ let

$$\rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega) \triangleq \exp\Big\{r\tau\Big\}\mathfrak{E}_1(\boldsymbol{\zeta}, t)(\omega)\rho_0\big(\mathfrak{E}_2(\boldsymbol{\zeta}, t)(\omega) - 1\big)$$

where

$$\mathfrak{E}_1(\boldsymbol{\zeta}, t)(\omega) \triangleq \exp\Big\{\tau\big(\boldsymbol{\zeta}x - \frac{1}{2}||\boldsymbol{\zeta}y||^2\big)\Big\}\Big|_{x=\boldsymbol{\mu}(\omega,t), y=\boldsymbol{\sigma}(\omega,t)}$$

and

$$\rho_0\big(\mathfrak{E}_2(\boldsymbol{\zeta}, t)(\omega) - 1\big) \triangleq \rho_0\Big(\exp\big\{xy\Delta_\tau W_0\big\} - 1\Big)\Big|_{x=\boldsymbol{\zeta}(\omega,t), y=\boldsymbol{\sigma}(\omega,t)}.$$

Obviously $(\rho_t)_{t\in[0,T]}$ satisfies Assumption 2.5.

Ex. 2: (Shortfall risk measures) Let $l : \mathbb{R} \to \mathbb{R}$ be a convex, non–decreasing continuous and non–constant function[2] with $|l(-\infty)| < +\infty$. Assume that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is generated by the Brownian motion $\{\boldsymbol{W}(t)\}_{t\in[0,T]}$ and that $\sigma_{i,j}(t) := \sigma_{i,j}(t, W_t)$ and $\mu(t) := \mu(t, W_t)$ where $\sigma_{i,j}$ and μ are deterministic Borelian functions. We set

$$\rho_t(-\mathfrak{E}(\zeta, t) - 1) \triangleq \mathbb{E}[l(\exp(r(\tau + \zeta x\frac{1}{2}\|\zeta y\|^2) + xy\Delta_\tau W_0))]_{x=\mu(t,W_t), y=\sigma(t,W_t)}$$

so that $\rho_t(-\mathfrak{E}(\zeta, t)-1) = \mathbb{E}[l(-\mathfrak{E}(\zeta, t)-1)|\mathcal{F}_t]$, $\mathbb{P}-a.s.$. Then the family $(\rho_t)_{t\in[0,T]}$ satisfies Assumption 2.5. Indeed, fix $\boldsymbol{a}$ in $\mathbb{R}^n$ and let $\boldsymbol{\zeta}$ in $\mathbb{R}^n$. Then, by monotonicity of the exponential and l we have that:

$$l(-\infty) \le l(-\mathfrak{E}(\boldsymbol{\zeta}, t) + 1) \le l(1).$$

Hence Lebesgue'e dominated convergence Theorem implies that:

$$\lim_{\boldsymbol{\zeta}\to\boldsymbol{a}} \rho_t(-\mathfrak{E}(\zeta, t) - 1) = \rho_t(-\mathfrak{E}(\boldsymbol{a}, t) - 1), \quad \forall t \in [0, T].$$

Finally, since the filtration we consider is the Brownian filtration, the stochastic process $(\rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1))_{t\in[0,T]}$ is predictable.

Remark 2.6. If we were to consider risk constraints based not on the relative projected wealth loss, but only on the quantities $\Delta_\tau X_t^{\boldsymbol{\zeta}}$, then the acceptance sets defined in Expression 2.8 would depend on $(\boldsymbol{\zeta}(s))_{s\in[0,t)}$. More precisely, the set of risk–admissible strategies would be

$$\mathcal{A} \triangleq \Big\{\boldsymbol{\zeta} = (\boldsymbol{\zeta}(s))_{s\in[0,T]} \,\big|\, \boldsymbol{\zeta} \text{ is admissible and } \boldsymbol{\zeta}(t) \in \mathcal{A}_t^{\rho, \boldsymbol{\zeta}\mathbf{1}_{[0,t)}}\Big\}$$

[2]Such functions are usually referred to as "loss functionals".

In the case where ρ_t is a $\mathcal{F}_{t-}$–coherent family, i.e. if $\rho_r(XY) = X\,\rho_t(Y)$ for all $X \in \mathcal{F}_{t-}$, then risk constraints in absolute terms are generated by inequalities of the form

$$X_t^{\bar{\boldsymbol{\zeta}}} \rho_t\Big(\frac{\Delta_\tau X_t^{\bar{\boldsymbol{\zeta}}}}{X_t^{\bar{\boldsymbol{\zeta}}}}\Big) \leq K_t.$$

This follows from the fact that the wealth level at time t is a $\mathcal{F}_{t-}$–measurable random variable. The structure then reverts to that of risk constraints in relative terms, except for a redefinition of the risk bound as $\tilde{K}_t(\omega) \triangleq K_t(\omega)/X_t(\omega)$. Notice that if $K_t \equiv K \in \mathbb{R}_+$, then $\tilde{K}_t$ would be a decreasing function of wealth. In other words, highly capitalized investors would face more stringent constraints. This could lend an approach to dealing with the too–big–to–fail problem, and could be further tweaked by allowing K_t to depend on the state of nature. It is, however, beyond the scope of this paper to discuss such policy–making issues, and we shall stick to the relative–measures–of–risk framework.

Remark 2.7. Note that $(\rho_t)_{t\in[0,T]}$ is not *stricto sensu* a dynamic risk measure, since every ρ_t is a priori not defined on the whole space $L^2(\mathcal{F}_T, P)$. As we we have seen in the previous lines, defining the risk of every random variable in $L^2(\mathcal{F}_T, P)$ is not relevant for us, since we only need to evaluate the risk of the very specific random variables $\Delta_\tau X_t^{\boldsymbol{\zeta}}$.

2.5. **The optimization problem.** We finish the section by formulating our central problem. Given a choice of a dynamic risk measure ρ satisfying Assumption 2.5 and a final date T, we are searching for a portfolio–proportion process $\boldsymbol{\zeta}^*(t) \in \mathcal{A}_t^\rho$ which maximizes the p−CRRA utility $U_p(x) = \frac{x^p}{p}, p > 0$, of the final wealth among all the portfolios satisfying the same constraint. In other words, for all $t \in [0,\infty)$ and $\boldsymbol{\zeta}(t) \in \mathcal{A}_t^\rho = \Big\{\boldsymbol{\zeta} \in \mathbb{R}^n \,\Big|\, \rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) \leq K_t\Big\}$

$$\mathbb{E}\big[U_p(X^{\boldsymbol{\zeta}^*}(T))\big] \geq \mathbb{E}\big[U_p(X^{\boldsymbol{\zeta}}(T))\big]. \tag{2.9}$$

3. Analysis

3.1. **The optimal policy.** In this section we prove the existence of an optimal investment strategy. In order to do so, we make use of the powerful theory of backward stochastic differential equations (BSDEs). Let

$$\mathcal{A}^\rho \triangleq \big\{\boldsymbol{\zeta} = (\boldsymbol{\zeta}(t))_{t\in[0,T]} \in \mathcal{A} \,\big|\, \boldsymbol{\zeta}(t) \in \mathcal{A}_t^\rho,\ \forall t \in [0,T]\big\}$$

where $\mathcal{A}$ is the set of admissible strategies in the sense of Definition 2.4. We recall that we consider the maximization problem

$$\max_{\boldsymbol{\zeta}\in\mathcal{A}^\rho} \mathbb{E}(U_p(X^{\boldsymbol{\zeta}}(T))).$$

By means of (2.4) we may write

$$U_p(X^{\boldsymbol{\zeta}}(t)) = U_p(X(0)) \exp\left(\int_0^t p\Big(r + \zeta_{\boldsymbol{\mu}}(u) - \tfrac{1}{2}\zeta_{\boldsymbol{\sigma}}(u)^2\Big)\,du + \int_0^t p\,\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)\,d\boldsymbol{W}(u)\right).$$

In analogous fashion as done in [HIM05], let us introduce the auxiliary process

$$R^{\boldsymbol{\zeta}}(t) \triangleq U_p(X(0)) \exp\left(Y(t) + \int_0^t p\Big(r + \zeta_{\boldsymbol{\mu}}(u) - \tfrac{1}{2}\zeta_{\boldsymbol{\sigma}}(u)^2\Big)\,du + \int_0^t p\,\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)\,d\boldsymbol{W}(u)\right),$$

where (Y, Z) is a solution to the BSDE

$$Y(t) = 0 - \int_t^T Z(u) d\boldsymbol{W}(u) - \int_t^T h(u, Z(u)) du, \qquad t \in [0, T]. \tag{3.1}$$

The function $h(t, z)$ should be chosen in such a way that

a) the process $R^{\boldsymbol{\zeta}}$ is a supermartingale, $R^{\boldsymbol{\zeta}}(T) = U_p(X^{\boldsymbol{\zeta}}(T))$ and $R^{\boldsymbol{\zeta}}(0) = \frac{(X(0))^p}{p}$ for every $\boldsymbol{\zeta} \in \mathcal{A}^\rho$,

b) there exists at least one element $\boldsymbol{\zeta}^*$ in $\mathcal{A}^\rho$ such that $R^{\boldsymbol{\zeta}^*}$ is a martingale.

We shall verify ex–post that the function $h(t, z)$ in question satisfies the measurability and growth conditions required to guarantee existence of solutions to Equation (3.1). Before going further we explain why achieving this would provide a solution to the optimization problem (2.9). Assume we were able to construct such a family of processes $R^{\boldsymbol{\zeta}}$, then we would obtain that $\boldsymbol{\zeta}^*$ is an optimal strategy for the utility maximization problem (2.9) with initial capital $X(0) > 0$ independent of $\boldsymbol{\zeta}$. Indeed let $\boldsymbol{\zeta}$ any element of $\mathcal{A}^\rho$, then using (a) and (b) we have

$$\mathbb{E}(U_p(X^{\boldsymbol{\zeta}}(T)) = \mathbb{E}(R^{\boldsymbol{\zeta}}(T)) \leq R^{\boldsymbol{\zeta}}(0) = \frac{(X(0))^p}{p} = \mathbb{E}(R^{\boldsymbol{\zeta}^*}(T)).$$

This method is known as the *martingale optimality principle*. Let us now perform a multiplicative decomposition of $R^{\boldsymbol{\zeta}}$ into martingale and an increasing process. To this end, given a continuous process M, we denote by $\mathcal{E}(M)$ its stochastic exponential:

$$\mathcal{E}(M(t)) \triangleq \exp\left(M(t) - \frac{1}{2}\langle M \rangle_t\right),$$

where $\langle M \rangle$ denotes the quadratic variation. Then

$$R^{\boldsymbol{\zeta}}(t) = \frac{(X(0))^p}{p} \mathcal{E}\left(\int_0^t (p\,\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u) + Z(u))\, d\boldsymbol{W}(u)\right) \exp\left(\int_0^t g(u, Z(u))\, du\right),$$

where

$$g(u, z) \triangleq h(u, z) + \frac{1}{2}||z||^2 + pr + p\boldsymbol{\zeta}'(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z) + \frac{p^2 - p}{2}||\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)||^2.$$

Since $R^{\boldsymbol{\zeta}'}$ should be a supermartingale for every admissible $\boldsymbol{\zeta}(u)$ (and a martingale for some element $\boldsymbol{\zeta}^*(u)$), then g has to be a non–positive process. With this in mind, a suitable candidate would be

$$h(u, z) \triangleq -pr - \frac{1}{2}||z||^2 + \inf_{\boldsymbol{\zeta}(u) \in \mathcal{A}(u)} \left\{ -p\boldsymbol{\zeta}'(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z) - \frac{p - p^2}{2}||\boldsymbol{\zeta}'(u)\boldsymbol{\sigma}(u)||^2 \right\},$$

which leads to

$$\begin{aligned} h(u, z) &= -pr - \frac{1}{2}||z||^2 + \frac{p}{2(p-1)}||\boldsymbol{\sigma}'(u)(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z)||^2 \\ &+ \frac{p(1-p)}{2}\text{dist}\left(\frac{\boldsymbol{\sigma}'(u)(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z)}{1-p}; \mathcal{A}_u^\rho \boldsymbol{\sigma}(u)\right)^2. \end{aligned} \tag{3.2}$$

If in addition we let

$$\tilde{z} \triangleq \frac{\boldsymbol{\sigma'}(u)(\boldsymbol{\sigma\sigma'})^{-1}(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z)}{1-p} \quad \text{and} \quad \tilde{\mathcal{A}}_u^{\rho} \triangleq \mathcal{A}_u^{\rho}\boldsymbol{\sigma}(u), \tag{3.3}$$

then

$$\text{dist}\left(\frac{\boldsymbol{\sigma'}(u)(\boldsymbol{\sigma\sigma'})^{-1}(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z)}{1-p}; \mathcal{A}_u^{\rho}\boldsymbol{\sigma}(u)\right)^2 = \|\frac{\boldsymbol{\sigma'}(u)(\boldsymbol{\sigma\sigma'})^{-1}(u)(\boldsymbol{\mu}(u) + p\boldsymbol{\sigma}(u)z)}{1-p} - \boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u)\|^2$$

with

$$\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u) \in \text{Proj}(\tilde{\text{Z}}(\text{u}), \tilde{\mathcal{A}}_{\text{u}}^{\rho}). \tag{3.4}$$

The available results on existence of solutions to BSDEs require, to begin with, the predictability of the driver h. In our case this is closely related to the predictability of $\boldsymbol{\zeta}^*$, in other words, to whether or not the candidate for an optimal strategy is acceptable.

Theorem 3.1. *Let Z be a predictable process such that*

$$\mathbb{E}\left(\int_0^T ||Z(u)||^2\, du\right)^{\frac{1}{2}} < \infty,$$

then for $(t,\omega) \in [0,T] \times \Omega$, the mapping

$$(t,\omega) \mapsto dist(\tilde{Z}_t(\omega), \tilde{\mathcal{A}}_t^{\rho}(\omega)),$$

where $\tilde{Z}$ is as in Equation (3.3), is predictable. In addition there exists a predictable process $\boldsymbol{\zeta}^*$ in $\mathbb{R}^n$ such that

$$\mathbb{E}\left(\int_0^T ||\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u)||^2\, du\right)^{\frac{1}{2}} < \infty$$

and

$$dist\big(\tilde{Z}_t, \tilde{\mathcal{A}}_t^{\rho}\big) = dist(\tilde{Z}_t, \boldsymbol{\zeta}^{*T}(t)\boldsymbol{\sigma}(t)), \quad \forall t \in [0,T],\ P-a.s..$$

Proof. Let us define for $k \in \mathbb{N}$

$$\mathcal{A}_{t,k}^{\rho}(\omega) \triangleq \left\{\boldsymbol{\zeta} \in [-k,k]^n \,\middle|\, \rho_t(\mathfrak{E}(\zeta,t))(\omega) - K_t(\omega) \leq 0\right\}.$$

The purpose of artificially bounding the values of $\mathcal{A}^{\rho}$ is to make use of the theory of compact–valued correspondences (see Appendix A). It follows from Lemma A.1 that for all $k \in \mathbb{N}$ and for all (t,ω), the set $\mathcal{A}_{t,k}^{\rho}(\omega)$ is non–empty and compact. Moreover, Proposition A.3 guarantees that for all $t \in [0,T]$ and $k \in \mathbb{N}$, the correspondence $(\omega,t) \mapsto \tilde{\mathcal{A}}_{t,k}^{\rho}(\omega)$ is weakly $\mathcal{P}$–measurable (see Definition *A*.2 in the Appendix for the definition of weakly measurability). Let $(C(\mathbb{R}^m), \mathcal{H})$ denote the space of non–empty, compact subsets of $\mathbb{R}^m$, equipped with the Hausdorff metric. This is a complete, separable metric space, in which $\tilde{\mathcal{A}}_{t,k}^{\rho}(\cdot)$ takes its values. Theorem A.4 then states that for $z \in \mathbb{R}^m$ and $t \in [0,T]$, the distance mapping

$$\delta(\omega,z) = \text{dist}\big(z,\, \mathcal{A}_{t,k}^{\rho}(\omega)\boldsymbol{\sigma}(t)\big)$$

is a Carathéodory one. Since the process $\tilde{Z}_t$ is predictable and $z \mapsto \delta(z, \omega)$ is continuous for all $\omega \in \Omega$, the map

$$(\omega, t) \mapsto \operatorname{dist}\big(\tilde{Z}_t(\omega),\, \mathcal{A}^{\rho}_{t,k}(\omega)\boldsymbol{\sigma}(t)\big)$$

is $\mathcal{P}$–measurable. Finally

$$\operatorname{dist}\big(\tilde{Z}_t(\omega), \tilde{\mathcal{A}}^{\rho}_t(\omega)\big) = \inf_{k\in\mathbb{N}} \big\{\operatorname{dist}\big(\tilde{Z}_t(\omega),\, \mathcal{A}^{\rho}_{t,k}(\omega)\boldsymbol{\sigma}(t)\big)\big\},$$

thus the mapping $\omega \mapsto \operatorname{dist}(\tilde{Z}_t(\omega), \tilde{\mathcal{A}}^{\rho}_t(\omega))$ is predictable as the pointwise infimum of predictable ones. We now turn our attention to the second claim. First we observe that since $\tilde{\mathcal{A}}^{\rho}_t(\omega)$ is closed (and contained in $\mathbb{R}^m$), the set

$$\overline{\mathcal{A}}^{\rho}_t(\omega) \triangleq \operatorname{argmin}_{a\in\tilde{\mathcal{A}}^{\rho}_t(\omega)} \Big\{\operatorname{dist}(\tilde{Z}_t(\omega), a)\Big\}$$

is compact. It follows from the Measurable Maximum Theorem ([AB06], page 605) that the correspondence $(t, \omega) \twoheadrightarrow \overline{\mathcal{A}}^{\rho}_t(\omega)$ is weakly $\mathcal{P}$–measurable. It is then implied by the Kuratowski–Ryll–Nardzewski Selection Theorem that $\overline{\mathcal{A}}^{\rho}_{\cdot}(\cdot)$ admits a measurable selection $\boldsymbol{\zeta}^{*T}\boldsymbol{\sigma}$; in other words, there exists a predictable process $\boldsymbol{\zeta}^* : [0, T] \times \Omega \to \mathbb{R}^n$ such that

$$\operatorname{dist}(\tilde{Z}_t(\omega), \tilde{\mathcal{A}}^{\rho}_t(\omega)) = \operatorname{dist}(\tilde{Z}_t(\omega), \boldsymbol{\zeta}^*(t, \omega)) \quad \text{and} \quad \boldsymbol{\zeta}^{*T}(t, \omega)\boldsymbol{\sigma}(t, \omega) \in \tilde{\mathcal{A}}^{\rho}_t(\omega).$$

Finally using the fact that the strategy $(0, \ldots, 0)$ belongs to $\tilde{\mathcal{A}}^{\rho}_{\cdot}$ we have that

$$\begin{aligned}
\int_0^T ||\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u)||^2 du &\leq 2\int_0^T ||\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u) - \tilde{Z}_u||^2 du + 2\int_0^T ||\tilde{Z}_u||^2 du \\
&= 2\int_0^T \operatorname{dist}(\tilde{Z}_u, \tilde{\mathcal{A}}^{\rho}_u)^2 du + 2\int_0^T ||\tilde{Z}_u||^2 du \\
&\leq 4\int_0^T ||\tilde{Z}_u||^2 du < \infty.
\end{aligned}$$

□

To finalize, we must show that the quadratic–growth BSDE (3.1) admits a solution. To this end we need the following result of Briand and Hu [BriandHu08] , which extends the results of Kobylanski [Ko00]:

Theorem 3.2. *Let $h : [0, T] \times \Omega \times \mathbb{R}^m \to \mathbb{R}$ be measurable. Assume that there exists a predictable process α and positive constants C_1, C_2 satisfying $\alpha \geq 0$ and*

$$\mathbb{E}\left[\exp\left(C_1 \int_0^T \alpha_s ds\right)\right] <\infty.$$

Then if h is such that

(1) *$z \mapsto h(u, z)$ is continuous*
(2) $|h(u, z)| \leq C_2\|z\|^2 + \alpha_u$,

then the BSDE (3.1) *with driver* h *admits a solution* (Y, Z) *where* Y *and* Z *are predictable processes with* Y *bounded and* Z *satisfying* $\mathbb{E}\left(\int_0^T \|Z(t)\|^2 dt\right)^{\frac{1}{2}} < \infty$.

The previous result allows us to show that the BSDE (3.1) with driver (3.2) admits a unique solution.

Corollary 3.3. *There exists a unique pair of predictable processes* (Y, Z) *with* Y *bounded and* Z *satisfying* $\mathbb{E}\left(\int_0^T \|Z(t)\|^2 dt\right) < \infty$ *solution to the BSDE* (3.1) *with driver* (3.2).

Proof. We apply Theorem 3.2, and measurability of h is guaranteed by Theorem 3.1 The continuity in z of the driver is straightforward, as are the growth conditions, given Assumption 2.3.

□

We conclude with the existence of an optimal strategy to the optimization problem (2.9).

Theorem 3.4. *Under the assumptions made above there exists an acceptable strategy* $\boldsymbol{\zeta}^*$ *that solves the power utility maximization problem* (2.9)*. If we define the value function* $v(x)$ *as:*

$$v(x) \triangleq max_{\boldsymbol{\zeta}\in\mathcal{A}^\rho}\mathbb{E}(U_p(X^{\boldsymbol{\zeta}}(T))), \quad x > 0$$

with $\mathcal{A}^\rho$ *the set of admissible* $\mathbb{R}^n$*-valued predictable processes* $\boldsymbol{\zeta}$ *with* $\boldsymbol{\zeta}(t) \in \mathcal{A}_t^\rho$ *for all* t *in* $[0, T]$ *and* $X^{\boldsymbol{\zeta}}(0) = x$*, then it holds that*

$$v(x) = U_p(x)\exp(Y_0),$$

where (Y, Z) *is a solution to the BSDE* (3.1) *with driver* (3.2) *and*

$$\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u) \in \mathrm{Proj}(\tilde{\mathrm{Z}}(\mathrm{u}), \tilde{\mathcal{A}}_{\mathrm{u}}^\rho).$$

Proof. The existence of a solution to the BSDE (3.1) is guaranteed by Corollary 3.3. Hence using the martingale optimality principle, the processes $R^{\boldsymbol{\zeta}}$ are well–defined and satisfy requirements (a) and (b). In addition, by construction, the processes $\boldsymbol{\zeta}^*$ such that $R^{\boldsymbol{\zeta}^*}$ is a martingale are those such that $\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u) \in \mathrm{Proj}(\tilde{\mathrm{Z}}(\mathrm{u}), \tilde{\mathcal{A}}_{\mathrm{u}}^\rho)$. Theorem 3.1 yields that these elements $\boldsymbol{\zeta}^*$ are admissible strategies, thus optimal. Take such an optimal strategy $\boldsymbol{\zeta}^*$. We have that

$$v(x) = \mathbb{E}(U_p(X^{\boldsymbol{\zeta}^*}(T)) = \mathbb{E}(U_p(R^{\boldsymbol{\zeta}^*}(T)) = R^{\boldsymbol{\zeta}^*}(0) = U_p(x)\exp(Y_0).$$

□

The previous result admits a dynamic version:

Theorem 3.5. *Let* $v(t, x)$ *be the dynamic value function defined as:*

$$v(t,x) := \operatorname*{esssup}_{\boldsymbol{\zeta}\in\mathcal{A}^t} \mathbb{E}\left(U_p\left(x + \int_t^T \boldsymbol{\zeta}(s)X_s^{\boldsymbol{\zeta}}\frac{dS_s}{S_s}\right)\Big|\mathcal{F}_t\right) \quad t \in [0,T],\ x > 0,$$

where $\mathcal{A}^t := \{\boldsymbol{\zeta} \in \mathcal{A}^\rho,\ \boldsymbol{\zeta}(s) = 0,\ s < t\}$*. Then*

$$v(t, x) = U_p(x)\exp(Y_t),$$

where (Y, Z) *is a solution to the BSDE* (3.1) *with driver* (3.2) *and*

$$\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u) \in \mathrm{Proj}(\tilde{\mathrm{Z}}(\mathrm{u}), \tilde{\mathcal{A}}_{\mathrm{u}}^\rho).$$

Proof. Let $\boldsymbol{\zeta}$ any element of $\mathcal{A}$ and $\boldsymbol{\zeta}^*$ such that the associated $R^{\boldsymbol{\zeta}^*}$ is a martingale. Then by definition of the $R^{\boldsymbol{\zeta}}$ processes, we have that $R^{\boldsymbol{\zeta}}(t) = U_p(x)\exp(Y_t)$ since $\boldsymbol{\zeta}(s) = 0$ for $s < t$ and so

$$\begin{aligned} & \mathbb{E}\left(U_p\left(x + \int_t^T \boldsymbol{\zeta}(s) X_s^{\boldsymbol{\zeta}} \frac{dS_s}{S_s}\right)\Big|\mathcal{F}_t\right) \\ = \; & \mathbb{E}\left(R^{\boldsymbol{\zeta}}(T)|\mathcal{F}_t\right) \\ \leq \; & R^{\boldsymbol{\zeta}}(t) = U_p(x)\exp(Y_t) = \mathbb{E}\left(R^{\boldsymbol{\zeta}^*}(T)|\mathcal{F}_t\right) = \mathbb{E}\left(U_p\left(x + \int_t^T \boldsymbol{\zeta}^*(s) X_s^{\boldsymbol{\zeta}^*} \frac{dS_s}{S_s}\right)\Big|\mathcal{F}_t\right). \end{aligned}$$

Hence, $v(t,x) = U_p(x)\exp(Y_t)$.

□

Remark 3.6. Sometimes one might be interested in another version of the dynamic value function above. Given an element $\boldsymbol{\zeta}$ in $\mathcal{A}^\rho$ they may consider the quantity

$$v(t, X_t^{\boldsymbol{\zeta}}) := \operatorname*{esssup}_{\tilde{\boldsymbol{\zeta}} \in \mathcal{A}^{t,\boldsymbol{\zeta}}} \mathbb{E}\left(U_p\left(X_t^{\boldsymbol{\zeta}} + \int_t^T \tilde{\boldsymbol{\zeta}}(s) X_s^{\tilde{\boldsymbol{\zeta}}} \frac{dS_s}{S_s}\right)\Big|\mathcal{F}_t\right), \quad t \in [0,T],$$

where $\mathcal{A}^{t,\boldsymbol{\zeta}} := \{\tilde{\boldsymbol{\zeta}} \in \mathcal{A}^\rho,\ \tilde{\boldsymbol{\zeta}}(s) = \boldsymbol{\zeta}(s),\ s \leq t\}$. Then we have that $v(t, X_t^{\boldsymbol{\zeta}}) = U_p(X_t^{\boldsymbol{\zeta}})\exp(Y_t)$ where (Y,Z) is the unique solution of the BSDE (3.1) with driver (3.2).

Remark 3.7. The stochastic process $\exp(Y_t)$ in the expression of the value function is sometimes called the *opportunity process*, since it gives the value of the optimal wealth with initial capital one unit of currency (see [N10]).

Remark 3.8. Note finally that for the sake of the explanation, we have chosen to fix the risk aversion coefficient p in $(0,1)$ but we can also consider the case where $p < 0$. Then the driver h in (3.2) has to be modified suitably.

4. Three Important Risk Measures

The purpose of this section is to show and exploit certain properties of VaR, TVaR and LEL. These are actually *families* of risk measures, parameterized by an (exogenously chosen) percentile parameter α, as well as the risk constraint parameters $K_V, K_T, K_L \in (0,1)$[3]. We will assume that α is fixed and constant and that it satisfies $\alpha \in (0, 1/2)$. This technical assumption relates well to the practice, where the typical values of α are 0.05 or 0.1. Following [PirZit09] we have the formal definitions:

Definition 4.1. *The* **value–at–risk** $\mathrm{VaR} = \mathrm{VaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ *- corresponding to the current wealth x, the portfolio rate of return $\zeta_{\boldsymbol{\mu}}$ and volatility $\zeta_{\boldsymbol{\sigma}}$ - is the positive part of the upper α-percentile of the projected loss distribution $L = L(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$, i.e.,*

$$\mathrm{VaR} = \gamma_\alpha^+ = \max(0, \gamma_\alpha), \ \textit{where } \gamma_\alpha \textit{ uniquely satisfies } \mathbb{P}[L \geq \gamma_\alpha] = \alpha.$$

[3]In this section we assume that, contingent on the choice of risk measure, the "risk limit" K_t remains constant over $[0,T]$.

Definition 4.2. *The* **tail value–at–risk** $\mathrm{TVaR} = \mathrm{TVaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})$ *is the positive part of the mean of the distribution of the projected loss distribution, conditioned on the loss being more severe than α–percentile, i.e.,*

$$\mathrm{TVaR} = w_\alpha^+, \ \textit{where } \gamma_\alpha \textit{ satisfies } \mathbb{P}[L \geq \gamma_\alpha] = \alpha, \ \textit{and } w_\alpha = \mathbb{E}[L|L \geq \gamma_\alpha].$$

The third measure of risk - LEL - is similar to TVaR, except that it does not take the market rate–of–return in consideration. More precisely, we have the following definition:

Definition 4.3. *The* **limited expected loss** $\mathrm{LEL} = \mathrm{LEL}(x, \zeta_{\boldsymbol{\sigma}})$ *is the tail value–of–risk corresponding to the loss distribution $L = L(x, 0, \zeta_{\boldsymbol{\sigma}})$ in which the portfolio rate of return is set to zero.*

All three VaR, TVaR and LEL measure the risk of a large loss in absolute terms. If we define the *relative projected wealth loss* as the distribution of the quantity $\frac{X^{\boldsymbol{\zeta}}(t_0) - X^{\boldsymbol{\zeta}}(t_0+\tau)}{X^{\boldsymbol{\zeta}}(t_0)}$ (under the simplifying assumptions 1. and 2. from Subsection 2.3 above), definitions of the analogous relative quantities VaR_r, TVaR_r and LEL_r can readily be given. In fact, due to the multiplicative structure of the wealth equations (2.3) and (2.4), given that the wealth at t is x, we have the following expressions:

$$\rho_t^{\mathrm{var}}(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) = \frac{\mathrm{VaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})}{x}, \qquad \rho_t^{\mathrm{tvar}}(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) = \frac{\mathrm{TVaR}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})}{x}, \text{ and}$$
$$\rho_t^{\mathrm{lel}}(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) = \frac{\mathrm{LEL}(x, \zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})}{x}.$$

As we would expect, the relative risk limits VaR_r, TVaR_r and LEL_r no longer depend on the current level of wealth x.

4.1. **Some explicit expressions.** As a consequence of the fact that the distribution appearing in (2.7) is normal, explicit formulae can be given for the values of all three risk measures appearing above.

Proposition 4.4. *For $\zeta_{\boldsymbol{\mu}} \in \mathbb{R}$ and $\zeta_{\boldsymbol{\sigma}} > 0$, we have*

$$\rho_t^{var}(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) = \left[1 - \exp\left(\tilde{Q}(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}})\tau + N^{-1}(\alpha)\zeta_{\boldsymbol{\sigma}}\sqrt{\tau}\right)\right]^+ \tag{4.1}$$

$$\rho_t^{tvar}(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) = \left[1 - \tfrac{1}{\alpha} e^{\tau(r+\zeta_\mu)} N(N^{-1}(\alpha) - \zeta_{\boldsymbol{\sigma}}\sqrt{\tau})\right]^+, \ \textit{and} \tag{4.2}$$

$$\rho_t^{lel}(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1) = \left[1 - \tfrac{1}{\alpha} e^{r\tau} N\left(N^{-1}(\alpha) - \zeta_{\boldsymbol{\sigma}}\sqrt{\tau}\right)\right]^+, \tag{4.3}$$

where $N : \mathbb{R} \to (0, 1)$ is the cumulative distribution function of a standard normal random variable.

Proof. See Proposition 2.16 in [PirZit09].

□

In the light of this result it becomes clear that the risk measures considered in this section meet the Assumption 2.5.

4.2. **A common form of the risk constraints.** In this section we find some properties of the constraint sets $\mathcal{A}^{\rho_k}$, $k \in \{V, T, L\}$. The following result follows from straightforward computations.

Lemma 4.5. *Each constraint set $\mathcal{A}_t^{\rho_k}$, $k \in \{V, T, L\}$, can be expressed as*

$$\mathcal{A}_t^{\rho_k} = \left\{ \boldsymbol{\zeta} \in \mathbb{R}^m \,:\, f_k(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'(t)\boldsymbol{\sigma}(t)||) \le K_k \right\},$$

for some function $f_k : \mathbb{R} \times [0, \infty) \to \mathbb{R} \cup \{\infty\}$, and positive K_k which satisfies the following properties:

(1) *$f_k \in C^1(\mathbb{R} \times [0, \infty))$ is jointly convex, $f_k(0, 0) \le 0$.*
(2) *For each $(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \in \mathbb{R} \times [0, \infty)$, the sections $f_k(\zeta_{\boldsymbol{\mu}}, \cdot)$ and $f_k(\cdot, \zeta_{\boldsymbol{\sigma}})$ are (respectively) strictly increasing and decreasing.*
(3) *$f_k(0, 0) < 0$ and there exist constants $\kappa_i > 0$, $i \in \{1, 2, 3\}$ such that for all $(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \in \mathbb{R} \times [0, \infty)$*

$$f_k(\zeta_{\boldsymbol{\mu}}, \zeta_{\boldsymbol{\sigma}}) \ge \kappa_1 \zeta_{\boldsymbol{\sigma}}^2 - \kappa_2 \zeta_{\boldsymbol{\mu}} - \kappa_3$$

As consequences of Lemma 4.5 we have the convexity and compactness of the constraint sets considered in this section. The following result is Proposition 4.3 from [PirZit09].

Proposition 4.6. *The constraint set $\mathcal{A}^{\rho_k}$, $k \in \{V, T, L\}$ is convex and compact.*

4.3. **A Three–Fund Separation Result.** In this section we want to further characterize the optimal investment strategy. In order to ease the exposition we drop the subscript k. The convexity of $\mathcal{A}^\rho$ will imply the uniqueness of optimal trading strategy $\boldsymbol{\zeta}^*$, this fact turns out to be useful in numerical implementations. Indeed, let us recall that $\boldsymbol{\zeta}^*$ is given by

$$\boldsymbol{\zeta}^{*T}(u)\boldsymbol{\sigma}(u) \in \mathrm{Proj}(\tilde{Z}(u), \tilde{\mathcal{A}}_u^\rho), \quad u \in [0, T].$$

The convexity of $\mathcal{A}^\rho$ leads to the likewise property of $\tilde{\mathcal{A}}^\rho$ which in turn yields the uniqueness of the projection.

Theorem 4.7. *There exist two stochastic processes β_1^* and β_2^* such that the optimal strategy $\boldsymbol{\zeta}^*$ can be decomposed as*

$$\boldsymbol{\zeta}^*(t) = \frac{\beta_1^*(t)}{1-p}\boldsymbol{\zeta}_M(t) + \beta_2^*(t)(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t))^{-1}\boldsymbol{\sigma}(t)Z(t), \quad 0 \le t \le T, \tag{4.4}$$

where $Z(t), 0 \le t \le T$ is part of the (Y, Z) solution of BSDE (3.1) *with driver* (3.2).

Proof. We cover the case $p \ge 0$ only (the case $p < 0$ can be obtained by a similar argument). Let us recall that for a fixed path ω, the optimal strategy $\boldsymbol{\zeta}^*(t)$ solves

$$\boldsymbol{\zeta}^*(t) = \arg\min_{\boldsymbol{\zeta} \in \mathcal{A}(t)} \left\{ -p\boldsymbol{\zeta}'(\boldsymbol{\mu}(t) + p\boldsymbol{\sigma}(t)Z(t)) - \frac{p-p^2}{2}||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||^2 \right\}.$$

The convex, quadratic functional

$$\boldsymbol{\zeta} \to H(t, \boldsymbol{\zeta}) \triangleq -p\boldsymbol{\zeta}'(\boldsymbol{\mu}(t) + p\boldsymbol{\sigma}(t)Z(t)) - \frac{p-p^2}{2}||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||^2$$

is minimized over the constraint set $\mathcal{A}(t)$ at the unique point $\boldsymbol{\zeta}^*(t)$, which is on the boundary of $\mathcal{A}(t)$. Thus, for a fixed path, $\boldsymbol{\zeta}^*(t)$ minimizes $H(t, \boldsymbol{\zeta})$ over the constraint $f(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||) = K$, (see Proposition 4.5). The solution $\boldsymbol{\zeta}^*(t)$ is not the zero vector, since $f(0,0) \le 0$. For $\boldsymbol{\zeta} \neq 0$, it follows that

$$\nabla f(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||) = f_1(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||)\boldsymbol{\mu}(t) - \frac{f_2(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||)}{||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||}\boldsymbol{\sigma}(t)\boldsymbol{\sigma}'(t)\boldsymbol{\zeta},$$

where f_1 and f_2 stand for the partial derivatives of function f. According to the Karush–Kuhn–Tucker conditions, either $\nabla f(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||) = 0$ or else there is a positive λ such that

$$\nabla H(t, \boldsymbol{\zeta}) = \lambda \nabla f(\boldsymbol{\zeta}'\boldsymbol{\mu}(t), ||\boldsymbol{\zeta}'\boldsymbol{\sigma}(t)||). \tag{4.5}$$

In both cases, straightforward computations show that $\boldsymbol{\zeta}^*(t)$ is of the form given in (4.4).

□

Theorem 4.7 is a three–fund separation result. It simply says that a utility–maximizing investor subject to regulatory constraints will invest his wealth into three funds: 1. the savings account; 2. a risky fund with return $\boldsymbol{\zeta}_M(t), t \in [0,T]$; 3. a risky fund with return $(\boldsymbol{\sigma}(t)\boldsymbol{\sigma}^T(t))^{-1}\boldsymbol{\sigma}^T(t)Z(t), t \in [0,T]$. Most of the results in financial literature are usually two–funds separation ones (optimal wealth being invested into a savings account and a risky fund). We would obtain such a result if we restricted ourselves to the more simplistic model in which stocks returns and volatilities are deterministic. It is for the randomness of stocks returns and volatilities that the optimal investment consists of an extra risky fund. Investment in this fund can be regarded as a hedge against risk carried in stochastic stock returns and volatilities.

5. A numerically implemented example

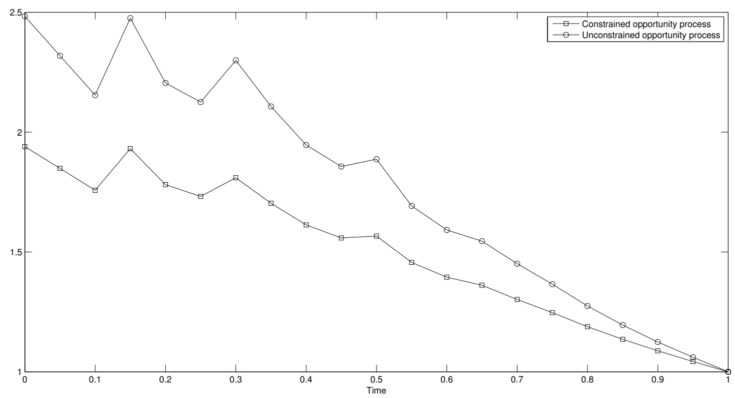
In this section we present numerical simulations for the constrained optimal strategies and the associated constrained *opportunity processes*. Recall that by opportunity process, we mean the process $\exp(Y_t)$ which appears in the value function $v(t,x)$ given in Theorem 3.5, that is $v(t,x) = \frac{x^p}{p}\exp(Y_t)$. The opportunity process represents the value function of an investor with initial capital one dollar; it is a stochastic process and in the figures below we present one sample path. For simplicity and the numerical tractability of the analysis we assume that we deal with one risky asset ($n = 1$), one bond with rate zero ($r = 0$) and one Brownian motion ($m = 1$). In addition, we assume that the risky asset is given by the following SDE:

$$dS_t = S_t(\mathbf{1}_{[-1,1]}(W_t)dt + dW_t), \quad t \in [0,1] \quad (T = 1), \quad S_0 = 1.$$

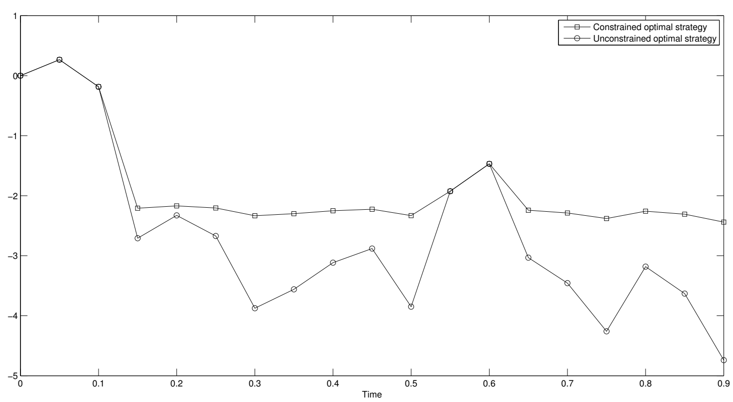
Our simulation relies on numerical schemes for quadratic growth BSDEs. We use the scheme of Dos Reis and Imkeller [DRI10, DR10], which in a nutshell relies on a truncation argument of the driver, and it reduces the numerical simulation problem to the one of a BSDE with a Lipschitz–growth driver . For the latter we use the so–called *forward scheme* of Bender and Denk [BenDen07].

5.1. **VaR.** In Figure 5.1, we consider the risk measure VaR (given in (4.1)) with the following set of parameters: p=`0.85`, α=`0.10`, K=`0.3`. The time discretization is $1/N$ with N=`15` and τ=`1/15`.

5.2. **TVar.** In Figure 5.2, we consider the risk measure TVar (given in (4.2)) with the same set of parameters: p=`0.85`, α=`0.10`, K=`0.3`. The time discretization is $1/N$ with N=`15` and τ=`1/15`.



(a) Constrained and unconstrained opportunity processes.



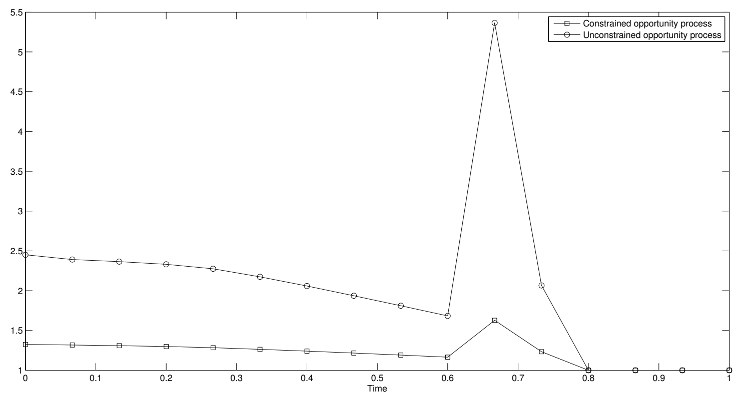
(b) Constrained and unconstrained optimal strategies.

FIGURE 1. Plots of the constrained and unconstrained opportunity processes and optimal strategies for VaR.

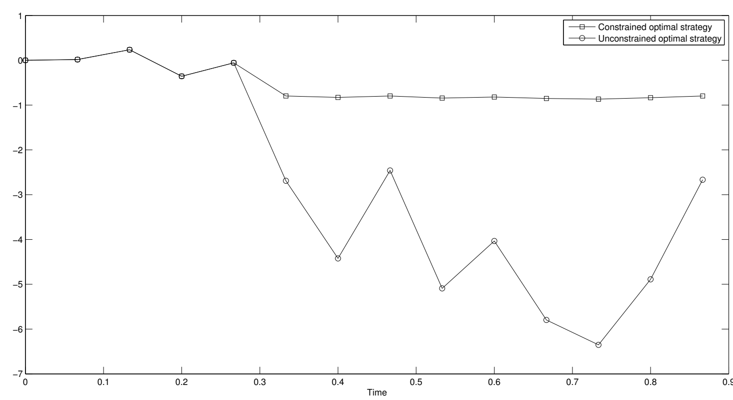
5.3. **LEL.** In Figure 5.3, we consider the risk measure LEL (given in (4.3)) with the same set of parameters: p=`0.85`, α=`0.10`, K=`0.3`. The time discretization is $1/N$ with N=`15` and τ=`1/15`.

6. CONCLUSIONS

We have analyzed, within an incomplete–market framework, the portfolio–choice problem of a risk averse agent (who is characterized by CRRA preferences), when risk constraints are imposed continuously throughout the investment phase. Using BSDE technology, in the spirit of [HIM05], has enabled us to allow for a broad range of risk measures that give rise to the risk constraints, the latter being (possibly) time–dependent. In order to use such technology, we have made use



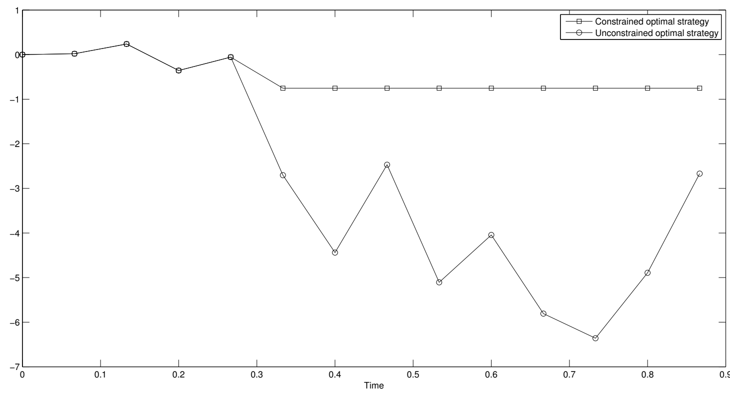
(a) Constrained and unconstrained opportunity processes.



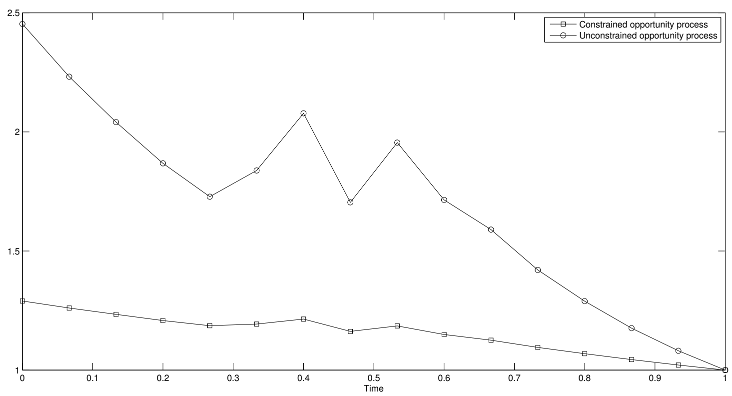
(b) Constrained and unconstrained optimal strategies.

Figure 2. Plots of the constrained and unconstrained opportunity processes and optimal strategies for TVar.

of Measurable Selections theory, specifically when addressing the issue of the driver of the BSDE at hand. We have characterized the optimal (constrained) investment strategies, and in the case of VaR_r, TVaR_r and LEL_r we have provided explicit (unique) expressions for them. Here we have shown that optimal strategies may be described as investments in three funds, which is in contrast with the classical two–fund separation theorems. Finally, using recent results in [DRI10], we have provided some examples that showcase the way in which our dynamic risk constraints limit investment strategies and impact utility at maturity.



(a) Constrained and unconstrained opportunity processes.



(b) Constrained and unconstrained optimal strategies.

FIGURE 3. Plots of the constrained and unconstrained opportunity processes and optimal strategies for LEL.

APPENDIX A. PROPERTIES OF THE CONSTRAINT SETS $\mathcal{A}_t^\rho$

Several analytical properties of the (instantaneous) constraint sets $\mathcal{A}_t^\rho$ are established in this section. The analysis requires some core concepts of the theory of measurable correspondences[4]. We require the following auxiliary correspondences:

$$\mathcal{A}_{t,k}^\rho(\omega) \triangleq \left\{ \boldsymbol{\zeta} \in [-k,k]^n \,\middle|\, \rho_t(\mathfrak{E}(\boldsymbol{\zeta},t)-1)(\omega) - K_t(\omega) \le 0 \right\}, \quad k \in \mathbb{N}.$$

The purpose of artificially bounding the values of $\mathcal{A}_\cdot^\rho$ is to make use of the theory of compact–valued correspondences, which exhibit many desirable properties.

[4]For a comprehensive overview of the theory of measurable correspondences, we refer the reader to [AB06] and [].

Lemma A.1. *For any $m \in \mathbb{N}$, the correspondence $\mathcal{A}^{\rho}_{\cdot,k} : [0,T] \times \Omega \twoheadrightarrow \mathbb{R}^n$ is non–empty and compact valued for almost all $(t,\omega) \in [0,T] \times \Omega$.*

Proof. The non–vacuity follows from the fact that $\boldsymbol{\zeta} \equiv 0$, i.e. no wealth invested in risky assets, is an acceptable position. To show closeness of the sets $\mathcal{A}^{\rho}_{t,k}(\omega)$, fix $\omega \in \Omega$ and consider a sequence $\{\boldsymbol{\zeta}_n\} \subset \mathcal{A}^{\rho}_{t,k}(\omega)$ such that $\boldsymbol{\zeta}_n \to \boldsymbol{\zeta}$. Using Assumption 2.5 it holds that

$$\rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega) - K_t(\omega) = \lim_{n\to\infty} \rho_t(\mathfrak{E}(\boldsymbol{\zeta}_n, t) - 1)(\omega) - K_t(\omega) \le 0$$

holds for all $t \in [0,T]$ and which implies that $\boldsymbol{\zeta} \in \mathcal{A}_t(\omega)$. The latter, together with the fact that $\boldsymbol{\zeta} \in [-k,k]^n$ finalizes the proof.

□

Definition A.2. *A correspondence ϕ between a measurable space $(\Theta, \mathcal{G})$ and a topological space X is said to be weakly measurable if for all $F \subset X$ closed, the lower inverse of F, defined as*

$$\phi^l(F) \triangleq \{\theta \in \Theta \mid \phi(\theta) \cap F \neq \emptyset\},$$

belongs to $\mathcal{G}$.

In the case of compact–valued correspondences, weak–measurability and Borel measurability (in terms of the Borel σ–algebra generated by the Hausdorff metric) are equivalent notions. Given a correspondence $\phi : \Omega \times [0,T] \mapsto\!\!\!\!\rightarrow \mathbb{R}^n$ we define the corresponding closure correspondence via $\bar{\phi}(\omega,t) \triangleq \overline{\phi(\omega,t)}$. For notational purposes let

$$f\big((t,\omega), \boldsymbol{\zeta}\big) = \rho_t(\mathfrak{E}(\boldsymbol{\zeta}, t) - 1)(\omega) - K_t(\omega).$$

Recall that $\mathcal{P}$ denotes the predictable σ–algebra on $[0,T]\times\Omega$. The function $f\big((\cdot,\cdot),\cdot\big)$ is a Carathéodory function with respect to $\mathcal{P}$, i.e. it is continuous in $\boldsymbol{\zeta}$ and $\mathcal{P}$–measurable in (t,ω).

Proposition A.3. *For any $k \in \mathbb{N}$, the correspondence $\mathcal{A}^{\rho}_{\cdot,k} : [0,T] \times \Omega \twoheadrightarrow \mathbb{R}^n$ is weakly $\mathcal{P}$–measurable.*

Proof. Let $F \subset \mathbb{R}^n$ be closed and consider $\{\boldsymbol{\zeta}_m\}_{m=1}^{\infty} \subset F$ dense. For $\eta \in \mathbb{N}$ let

$${}^{\eta}\mathcal{A}^{\rho}_{t,k}(\omega) \triangleq \left\{ \boldsymbol{\zeta} \in [-k,k]^n \,\middle|\, f\big((t,\omega),\boldsymbol{\zeta}\big) < \frac{1}{\eta} \right\}.$$

We have that

$$\begin{aligned}
\big({}^{\eta}\mathcal{A}^{\rho}_{\cdot,k}\big)^l(F) &= \Big\{(t,\omega) \in [0,T] \times \Omega \,\Big|\, f\big((t,\omega),\boldsymbol{\zeta}\big) < \frac{1}{\eta} \quad \text{for some} \quad \boldsymbol{\zeta} \in F\Big\} \\
&= \Big\{(t,\omega) \in [0,T] \times \Omega \,\Big|\, f\big((t,\omega),\boldsymbol{\zeta}_m\big) < \frac{1}{\eta} \quad \text{for some} \quad m \in \mathbb{N}\Big\} \\
&= \bigcup_{m=1}^{\infty} f^{-1}\big((\cdot,\cdot),\boldsymbol{\zeta}\big)\big(-\infty, \frac{1}{\eta}\big).
\end{aligned}$$

The second equality holds because f is continuous in $\boldsymbol{\zeta}$, $\{\boldsymbol{\zeta}_m\}_{l=1}^{\infty}$ is dense and $(\infty, 1/\eta)$ is open. Since f is Carathéodory, then $f^{-1}\big((\cdot,\cdot),\boldsymbol{\zeta}\big)\big(-\infty,\frac{1}{\eta}\big) \in \mathcal{P}$, hence for all $\eta \in \mathbb{N}$, the correspondence

${}^{\eta}\mathcal{A}^{\rho}_{\cdot,k}$ is weakly $\mathcal{P}$–measurable. Next we have

$$\mathcal{A}^{\rho}_{t,k}(\omega) \subset \overline{{}^{\eta}\mathcal{A}^{\rho}_{t,k}(\omega)} \subset \left\{ \boldsymbol{\zeta} \in [-k,k]^n \,\middle|\, f\big((t,\omega),\boldsymbol{\zeta}\big) \leq \frac{1}{\eta} \right\},$$

where the second inclusion follows again from the continuity of f in $\boldsymbol{\zeta}$. This implies that

$$\mathcal{A}^{\rho}_{t,k}(\omega) = \bigcap_{\eta=1}^{\infty} \overline{{}^{\eta}\mathcal{A}^{\rho}_{t,k}}(\omega),$$

and

$$\operatorname{graph}\big(\mathcal{A}^{\rho}_{\cdot,k}(\cdot)\big) = \bigcap_{\eta=1}^{\infty} \operatorname{graph}\big(\overline{{}^{\eta}\mathcal{A}^{\rho}_{\cdot,k}}(\cdot)\big).$$

The graph of the closure of a weakly–measurable correspondence is measurable, hence $\operatorname{graph}\big(\mathcal{A}^{\rho}_{t,k}\big)$ is measurable, by virtue of being the (denumerable) intersection of measurable graphs. Since a compact–valued correspondence with a measurable graph is itself weakly–measurable (see Lemma 18.4 (part 3) and Corollary 18.8 in [AB06]), we conclude that the correspondence $(t,\omega) \mapsto\!\!\!\rightarrow \mathcal{A}^{\rho}_{t,k}(\omega)$ has such property.

□

The following theorem, whose proof can be found in [AB06], page 595, plays an important role in the proof of predictability of our BSDE's driver:

Theorem A.4. *A nonempty–valued correspondence mapping a measurable space into a separable, metrizable space is weakly–measurable if and only if its associated distance function is a Carathéodory function.*

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