Continuous Equilibrium under Base Preferences and Attainable Initial Endowments

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We consider a full equilibrium model in continuous time comprising a finite number of agents and tradable securities. We show that, if the agents' endowments are spanned by the securities and if the agents have entropic utilities, an equilibrium exists and the agents' optimal trading strategies are constant. Affine processes, and the theory of information-based asset pricing are used to model the endogenous asset price dynamics and the terminal payoff. Semi-explicit pricing formulae are obtained and applied to numerically analyze the impact of the agents' risk aversion on the implied volatility of simultaneously-traded European-style options.[1]

Keywords: Continuous-time equilibrium, CAPM, affine processes, information-based asset pricing, implied volatility.

JEL Classification: C62, D52, D53

Introduction

The standard Arrow-Debreu theory asserts that equilibria exist when markets are complete, that is, when all uncertainty can be hedged using the traded assets. In the framework of complete markets equilibria are Pareto efficient and competitive equilibria are those of a suitably defined representative agent economy. In incomplete markets, the situation is more complex. Competitive equilibria may fail to exist and even if they exist, they may neither be Pareto efficient nor equilibria of a representative agent economy. As a result, when incomplete markets are studied, the analysis is confined to special cases, e.g. to single agent models neglecting the problem of aggregation and disaggregation in equilibrium (He and Leland [21], 1993; Gârleanu et al. [20], 2009), multiple agent models of complete markets (Dana and Jeanblanc [9], 2002; Duffie and Huang [12], 1985; Horst and Müller [23], 2007; Horst et al. [24], 2010; Karatzas et al.

[27], 1990) or to models with particular classes of goods (Jofre et al. [26], 2010) or preferences (Carmona et al. [7], 2010; Carmona and Hinz [6], 2011). Cheridito, Horst, Kupper and Pirvu [8], 2010, (henceforth CHKP) established existence, uniqueness and characterization of equilibrium results for dynamically incomplete financial market models with general sample spaces in discrete time when the agents' preferences are translation invariant. Preferences are called translation invariant or semi-linear if an amount of money added to a consumption bundle or financial position increases the utility of the bundle/position by that amount. The assumption of translation invariance is often made in models of asymmetric information and in many financial mathematics models. Typical examples are preferences described by (conditional) expected values or, more generally, (dynamic) convex risk measures.

CHKP showed that under translation invariance the problem of finding equilibria can be reduced to solving a constrained optimization problem of a representative agent and that equilibrium prices and trading strategies can be characterized in terms of the solution to a system of fully coupled forward-backward stochastic difference equations when the flow of information is generated by finitely many random walks (see Dumas and Lyasoff [15], 2009, for a related result). The forward part of their system describes the dynamics of the random walks (or factors driven thereby) while the backward part describes the joint dynamics of the equilibrium prices and all the agents' equilibrium utilities. In particular, the dimensionality of the forward-backward system depends on the number of market participants. As a result, the system is almost always high dimensional, and equilibrium quantities are hard to compute and calibrate. There is one important special case (other than the benchmark case of a complete market) though, where the system reduces to a one dimensional equation and which is thus potentially amenable to an empirical analysis: a generalized Capital Asset Pricing Model (CAPM) where all agents share the same base preferences and where their endowments are attainable in some sense.

In this paper we study a class of generalized CAPMs in continuous time. The advantage of our continuous time framework over the corresponding discrete time model of CHKP is that we are able to obtain (semi-) explicit formulae for equilibrium prices. If not explicitly computable, key equilibrium quantities can be computed using numerical integration only. In particular, no Monte Carlo methods are needed. This makes the computation of equilibrium prices and functions fast, efficient and stable. Implied volatility surfaces, for instance, can easily be computed.

Specifically, we consider a financial market model with a finite set of agents that differ with respect to their risk aversion and initial endowment. At any point in time, our agents maximize expected exponential utility from trading in a financial market up to some terminal time T. The financial market is incomplete.

The agents can trade finitely many securities. Securities entitle their holders to an uncertain payoff/dividend at the terminal time and are priced by demand and supply. Terminal payoffs are determined by finitely many factors of which dynamics follow affine processes. Affine processes are widely used in mathematical finance (see for instance Duffie et al. [14], Keller-Ressel [28] and references therein) as they lend themselves to a transparent mathematical analysis and efficient numerical methods. Within the present affine framework, equilibrium securities prices are given by the quotient of two integrals. Both integrals are integrals of the product of an exponential function evaluated at the current state of an affine process and the Fourier transform of a smooth function. The benchmark cases of pricing a single call option and multiple call options in zero net supply are particularly transparent.

Representing equilibrium prices in terms of deterministic integrals allows for a fast and efficient numerical analysis of equilibrium quantities such as option implied volatilities. We analyze implied volatilities for two single-security benchmark models: an additive Heston stochastic volatility model where the payoff of the security at the terminal time is given through an additive Heston model, and an Ornstein-Uhlenbeck model with jumps where the dynamics of the terminal payoff follows a mean-reverting process with jumps. Both models can be solved in closed form, and reproduce the well documented smile-effect

of implied volatilities. We identify the risk aversion of investors and jump intensities as important drivers of implied volatilities.

An alternative approach to the pricing of securities in the context of the present paper is the theory of information-based asset pricing (Brody, Hughston and Macrina [4, 5], 2008; Hoyle, Hughston and Macrina [25], 2011). Within this approach, the cash flows are explicitly modeled by functions of independent random variables, and the asset price dynamics is explicitly generated by taking the conditional expectation of the future cash flows, which are multiplied by the pricing rule, given the partial information about the market factors that is available to market agents. The partial information is modeled by stochastic processes, which (i) carry information about the a priori distribution of the market factors used to model the asset's cash flows, and (ii) embody pure noise preventing market participants from accessing full knowledge as to what is the "true" value of the asset at any time before the cash flows occur. Since the pricing rules obtained in this paper depend on the cash flow of the asset under consideration, the dynamics of the agent's preference depend on the market information available at each point in time. In other words, the agent's preferences change over time given the noisy information about the market factors on which the asset's cash flows depend. The flexibility of the information-based framework for asset pricing allows for semi-explicit pricing formulae for assets including vanilla options. We also show that it is possible to address the situation when market factors are taken to be dependent random variables while maintaining a high degree of tractability in deriving the asset price processes along with the related pricing rules subject to the information accessible to market participants.

The paper is structured as follows: A general existence result along with a proof is given in Section 1. In Section 2, we present two asset pricing methods, namely the affine processes technique and the theory of information-based asset pricing. Finally, addenda and proofs can be found in the appendix.

1. General framework

We consider a full equilibrium model in continuous time with a finite number of agents belonging to the set $\mathbb{A}$. The proposed approach is developed based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where the filtration $(\mathcal{F}_t)$ satisfies the usual conditions. In the following, all equalities and inequalities are to be understood in the P-almost sure sense. We fix a finite time horizon $T > 0$. The agents are trading K securities that are in net supply $n = (n^1, \cdots, n^K) \in \mathbb{R}^K$, of which discounted price process $S = (S^1, \cdots, S^K)$ are determined endogenously in equilibrium. The assets S^i are characterised a priori only by their terminal payoffs S_T^i, which we regard as given. Each agent a is initially endowed with some $\mathcal{F}_T$-measurable random endowment H^a. Let us assume that the agents measure their financial positions with an entropic utility function. Thus, at each $t \in [0, T]$, agent a maximizes the functional given by

$$U_t^a(X) = -\frac{1}{\gamma^a} \log\left(E\left[\mathrm{e}^{-\gamma^a X} \mid \mathcal{F}_t \right] \right), \tag{1.1}$$

where $\gamma^a > 0$ is the risk aversion parameter. At each time $t \in [0, T]$, agent a faces the optimization problem

$$\sup_{\vartheta \in \Theta} U_t^a\left(H^a + \int_t^T \vartheta_t dS_t \right), \tag{1.2}$$

where the set of admissible trading strategies Θ is given by

$$\Theta = \left\{ \vartheta \in L(S) \, : \, G(\vartheta) \text{ is a } \tilde{Q}\text{-supermartingale, for all } \tilde{Q} \in \mathcal{P} \right\}. \tag{1.3}$$

Here, $L(S)$ and $G_t := \int_0^t \vartheta_u dS_u$ denote the set of S-integrable[2] predictable processes and the gains process, respectively, whereas $\mathcal{P}$ denotes the set of all equivalent sigma-martingale measures (ESMM) for S. It is known from [8, Theorem 5.1] that in discrete time and for a special class of endowments[3], the agents' optimal trading strategies are constant and define an equilibrium pricing measure. A similar result in continuous time follows:

Theorem 1.1. *Suppose that the endowments are of the form*

$$H^a = c^a + \eta^a \cdot S_T , \tag{1.4}$$

for constants $c^a \in \mathbb{R}$ and $\eta^a \in \mathbb{R}^K$. Moreover, assume that $S_T \in L^1(Q)^K$ and that

$$\exp\left(-\gamma(n+\eta) \cdot S_T\right) \in L^1(P) \ , \quad where \quad \gamma := \left(\sum_{a\in\mathbb{A}} \frac{1}{\gamma^a}\right)^{-1} \quad and \quad \eta := \sum_{a\in\mathbb{A}} \eta^a , \tag{1.5}$$

and Q is an equivalent probability measure with density

$$\frac{dQ}{dP} = \frac{\exp(-\gamma(n+\eta) \cdot S_T)}{E\left[\exp(-\gamma(n+\eta) \cdot S_T)\right]} . \tag{1.6}$$

Then, the price process S defined by

$$S_t = E_Q\left[S_T \,|\, \mathcal{F}_t\right] , \quad t \in [0, T] , \tag{1.7}$$

together with agent a's constant optimal trading strategy

$$\hat{\vartheta}_t^a \equiv \frac{\gamma}{\gamma^a}(n+\eta) - \eta^a \tag{1.8}$$

constitutes an equilibrium.

Remark 1.2. Apart from the full equilibrium case, partial equilibrium could also be covered in the present approach. A partial equilibrium includes additional assets with exogenously given price process, which can be used as additional hedging instruments. The theory provided in [8] yields the existence of an optimal portfolio in the exogenous assets also influencing the equilibrium pricing measure (1.6). ♦

Proof (of Theorem 1.1). Due to the time-consistency and strict monotonicity of the entropic preferences, it suffices to show that the strategies $\hat{\vartheta}^a$ are optimal for the utility maximization in $t = 0$. Note first that (1.5) and $S_T \in L^1(Q)^K$ ensure that (1.6) and (1.7) are well-defined, respectively. In particular, $Q \in \mathcal{P}$, the price process S is a Q-martingale and $\hat{\vartheta}^a$ lies in Θ, since for any $\tilde{Q} \in \mathcal{P}$, the process $G_t(\hat{\vartheta}^a) = \hat{\vartheta}^a \cdot (S_t - S_0)$ is by assumption a $\tilde{Q}$-sigma-martingale, which is non-negative, and hence a $\tilde{Q}$-supermartingale, see Delbaen and Schachermayer [10, Section 8.3].

We now show that the aggregated risk aversion γ in (1.5) can be seen as the risk aversion of some *representative agent* maximizing utility of terminal wealth against the aggregated initial endowments and that the optimal utility is attained at the constant strategy $\vartheta^* \equiv n$. Indeed, since S is a Q-martingale, with

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[2] In equilibrium the price process S will satisfy (NFLVR) and is in particular a semimartingale.

[3] More specifically, attainable endowments are considered, a model similar to the Capital Asset Pricing Model (CAPM) first introduced in Markowitz [31].

Q defined by (1.6), and $n \cdot S_T \in L^1(Q)$, it holds

$$\sup_{\vartheta \in \Theta, E_Q[G_T(\vartheta)] \leq E_Q[n \cdot S_T]} \{U_0^\gamma (\eta \cdot S_T + G_T(\vartheta))\}$$

$$= \sup_{\vartheta \in \Theta} \{U_0^\gamma (\eta \cdot S_T + G_T(\vartheta) - E_Q[G_T(\vartheta)] + E_Q[n \cdot S_T])\}$$

$$= \sup_{\vartheta \in \Theta} \{U_0^\gamma (G_T(\vartheta)) - E_Q[G_T(\vartheta)]\} + E_Q[(n + \eta) \cdot S_T]$$

$$\leq \frac{1}{\gamma} H(Q|P) + E_Q[(n + \eta) \cdot S_T], \tag{1.9}$$

where the last equality is derived from the dual representation of U_0^γ and the relative entropy is given by $H(Q\,|\,P) = E[\frac{dQ}{dP} \log(\frac{dQ}{dP})]$. But $G_T(\vartheta^*)$ with $\vartheta^* \equiv n$ plugged into the representative agent's utility $U_0^\gamma(\eta \cdot S_T + \cdot)$ yields

$$U_0^\gamma ((n + \eta) \cdot S_T) = \frac{1}{\gamma} H(Q|P) + E_Q[(n + \eta) \cdot S_T].$$

Comparing this with (1.9) shows that $\vartheta^* \equiv n$ is indeed optimal for the representative agent when the price process S is given by (1.7). Indiviual optimality of $\hat{\vartheta}^a$ for the single agents now follows by a scaling argument and the specific form of the aggregated endowment. Note to this end that, for all $a \in \mathbb{A}$,

$$\vartheta^* + \eta = \underset{\vartheta \in \Theta, E_Q[G_T(\vartheta)] \leq E_Q[n \cdot S_T]}{\arg\max} \{U_0^\gamma (G_T(\vartheta))\}$$

is equivalent to

$$\frac{\gamma}{\gamma^a}(\vartheta^* + \eta) = \underset{\vartheta \in \Theta, E_Q[G_T(\vartheta)] \leq E_Q[\eta^a \cdot S_T]}{\arg\max} \{U_0^a (G_T(\vartheta))\},$$

which in turn is equivalent to

$$\frac{\gamma}{\gamma^a}(\vartheta^* - \eta) - \eta^a = \underset{\vartheta \in \Theta, E_Q[G_T(\vartheta)] \leq 0}{\arg\max} \{U_0^a (H^a + G_T(\vartheta))\}. \tag{1.10}$$

But (1.10) shows that $\hat{\vartheta}^a$ as in (1.8) is the optimal strategy for the utility maximization problem of agent a. Since in addition the strategies $(\hat{\vartheta}^a)_{a \in \mathbb{A}}$ add up to n, the market clears at any time, hence $((S_t)_{t \in [0,T]}, (\hat{\vartheta}^a)_{a \in \mathbb{A}})$ forms an equilibrium and we are done. $\qquad\square$

2. Specification of the market filtration

So far, we established existence of a continuous equilibrium within a potentially incomplete financial market by working on the abstract probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ satisfying the usual conditions, and without having said anything explicitly yet about the underlying filtration $(\mathcal{F}_t)$. Representing the flow of information available to the agents trading in S though, the nature of $(\mathcal{F}_t)$ is of vital importance to an equilibrium model. Let us recall that all it needed for the results of Section 1 to be applicable, was the measurability of the terminal payoff S_T with respect to the σ-algebra $\mathcal{F}_T$. In this section, we present two different approaches to model the random variable S_T and for the construction of the filtration $(\mathcal{F}_t)$. Furthermore, semi-explicit pricing formulas for S will be established. Each of them is thus suited to explain respective different aspects of market price behavior observed in real world. We will consider:

1. An approach based on the the theory of *affine processes*, a class of Markov processes lending itself to advanced, but still tractable, modeling of phenomena appearing within the field of mathematical

finance. Their importance and properties are discussed extensively in the literature, see for instance [13], [14] or [28], also for numerous applications. We model the payoff S_T by a function of the terminal value of an affine process, the analytical properties of which will be used to discuss the role of the underlying parameters on the equilibrium dynamics.

2. An approach based on the theory of *information-based asset pricing* as established in [4], [5] and [25]. The market's filtration, to which the equilibrium price process will be adapted, is explicitly constructed by information processes, which provide partial information about the random cash flow S_T at times $t < T$, and then reveal the value of S_T at the terminal date T.

2.1. Affine equilibrium framework

After a short introduction into the theory of affine processes, the results from Section 1 are used to derive equilibrium pricing formulas in Section 2.1.1. This is followed by Sections 2.1.2, 2.1.3 and 2.1.4, where the payoff structure as well as the underlying models are specified. The case of simultaneously traded European options along with two specific affine models are considered. The examples range from pricing a single security in an incomplete financial market to simultaneously pricing stocks and multiple options written on them in the case of zero option supply as well as in the case of non-zero supply.

2.1.1. Setup and equilibrium pricing formulae

In this section we specify the terminal payoff S_T by means of affine processes. In particular, we focus on the case $S_T = f(X_T)$, where X will be a component of some affine process Y and $f : \mathbb{R} \to \mathbb{R}^K$ a well-behaved function. By this, the equilibrium price of each S^k can be described in terms of the functions characterizing the log-characteristic function of Y. We begin with some useful definitions and results on affine processes, the details of which can be found in Duffie et al. [14] or in Keller-Ressel [28].

In order to consider a Markov process Y with (polish) state space D, we need an underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, on which this process is defined. If not stated otherwise, we consider the canonical probability space for Y, compare [33, Chapter III]. Associated to Y there is a family of probability measures $(P^x)_{x \in D}$ representing the law of the process Y starting at $x \in D$[4]. The filtration $(\mathcal{F}_t)$ can then be completed with respect to the family $(P^x)_{x \in D}$, as described in [33, Section III.2], and hence we find ourselves in accordance with the setting of Chapter 1.

We focus on the case $D = \mathbb{R}^n \times \mathbb{R}^m_+$ and further assume that Y^T, the Markov process stopped at T, is conservative, meaning that up to time T there are no explosions or absorbing states. We define $I = \{1, \cdots, m\}$ and $J = \{m+1, \cdots, m+n\}$, the index sets of the $\mathbb{R}_+$-valued and $\mathbb{R}$-valued parts of Y, respectively. For $d := m + n$ and any d-dimensional vector x, let x_I and x_J denote the respective projections on the components of the index.

Definition 2.1 (Regular Affine Process). An affine process is a stochastically continuous[5], time-homogeneous Markov process (Y, P^x) with state-space $D = \mathbb{R}^n \times \mathbb{R}^m_+$, whose log-characteristic function is an affine function of the state vector. This means that there exist functions $\phi : \mathbb{R}_+ \times i\mathbb{R}^d \to \mathbb{C}$ and $\psi : \mathbb{R}_+ \times i\mathbb{R}^d \to \mathbb{C}^d$ such that

$$E^x \left[\exp\left(u \cdot Y_t\right) \right] = \exp\left[\phi(t, u) + x \cdot \psi(t, u) \right], \tag{2.1}$$

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[4]That is, $Y_0 = x$, P^x-almost surely.

[5]A stochastic process Y is stochastically continuous, if for any sequence $t_m \to t$ in $\mathbb{R}_+$, Y_{t_m} converges to Y_t in probability.

for all $x \in D$ and $(t, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$. An affine process Y is called *regular*, if the derivatives

$$F(u) := \partial_t \phi(t, u)|_{t=0^+} \quad , \quad R(u) := \partial_t \psi(t, u)|_{t=0^+}$$

exist for all $u \in \mathcal{U} := \left\{ u \in \mathbb{C}^d : \mathrm{Re}(u_I) \leq 0, \ \mathrm{Re}(u_J) = 0 \right\}$ and are continuous in $u = 0$.[6]

Remarks 2.2. (i) We note that whenever $T \geq t$, the definition of an affine process Y implies that the $\mathcal{F}_t$-conditional characteristic function of Y_T is an affine function of Y_t, that is,

$$E\left[\exp\left(u \cdot Y_T\right) \mid \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + Y_t \cdot \psi(\tau, u)\right], \tag{2.2}$$

for all $(\tau, u) \in \mathbb{R}_+ \times i\mathbb{R}^d$, where $\tau := T - t$. The affine property will be used in this form in our argumentations. (ii) We can always choose a càdlàg version of Y [7], and a regular affine process stopped at T is in particular a classical semimartingale.[8] Hence the local Q-martingale S from (1.7) is well defined as an integrator in the sense of [32, Chapter II and IV]. (iii) The functions F and R are called functional characteristics of Y. They completely characterize a regular affine process and the functions ϕ and ψ satisfy so-called generalized Riccati equations strongly connected to F and R. This holds, because F and R are of the well-known Lévy-Khintchine form[9]. Also affiliated with an affine process Y are its so-called admissible parameters[10] determining its generator and its functional characteristics. ◆

The rather technical proposition stating the details of the last point in Remark 2.2 can be found in Section A.1 of the appendix. We omit it here, since it is only needed for the proofs, which can be found in the appendix as well.

Although the special form of the log-characteristic function of an affine process perfectly lends itself to tractable computations, we need to consider a class of processes for which formulas (2.1) or (2.2) extend to a broader subspace of $\mathbb{C}^d$ than $i\mathbb{R}^d$.[11] Therefore, we introduce the notion of an analytic affine process:

Definition 2.3 (Analytic affine process). For Y a regular affine process, let us, for each $t \geq 0$, define the following sets:

$$\mathcal{D}_t := \left\{ y \in \mathbb{R}^d : \sup_{0 \leq s \leq t} E^x\left[\exp(y \cdot Y_s)\right] < \infty \ \text{for all} \ x \in D \right\},$$

$$\mathcal{D}_{t^+} := \bigcup_{s > t} \mathcal{D}_s \quad \text{and} \quad \mathcal{D} := (D_{0^+})^\circ \cup \{0\}. \tag{2.3}$$

$\mathcal{D}$ is called the *real domain* of Y; Y is said to be *analytic*, if the interior of $\mathcal{D}$ is non-empty.

Remark 2.4. From now on, we focus on the case where X is the first component of a certain class of d-dimensional affine processes $Y = (X, V^1, \cdots, V^m) \in \mathbb{R} \times \mathbb{R}_+^m$. This means, the earlier introduced index sets I and J equal $J = \{1\}$ and $I = \{2, \cdots, d\}$, respectively. ◆

We further introduce the following notation. Whenever we consider some subset $\mathcal{S}$ of $\mathbb{R}_+ \times \mathbb{R}^d$, the so-called *tube domain* $\mathcal{S}_\mathbb{C}$ affiliated with $\mathcal{S}$ is given by

$$\mathcal{S}_\mathbb{C} := \left\{ (t, u) \in \mathbb{R}_+ \times \mathbb{C}^d : (t, \mathrm{Re}(u)) \in \mathcal{S} \right\}.$$

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[6] In the recent work [30], the authors actually showed that each affine process as defined above is regular, whereas in [14] and [28] regularity was still an assumption on Y.

[7] It was shown in [28] that an affine process is a Feller process and hence admits for a càdlàg version. Compare [33, Chapter III].

[8] Compare [14, Theorem 2.12].

[9] See [32, Chapter I, Theorem 43].

[10] A definition can be found in [14, Page 991].

[11] By extension it is meant that the functions ϕ and ψ can be uniquely analytically extended to a suitable subspace of $\mathbb{R}_+ \times \mathbb{C}^d$.

We chose to introduce this notation, because it was shown in [28, Chapter 3] that the functions ϕ and ψ characterizing the process Y have unique extensions to analytic functions on the interior $\mathcal{E}_{\mathbb{C}}^{\circ}$ of the tube domain $\mathcal{E}_{\mathbb{C}}$, where $\mathcal{E}$ is defined by

$$\mathcal{E} := \{(t,v) \in \mathbb{R}_+ \times \mathbb{R}^d \ : \ v \in \mathcal{D}_{t+}\}.$$

Those extensions still satisfy the aforementioned Riccati equations and (2.1) and (2.2) extend to $\mathcal{E}_{\mathbb{C}}$.[12] We are now ready to state the main result of this section—a combination of the properties of analytic affine processes, and the earlier provided equilibrium pricing density, which yields a semi-explicit representation for the equilibrium price process S.

Theorem 2.5 (Main Theorem). *Let $Y = (X, V^1, \cdots, V^m)$ be an analytic affine process with state space $\mathbb{R} \times \mathbb{R}_+^m$, as in Definitions 2.1 and 2.3. Suppose that there are K securities $S^1, \cdots, S^K$ with terminal conditions*

$$S_T^k = f^k(X_T),\tag{2.4}$$

for payoff functions $f^k : \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, K$. Suppose further that the functions f^k and the parameters n^k, η^k and γ are such that there exist $(\alpha^k)_{k=1}^K, \beta \in \mathbb{R}$ satisfying:

1. *The functions*

$$g^k(x) := \exp\left(\alpha^k x\right) f^k(x) \exp\left(-\gamma \sum_{k=1}^K (n^k + \eta^k) f^k(x)\right),\tag{2.5}$$

$$h(x) := \exp\left(\beta x\right) \exp\left(-\gamma \sum_{k=1}^K (n^k + \eta^k) f^k(x)\right),\tag{2.6}$$

and

$$\hat{g}^k(s) = \int_{\mathbb{R}} \mathrm{e}^{-isy} g^k(y) dy \qquad \text{and} \qquad \hat{h}(s) = \int_{\mathbb{R}} \mathrm{e}^{-isy} h^k(y) dy$$

are integrable for all k. Here, $\hat{g}^k$ and $\hat{h}^k$ denote the Fourier transforms of the functions g^k and h^k, respectively.

2. *In addition,*

$$\left(T, (-\alpha^k, 0_{d-1})\right) \in \mathcal{E} \quad , \quad (T, (-\beta, 0_{d-1})) \in \mathcal{E},\tag{2.7}$$

where, for $z \in \mathbb{R}$, we denote by $(z, 0_{d-1})$ the d-dimensional vector $(z, 0, \cdots, 0)$.

Then we have that, with $\tau := T - t$, the equilibrium price of S at time t is a function of τ and the current state of the process Y and the price of the k-th security at time $t \in [0, T]$ is given by

$$S_t^k = \frac{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (-\alpha^k + is, 0_{d-1})\big) + \psi\big(\tau, (-\alpha^k + is, 0_{d-1})\big) \cdot Y_t\right] \hat{g}^k(s)\, ds}{\int_{\mathbb{R}} \exp\left[\phi\big(\tau, (-\beta + is, 0_{d-1})\big) + \psi\big(\tau, (-\beta + is, 0_{d-1})\big) Y_t\right] \hat{h}(s)\, ds}\, .\tag{2.8}$$

Here, ϕ and ψ denote the analytic extensions of the functions introduced in Definition 2.1.

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[12]More precisely, it is first shown [28, Lemma 3.12] that this holds on the set $\{(t,u) \in \mathcal{E}_{\mathbb{C}} \ : \ \left|E^0\left[\exp(u \cdot Y_s)\right]\right| \neq 0 \ \forall s \in [0,t)\}$, whereas [28, Lemma 3.19] then states that both sets coincide.

Remarks 2.6. (i) It will be clear from the proof of Theorem 2.5 that (2.7) is sufficient for ensuring that (1.5) is satisfied and hence the stochastic integral with respect to (1.7) is well defined. (ii) Note that the choice of $(\alpha^k)_{k=1}^K$ and β does of course depend on γ, n and η, the aggregated parameters of the underlying model. ♦

Fourier transform methods are used for the calculation of both, the numerator and the denominator in the pricing formula (2.8). We now present a slightly different method using only the denominator and a partial derivative argument with respect to the following parameter:

Definition 2.7. We denote the *supply-adjusted risk aversion* of the representative agent $\tilde{\gamma} \in \mathbb{R}^K$ by

$$\tilde{\gamma} = (\tilde{\gamma}^1, \cdots, \tilde{\gamma}^K) := \gamma(n + \eta) = \big(\gamma(n^1 + \eta^1), \cdots, \gamma(n^K + \eta^K)\big) \,.$$

Proposition 2.8. *Let* (2.4)*,* (2.6) *and the second part of* (2.7) *from Theorem 2.5 hold true. In particular there exist a suitable β and accordingly defined functions h and $\hat{h}$, which, in view of Remark 2.6, depend on $\tilde{\gamma}$. That is,*

$$\beta = \beta(\tilde{\gamma}), \;\; h(x) = h(\tilde{\gamma}, x), \;\; \hat{h}(s) = \hat{h}(\tilde{\gamma}, s) \,.$$

Then the equilibrium price process of S at time t is given by

$$S_t^k = \frac{\partial}{\partial y^k}\, v(y)\big/v(-\tilde{\gamma})\,\bigg|_{y=-\tilde{\gamma}} \,, \tag{2.9}$$

where for, $y \in \mathbb{R}^K$, the function v is given by

$$v(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[\phi\Big(\tau, \big(-\beta(-y) + is, 0_{d-1}\big)\Big) + \psi\Big(\tau, \big(-\beta(-y) + is, 0_{d-1}\big)\Big) \cdot Y_t \right] \hat{h}(-y, s)\, ds \,.$$

The special case, where only one security S_T^k with functional relation $f^k(x) = x$ affects (1.6), is covered by the following corollary, with no need of Fourier transformation methods.

Corollary 2.9. *Let the process Y and the functions ϕ and ψ be as in Theorem 2.5. Let us further assume that there is only one security, denoted by S_T^1, affecting the equilibrium pricing density* (1.6) *with terminal payoff $S_T^1 = X_T$. If $\tilde{\gamma}$ satisfies $\big(T, (-\tilde{\gamma}^1, 0_{d-1})\big) \in \mathcal{E}$, then the equilibrium price process S^1 is given by*

$$S_t^1 = \big[\partial_{u_1} \phi(\tau, u) + \partial_{u_1} \psi(\tau, u) \cdot Y_t \big]\, \big|_{u=(-\tilde{\gamma}^1, 0, \cdots, 0)} \,, \quad t \in [0, T] \,. \tag{2.10}$$

2.1.2. Pricing of call options

In this section we establish semi-explicit pricing formulas for European call options. The main task will be to find suitable "flattening" functions such that the Fourier methods of Theorem 2.5 can be applied. We embed this in our framework by thinking of an underlying asset and (possibly multiple) options as $N + 1$ securities $S, S^1, \cdots, S^N$ with terminal conditions $S_T = X_T, S_T^1 = (X_T - K_1)^+, \cdots, S_T^N = (X_T - K_N)^+$, for strikes $K_1, \cdots, K_N$. There is no need for specifying the underlying affine process yet. We denote the equilibrium price process of the underlying security by S and the ones of the options by C^k for $k = 1, \cdots, N$. The securities in total net supplies $(n^S + \eta^S, n^1 + \eta^1, \cdots, n^N + \eta^N)$. We first state the pricing formula for the most general case of multiple simultaneously traded options in non-zero net supply, a direct application of Theorem 2.5. Following this, two special cases, namely of a single option

affecting the pricing measure and of all options in zero total net supply are dealt with.

1) *Multiple, simultaneously traded options*

We first examine the general case where there are a finite number $N > 0$ of call options traded. We assume that the total net supply of all options $n^C = \sum_{i=1}^N (n^i + \eta^i)$ is equally split among the options with different strikes such that the net supply of each option equals n^C/N.

Remark 2.10. For the rest of this section all the supply parameters are without loss of generality chosen such that $\tilde\gamma$ from Definition 2.7 satisfies $\tilde\gamma = (\gamma, \cdots, \gamma)$. ♦

We order the strikes of the non-redundant options such that, without loss of generality, $K_1 < \ldots < K_N$. We deduce from (1.6) and Remark 2.10 that the pricing measure is obtained by

$$\frac{dQ}{dP} = \frac{\exp\left(-\gamma\left(S_T + \sum_{i=1}^N (S_T - K_i)^+\right)\right)}{E\left[\exp\left(-\gamma\left(S_T + \sum_{i=1}^N (S_T - K_i)^+\right)\right)\right]}. \tag{2.11}$$

Theorem 2.5 immediately allows us to state the following:

Theorem 2.11. *Whenever α and β satisfy $\gamma < \alpha, \beta < (N+1)\gamma$, the equilibrium price of the underlying security S at time $t \in [0, T]$ is given by*

$$S_t = \frac{\int_{\mathbb{R}} \exp\left[\phi(\tau, (-\alpha + is, 0)) + \psi_1(\tau, (-\alpha + is, 0_m))X_t + \psi_I(\tau, (-\alpha + is, 0_m)) \cdot V_t\right] \hat g(s)ds}{\int_{\mathbb{R}} \exp\left[\phi(\tau, (-\beta + is, 0)) + \psi_1(\tau, (-\beta + is, 0_m))X_t + \psi_I(\tau, (-\beta + is, 0_m)) \cdot V_t\right] \hat h(s)ds},$$

and the price of the k-th call option is given by

$$C_t^k = \frac{\int_{\mathbb{R}} \exp\left[\phi(\tau, (is, 0_m)) + \psi_1(\tau, (is, 0_m))X_t + \psi_I(\tau, (is, 0_m)) \cdot V_t\right] \hat g^k(s)ds}{\int_{\mathbb{R}} \exp\left[\phi(\tau, (-\beta + is, 0_m)) + \psi_1(\tau, (-\beta + is, 0_m))X_t + \psi_I(\tau, (-\beta + is, 0_m)) \cdot V_t\right] \hat h(s)ds},$$

for $k = 1, \ldots, N$. Here the functions $\hat g$, $\hat g^k$ and $\hat h$ are given by

$$\hat g(s) = \sum_{j=1}^N \exp\left(\gamma \sum_{k=1}^{j-1} K_k\right) \exp\left[(-is + \alpha - j\gamma)K_j\right] \left[\left(\frac{-K_j\gamma}{(-is + \alpha - j\gamma)(-is + \alpha - (j+1)\gamma)}\right)\right.$$
$$\left. + \left(\frac{1}{(-is + \alpha - (j+1)\gamma)^2} - \frac{1}{(-is + \alpha - j\gamma)^2}\right)\right],$$

$$\hat h(s) = \sum_{j=1}^N \exp\left(\gamma \sum_{k=1}^{j-1} K_k\right) \exp\left[(-is + \beta - j\gamma)K_j\right] \left[\frac{-\gamma}{(-is + \beta - j\gamma)(-is + \beta - (j+1)\gamma)}\right],$$

$$\hat g^k(s) = \exp\left(\gamma \sum_{h=1}^{k-1} K_h\right) \exp\left[(-is - k\gamma)K_k\right] \left[\frac{1}{(-is - (k+1)\gamma)^2}\right]$$
$$+ \sum_{j=k+1}^N \exp\left(\gamma \sum_{h=1}^{j-1} K_h\right) \exp\left[(-is - j\gamma)K_j\right] \left[\left(\frac{-(K_j - K_k)\gamma}{(-is - j\gamma)(-is - (j+1)\gamma)}\right)\right.$$
$$\left. + \left(\frac{1}{(-is - (j+1)\gamma)^2} - \frac{1}{(-is - j\gamma)^2}\right)\right].$$

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Remark 2.12. Any suitable choice for the damping factors under the restriction $\gamma < \alpha, \beta < (N+1)\gamma$ and assuring that the functions g, g^k, h of (2.5) and (2.6) allow for an integrable Fourier transform, is possible. In all preceding equations involving the functions ϕ and ψ, the parameters of the underlying affine model have to be chosen such that the expressions are well defined in the sense that (2.7) is satisfied and hence (2.2) applies. The details will be discussed in Sections 2.1.3 and 2.1.4. ♦

2) *Case of a single option*

Let us briefly consider the special case where, there is only one option with strike $K > 0$ influencing the pricing measure. Expression (2.11) and the Fourier transforms from Theorem 2.11 then simplify. The equilibrium prices S_t and C_t at time $t \in [0, T]$ can be computed analogously to Theorem 2.11, where the functions $\hat{g}, \hat{g}^1$ and $\hat{h}$ are in this case given by

$$\hat{g}(s) = \exp\left[(\alpha - \gamma - is)K\right] \left[\frac{-K\gamma}{(-is - \gamma + \alpha)(-is - 2\gamma + \alpha)}\right.$$
$$\left. + \left(\frac{1}{(\alpha - 2\gamma - is)^2} - \frac{1}{(\alpha - \gamma - is)^2}\right)\right].$$

$$\hat{h}(s) = \exp\left[(\beta - \gamma - is)K\right] \left(\frac{-\gamma}{(\beta - \gamma - is)(\beta - 2\gamma - is)}\right).$$

$$\hat{g}^1(s) = \exp\left[-(is + \gamma)K\right] \frac{1}{(-is - 2\gamma)^2}.$$

3) *Options in zero-net supply*

Let us now consider the simplest case, where options are in zero net supply and not occurring in the agents' endowment. This ensures that they do not affect the pricing density (1.6) and hence do not alter the equilibrium price (2.10) of S in Corollary 2.9. So the task of pricing an arbitrary call option written on S with maturity T, strike K and payoff $C_T = (S_T - K)^+ = (X_T - K)^+$ consists of finding a suitable α corresponding to the weighted payoff function of the option (2.5) in Theorem 2.5. The simple choice $\alpha^1 = 0$ suffices for the Fourier-transform

$$\hat{g}^1(s) = \frac{1}{(is + \gamma)^2} \exp\left[-K(is + \gamma)\right]$$

of the function $g^1(x) := e^{-\gamma x}(x - K)^+$ to be integrable. Thus,

$$C_t(\tau, K) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left[\left(\Delta_\tau^{is,\gamma}(\phi) + \Delta_\tau^{is,\gamma}(\psi_1)X_t + \Delta_\tau^{is,\gamma}(\psi_I) \cdot V_t\right)\right] \hat{g}^1(s) ds,$$

for $t \in [0, T]$, and where $\Delta_\tau^{is,\gamma}(f) := f\left(\tau, (is, 0_m)\right) - f\left(\tau, (-\gamma, 0_m)\right)$, for functions on $\mathbb{R}_+ \times \mathbb{C}^{m+1}$.

2.1.3. The equilibrium dynamics in the Heston setting

By choosing the dynamics of Y according to the well-known Heston model, compare with [22], it is possible to derive explicit formulae for the equilibrium dynamics of S. Suppose $Y = (X, V)$ evolves according to

$$\begin{array}{rcll}
dX_t &=& \mu dt + \sqrt{V_t}\, dW_t^1, & X_0 = x_0 \\
dV_t &=& (\kappa - \lambda V_t)dt + \sigma\sqrt{V_t}\, dW_t^2, & V_0 = v_0.
\end{array} \qquad (2.13)$$

Here, $W = (W^1, W^2)$ denotes a two-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t \in [0,T]}$. The parameters $\mu, \kappa, \lambda, \sigma > 0$ will be chosen appropriately later on. We initially assume that the agents

are trading a single security S, the equilibrium dynamics of which we want to derive. The security is in net supply n and total endowment supply η. In view of Remark 2.10, we set $n = 1$ and $\eta = 0$ in the following. Since the additive Heston model is analytic affine and allows for explicit solutions of the functions ϕ and ψ, we can apply the results obtained in Sections 1 and 2.1 to compute the equilibrium price S_t at time $t \in [0, T]$ in closed form as a function of (X_t, V_t).

Theorem 2.13. *Let $\theta(\gamma)$ be defined by*

$$
\theta(\gamma) = \begin{cases} \sqrt{\lambda^2 - \sigma^2 \gamma^2} & \text{if } \gamma < \frac{\lambda}{\sigma} \\ i\sqrt{\sigma^2 \gamma^2 - \lambda^2} & \text{if } \gamma > \frac{\lambda}{\sigma} \end{cases} .
$$

We suppose that γ is such that T satisfies

$$
T < \begin{cases} +\infty & \gamma < \frac{\lambda}{\sigma} \\ \frac{2}{|\theta(\gamma)|}\left(\arctan\frac{|\theta(\gamma)|}{-\lambda} + \pi\right) & \gamma > \frac{\lambda}{\sigma} \end{cases} . \tag{2.14}
$$

Then we have that, with $\tau := T - t$, $\theta := \theta(\gamma)$ and $\theta' := \frac{\partial}{\partial \gamma}\theta(\gamma)$, the equilibrium price process S is given by

$$
S_t = T(\tau, \gamma) + X_t - \gamma \frac{N(\tau, \gamma)}{D(\tau, \gamma)} V_t, \tag{2.15}
$$

for $t \in [0, T]$, and where

$$
T(\tau, \gamma) = \frac{2\kappa}{\sigma^2 \theta\big(\theta(e^{\theta\tau} + 1) + \lambda(e^{\theta\tau} - 1)\big)}\left[\Big(\theta(e^{\theta\tau} + 1) + \lambda(e^{\theta\tau} - 1)\Big)\Big(\theta' - \frac{1}{2}\sigma^2\gamma\tau\Big)\right.
$$
$$
\left. - \theta\Big(\theta'(e^{\theta\tau} + 1) + \tau e^{\theta\tau}(\lambda\theta' - \gamma\sigma^2)\Big)\right],
$$

$$
N(\tau, \gamma) = \Big[\theta\left(e^{\theta\tau} + 1\right) + \lambda\left(e^{\theta\tau} - 1\right)\Big]\Big[2\left(e^{\theta\tau} - 1\right) - \gamma\tau\theta'e^{\theta\tau}\Big]
$$
$$
+ \gamma\left(e^{\theta\tau} - 1\right)\Big[\theta'\left(e^{\theta\tau} + 1\right) + \tau e^{\theta\tau}\left(\lambda\theta' + \gamma\sigma^2\right)\Big],
$$

$$
D(\tau, \gamma) = \Big[\theta\left(e^{\theta\tau} + 1\right) + \lambda\left(e^{\theta\tau} - 1\right)\Big]^2.
$$

Next we illustrate the effect of the parameters $\tilde{\gamma}$ and σ on implied volatilities in a setting with one underlying asset and fifteen simultaneously traded call options written on it. In Figure 1, four different implied volatility curves are shown, corresponding to four different values for the risk aversion $\tilde{\gamma}$. Especially on the left side, that is, for in-the-money options, higher risk aversion yields a higher level of implied volatility. The more risk averse the representative agent is, the more in-the-money options are appreciated as good hedges against possibly low values of the underlying. The implied voltatility curves for two different choices of the vol-of-vol parameter σ in (2.13) are shown in Figure 2. We observe a significant increase in implied volatility when changing from the low value (blue curve) to the higher one (red curve). That is due to the fact that a high value of σ increases the probability of S_T taking on extreme tail values and hence rendering even out-of-the-money options attractive instruments.[13] We emphasize here that, unlike many standard models, this setting enables one to considering the case of multiple simultanously traded options.

---

[13] For the Figures 1 and 2, the following parameters were used for the numerical computations: $\mu = 0.1$, $\kappa = 0.006$, $\lambda = 0.2$, $T = 0.5$, $t = 0$, $(x_0, v_0) = (1, 0.03)$. In Figure 1, we set $\sigma = 0.3$, whereas in Figure 2, $\tilde{\gamma} = 0.2$ was used. We considered an underlying together with 15 simultanously traded call options.
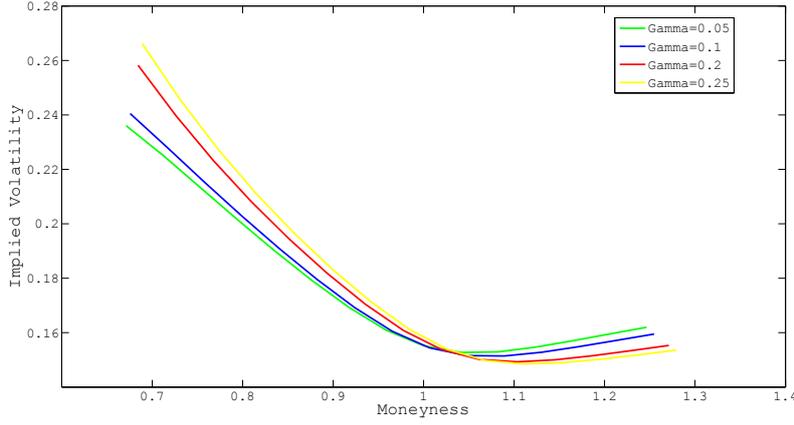
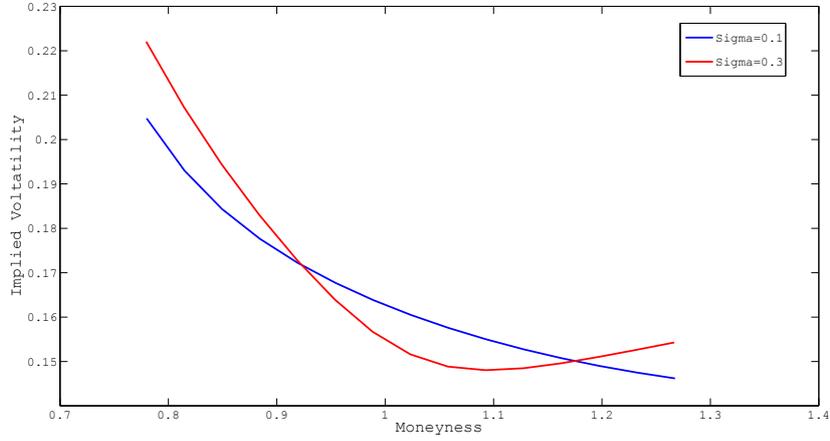Figure 1: **Implied volatility curves with varying risk aversion**

Figure 2: **Implied volatility curves with varying vol-of-vol**

2.1.4. Equilibrium dynamics in the pure jump Ornstein-Uhlenbeck setting

We consider $S_T = X_T$, where X is a one-dimensional analytic affine process. A classical example, which we want to include in our equilibrium framework, is that of an Ornstein-Uhlenbeck process with a pure jump component as Lévy part.[14] More precisely,

$$dX_t = -\lambda(X_t - \mu)dt + dJ_t \quad , \quad X_0 = x_0 , \tag{2.16}$$

where J is an adapted compound Poisson process with intensity $\kappa > 0$ and jump distribution $\nu(dx) = \frac{1}{2}\theta \exp(-\theta |x|)dx$.[15] Here, μ and λ describe the long term mean and the mean reversion rate, respectively. The security is in net supply n and total endowment supply η.

---

[14]This model was used among others in [13] or [29].

[15]More precisely, $J_t = \sum_{i=0}^{N_t} b_i D_i$, where N_t is a Poisson process with intensity κ, D_i are exponentially distributed i.i.d. random variables with jumps of mean $\frac{1}{\theta} > 0$, and b_i are i.i.d. Bernoulli random variables with $P[b_1 = 1] = P[b_1 = -1] = 0.5$.

It is well-known and easily computed that in this one-dimensional setting (A.1) and (A.2), the equations for the functional characteristics F and R, become

$$F(u) = \lambda\mu u + \frac{\kappa\, u^2}{\theta^2 - u^2} \qquad \text{and} \qquad R(u) = -\lambda u\,. \tag{2.17}$$

Combining (2.17) with (A.4) and (A.5) we deduce that the functions ϕ and ψ satisfy the following system of Riccati equations

$$\begin{aligned}
\partial_t\phi(t,u) &= \lambda\mu\psi(t,u) + \frac{\kappa\,\psi^2(t,u)}{\theta^2-\psi^2(t,u)} &,&\quad \phi(0,u)=0 \\
\partial_t\psi(t,u) &= -\lambda\psi(t,u) &,&\quad \psi(0,u)=u\,,
\end{aligned}$$

which allows for the explicit solutions

$$\phi(t,u) = \frac{\kappa}{2\lambda}\log\left(\frac{\theta^2 - u^2\mathrm{e}^{-2\lambda t}}{\theta^2 - u^2}\right) - \mu u(\mathrm{e}^{-\lambda t}-1) \qquad \text{and} \qquad \psi(t,u) = u\mathrm{e}^{-\lambda t}\,,$$

such that formula (2.1) holds, as long as $u \in \mathbb{R}\setminus\{-\theta,\theta\}$ and $T < t^*(u)$, with

$$t^*(u) = \begin{cases} +\infty & |u| < \theta \\ -\frac{1}{2\lambda}\log(\frac{\theta^2}{u^2}) & |u| > \theta \end{cases}\,. \tag{2.18}$$

This, together with Corollary 2.9 allows us to formulate the following:

Proposition 2.14. *If $-\tilde{\gamma} < \theta$ and $T < t^*(-\tilde{\gamma})$, where t^* is as in (2.18), then, with $\tau := T - t$, the equilibrium price process S is given by*

$$S_t = \left[\frac{\kappa\theta\tilde{\gamma}}{\lambda}\left(\frac{\mathrm{e}^{-2\lambda\tau}-1}{\theta^2-\tilde{\gamma}^2}\right) + \mu(1-\mathrm{e}^{-\lambda\tau})\right] + \mathrm{e}^{-\lambda\tau}X_t\,, \quad t \in [0,T]\,.$$

Note that the equilibrium market price of risk of S strongly depends on the jump parameter and the supply-adjusted risk aversion.

We briefly discuss the influence the jump parameters, like jump intensity and mean jump height, have on the implied volatility of call options written on an underlying, the payoff of which is modelled by (2.16). In Figure 3 two implied volatility curves are shown. The red curve corresponds to smaller jumps arriving at a high freqency $((\kappa, \frac{1}{\theta}) = (30, \frac{1}{30}))$, whereas the blue one was obtained considering higher jumps at a lower frequency $((\kappa, \frac{1}{\theta}) = (20, \frac{1}{20}))$.[16] Increasing the mean jump height distinctly lifts the level of implied volatility, since the probability of S_T taking on extreme values is higher that way. We further note that in general an affine model including jumps seems more suitable to reproduce the right-hand side smile observed in real market data.

2.2. Application of the information-based asset pricing

In recent years Brody et al. [4, 5] and Hoyle et al. [25] have developed an asset pricing approach, based on the modelling of cash flows and the explicit construction of market filtrations, predominantly in finite time. Such a framework can be naturally applied in the context of equilibrium pricing considered in the present paper. In what follows, we recall some of the features of the information-based pricing approach.

---

[16]The remaining parameters were chosen as $(\mu, \lambda, \tilde{\gamma}, T, t, x_0) = (1, 2, 0.2, 0.1, 0, 1)$. As before, we considered 15 simultanously traded call options.
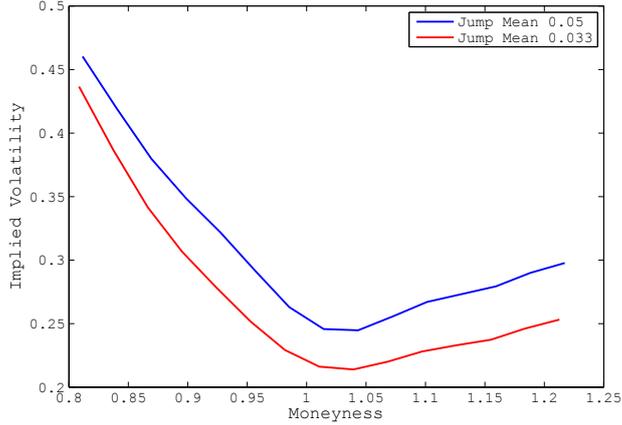
Figure 3: **Implied volatility curves with varying jump mean and intensity**

One of the characterising properties of information-based asset pricing is that, instead of assuming from the outset some abstract filtration representing the information available to the market, processes carrying market-relevant information are explicitly constructed, and a distinction between "genuine" information and market noise is made. The equilibrium dynamics S is then computed by using (i) the special form of the pricing measure Q obtained in Section 1, (ii) the *a priori* distribution of the terminal payoff S_T, and (iii) the updated *a posteriori* distribution of S_T obtained by a version of Bayes formula.

We assume that the probability space $(\Omega, \mathcal{F}, P)$ supports a N-dimensional Brownian motion B together with N independent random variables $(X_i)_{i=1}^N$, all independent of B. The X_i are so-called market factors determining the future cash flow of the assets at time T, that is, there are K functions $F^k : \mathbb{R}^N \to \mathbb{R}_+$ such that $S_T^k = F^k(X_1, \cdots, X_N)$.[17] As an example, we later consider $K = N = 1$ and $S_T = X$. We assume that the agents have an *a priori* idea of the probability distribution of the market factors (and hence of their terminal payoff) and denote the initial distributions of the X^i by ν^i, that is, $\nu^i(dx) = P[X_i \in dx]$.[18]

With each market factor X_i we associate an *information process* ξ^i on $[0,T]$ defined by

$$\xi_t^i = \sigma_i X_i t + \beta_t^i, \quad t \in [0,T], \tag{2.19}$$

where the independent standard Brownian bridges β^i on $[0,T]$ are defined in terms of B as solutions to the SDEs[19]

$$d\beta_t^i = \frac{-\beta_t^i}{T-t}dt + dB_t^i, \tag{2.20}$$

for $t \in [0,T)$ and $\beta_T^i = 0$. Looking at the different components of the processes (2.19), we can distinguish between the part $\sigma^i X_i t$ containing real information about the realization of a market factor revealed over

---

[17]Of course not all securities have to depend necessarily on all available market factors. One could also connect with each S^k the number m^k of market factors determining *exclusively* its payoff structure and then set $N = (\sum_k m^k + \#\text{common factors})$. Securities sharing common market factors are thereby correlated.

[18]Due to the independence of the single market factors the joint distribution factorizes and we get that $P[S_T^k \in dx] = \int \cdots \int F^k(x_1, \ldots, x_N)\nu^i(dx_1) \cdots \nu^N(dx_N)$. As soon as we consider correlated market factors, the agents should have a priori ideas about the joint distributions, see Section 2.2.4.

[19]This is a well defined continuous process allowing for the expression $\beta_t^i = (T-t)\int_0^t \frac{1}{T-s}dB_s^i$ on $[0,T)$, compare [32].

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time, and the bridge part representing market noise such as rumors, bubbles, innuendos et cetera. The speed at which the true outcome of X_i is revealed, is governed by the so-called *information rate* σ_i.

We assume that the information processes ξ^is describe the flow of information available to the market agents, and thus generate the market filtration. We set:

$$\mathcal{F}_t = \sigma\left(\xi_s^1, \ldots, \xi_s^N, s \le t\right), \quad t \in [0, T]. \tag{2.21}$$

By construction, the security vector S_T is $\mathcal{F}_T$-measurable, since the noise term in the information process vanishes at time T. Hence, at each time $t \in [0, T]$, the equilibrium price S_t will be determined based on the preference structure of the single agents as in (1.1), which in turn relies on their current state of information about the future outcome of the market factors.

Remark 2.15. The information process introduced in (2.19) is only one of many possible. More generally one could consider information processes of the form

$$I_t = \int_0^t v(s, X_1, \ldots, X_N)ds + B_t \tag{2.22}$$

and set $\mathcal{F}_t = \sigma(I_s, s \le t)$, see Filipović, Hughston and Macrina [18]. The function v corresponds to the signal carrying information about the market factors, whereas B represents market noise. The information process (2.19) is obtained by choosing $v_i(s, X_i) = \sigma_i T X_i / (T - s)$ and by setting

$$\xi_t^i = (T - t) \int_0^t \frac{1}{T - u} dI_u^i. \qquad \blacklozenge$$

The results obtained in Section 1 now yield the following Theorem.

Theorem 2.16. *Assume that all a priori distributions ν^i allow for a density with respect to the Lebesgue-measure denoted by $v^i(x)$, respectively. For $t < T$, the equilibrium price process of the k-th security is given by*

$$S_t^k = \frac{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} z(x_1, \ldots, x_N) F^k(x_1, \ldots, x_N) \pi_t^1(x_1) \cdots \pi_t^N(x_N) dx_1 \cdots dx_N}{\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} z(x_1, \ldots, x_N) \pi_t^1(x_1) \cdots \pi_t^N(x_N) dx_1 \cdots dx_N}, \tag{2.23}$$

where the function z is defined by

$$z(\cdot) = \exp\left[-\gamma \sum_k \left(n^k + \eta^k\right) F^k(\cdot)\right]. \tag{2.24}$$

The regular conditional density function π_t^i associated with the i-th market factor is given by

$$\pi_t^i(x) = \frac{v^i(x) \exp\left[\frac{T}{T-t}\left(\sigma_i x \xi_t^i - \frac{1}{2}(\sigma_i x)^2 t\right)\right]}{\int_{\mathbb{R}} v^i(y) \exp\left[\frac{T}{T-t}\left(\sigma_i y \xi_t^i - \frac{1}{2}(\sigma_i y)^2 t\right)\right] dy}. \tag{2.25}$$

Proof. Recall that the equilibrium price is obtained by the change of measure from P to Q, that is:

$$S_t^k = E_Q\left[S_T^k \mid \mathcal{F}_t\right] = E_Q\left[F^k(X_1, \cdots, X_N) \mid \mathcal{F}_t\right] = E\left[\frac{dQ}{dP} F^k(X_1, \cdots, X_N) \mid \mathcal{F}_t\right] E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right]^{-1}.$$

By (1.6), we know that $\frac{dQ}{dP}$ is a function of S_T and hence of $X_1, \ldots, X_N$, which is given in (2.24). It remains to compute the regular conditional distribution of $(X_1, \ldots, X_N)$ given $(\xi_t^1, \ldots, \xi_t^N)$. Using the independence of the market factors, the Markov property of ξ, the Bayes formula, and observing that, given $(X_1, \ldots, X_N) = (x_1, \ldots, x_N)$, ξ_t^i is Gaussian with mean $\sigma_i x_i t$ and variance $\frac{tT}{T-t}$, yields (2.25). $\square$

Remarks 2.17. (i) We note that the independence assumption on the market factors may be relaxed, see Section 2.2.4. A natural model for correlated securities can readily be obtained by letting them depend on common market factors. ii) The choice of Brownian bridges as noise processes may be generalized to so called Lévy Random Bridges, see [25], allowing for instance the use of increasing information processes built on gamma bridges and reflecting the current idea about an accumulated dividend stream. (iii) In most cases, the semi-explicit integral formulae (2.23) and (2.25) have to be computed numerically. An example where it can be worked out explicitly is considered in Section 2.2.3. $\blacklozenge$

2.2.1. Innovation Processes and Equilibrium Market Price of Risk

Let us from now on consider the case $K = N = 1$, and in particular the case $S_T = X$ with corresponding information process given by $\xi_t = \sigma X t + \beta_t$ for $t \in [0, T]$. We denote by $\tilde{\gamma}$ the supply adjusted aggregated risk aversion as in Definition 2.7. Formula (2.23) reduces to

$$S_t = \frac{E\left[S_T \exp\left(-\tilde{\gamma} S_T\right) \mid \mathcal{F}_t\right]}{E\left[\exp\left(-\tilde{\gamma} S_T\right) \mid \mathcal{F}_t\right]} = \frac{\int x \exp\left(-\tilde{\gamma} x\right) \pi_t(x) dx}{\int \exp\left(-\tilde{\gamma} x\right) \pi_t(x) dx} . \tag{2.26}$$

Results from general non-linear filtering theory assure the existence of a P-Brownian motion W on $[0, T)$, *adapted* to the filtration generated by ξ. Observe to this end that rearranging (2.20) leads to the following SDE satisfied by ξ on $[0, T)$

$$d\xi_t = \left[\frac{1}{T-t}\left(\sigma T E\left[X \mid \xi_t\right] - \xi_t\right)\right] dt + dB_t . \tag{2.27}$$

Hence W is the innovation process associated with the information ξ given by

$$W_t = \xi_t - \int_0^t \left[\frac{1}{T-s}\left(\sigma T E\left[X \mid \mathcal{F}_s\right] - \xi_s\right)\right] ds, \quad t < T . \tag{2.28}$$

Thus, instead of having to assume the existence of Brownian motions as drivers for the prices, they rather emerge naturally from within the information-driven structure. By the Fujisaki-Kallianpur-Kunita Therorem, see [2, Proposition 2.31], both expressions appearing in (2.26) allow for a representation with respect to the innovation Brownian motion. Furthermore, we know the structure of the integrands in the above representations. For every function $h : \mathbb{R} \to \mathbb{R}$ such that $h(X) \in L^2$ and for $t < T$ we obtain that

$$dE\left[h(X) \mid \mathcal{F}_t\right] = \frac{\sigma T}{T-t} V_t^h dW_t , \tag{2.29}$$

where
$$V_t^h = E\left[h(X) X \mid \mathcal{F}_t\right] - E\left[h(X) \mid \mathcal{F}_t\right] E\left[X \mid \mathcal{F}_t\right] \tag{2.30}$$

is the conditional covariance of the market factor with the function h. We apply the above to (2.26) with $f(x) = x e^{-\tilde{\gamma} x}$ and $g(x) = e^{-\tilde{\gamma} x}$ and use the Ito product rule to obtain

$$dS_t = \sigma^g \left[\sigma^g \left(S_t (V_t^g)^2 - V_t^f V_t^g\right) dt + \left(V_t^f - S_t V_t^g\right) dW_t\right] , \tag{2.31}$$

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where

$$\sigma^g = \frac{\sigma T}{(T-t)E\left[g(X)\,|\,\mathcal{F}_t\right]}. \tag{2.32}$$

The expressions V_T^f, V_t^g and $E\left[g(X)\,|\,\mathcal{F}_t\right]$ can be worked out semi-explicitly by means of (2.30), the integral formula (2.26), and the regular conditional density $\pi(x)$ defined in (2.25). Each is a function of t and ξ_t due to the Markov property of the information process[20].

2.2.2. Pricing credit-risky securities

In this section we consider a credit-risky security. We assume that the a-priori distribution of $S_T = X$ is discrete on $\{0, 1\}$ and use the following notation for the single a-priori probabilities $p_i = P[X = x_i]$, where $i = 0, 1$ and $x_0 = 0$. Hence, p_0 is the probability of total default. Due to the discrete payoff structure, formula (2.23) simplifies and allows us to examine the impact of model parameters such as the *information flow rate* or the risk aversion and supply on the equilibrium price of S, see also Brody et al. [3]. The price of the defaultable security can be obtained in closed form using (2.23) and is given by

$$S_t = \frac{p_1 x_1 \exp\left(-\tilde{\gamma} x_1\right) \exp\left[\frac{T}{T-t}\left(\sigma x_1 \xi_t - \frac{1}{2}(\sigma x_1)^2 t\right)\right]}{\sum_{i=0,1} p_i \exp\left(-\tilde{\gamma} x_i\right) \exp\left[\frac{T}{T-t}\left(\sigma x_i \xi_t - \frac{1}{2}(\sigma x_i)^2 t\right)\right]}, \quad t < T. \tag{2.33}$$

Applying Itô's product rule to (2.33) together with (2.28) yields the following dynamics[21] for S

$$dS_t = \frac{\sigma T}{T-t}\mathrm{Var}_t^Q(X)\left[\frac{\sigma T}{T-t}\left(E\left[X\,|\,\mathcal{F}_t\right] - S_t\right)dt + dW_t\right], \quad t < T. \tag{2.34}$$

Due to the Markov property of the information process, the terms $E\left[X\,|\,\mathcal{F}_t\right]$ and $\mathrm{Var}_t^Q(X)$ are functions of the triplet (t, T, ξ_t) and quadruplet $(t, T, \tilde{\gamma}, \xi_t)$, respectively. Figure 4 shows the impact of σ on the price of a defaultable bond, where the probability of default was chosen to be $p_0 = P[X = 0] = 0.2$. On the left-hand side the bond does not default, whereas on the right-hand side we considered the situation of a default. In both cases, a low information flow rate (green curve, $\sigma = 0.1$) leads to a rather late adjustment of the equilibrium price process towards the prevailing terminal value, while the red curve ($\sigma = 1$) reacts earlier to the information about the outcome of X.[22]

2.2.3. One-dimensional, exponentially-distributed terminal cash flow

We illustrate how, for particular choices of v and F, the formulae (2.25) and (2.26) can be worked out. We assume $F(x) = x$, corresponding to the assets payoff itself being the market factor. Moreover, the a priori distribution of S_T, the dividend at time T, is assumed to be exponential.

Corollary 2.18. *Assume that the a-priori distribution of $S_T = X$ is of the exponential form, that is, $v(x) = (1_{\{x \geq 0\}}/\kappa) \exp\left(-x/\kappa\right)$ for some $\kappa > 0$. Then the equilibrium price at time $t < T$ is given by*

$$S_t = \left[\frac{\exp\left(-\frac{1}{2}B_t^2/A_t\right)}{\sqrt{2\pi A_t}\,\mathcal{N}(B_t/A_t)} + \frac{B_t}{A_t}\right], \tag{2.35}$$

---

[20]Compare [25, Proposition 2.4].

[21]The expression $\mathrm{Var}_t^Q(X)$ denotes the conditional variance of X under Q.

[22]The following parameters were used for the simulations: $x_1 = 1$, $P[X = x_1] = 0.8$, $T = 5$, $\tilde{\gamma} = 0.6$. The price process is shown for $t \in [0, 4.9]$.
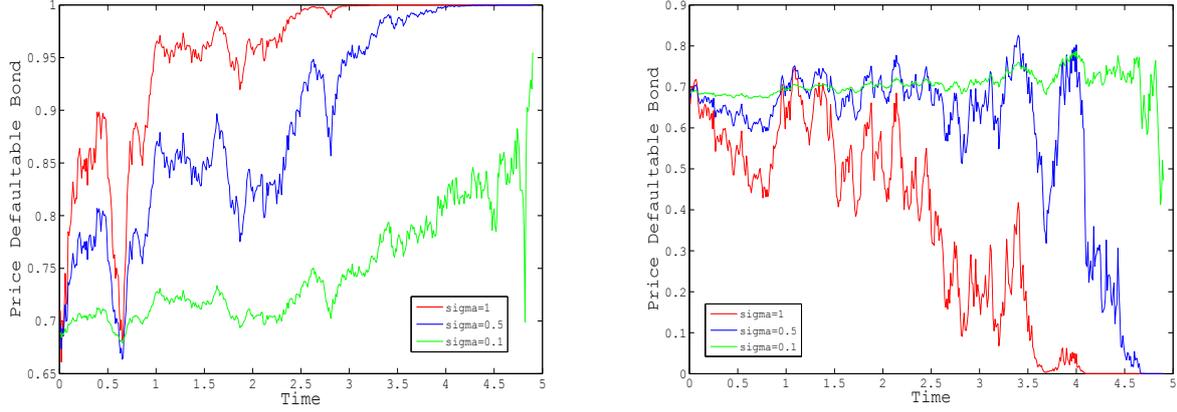
Figure 4: **Defaultable Bond Prices**

where

$$A_t = \sigma^2 t T/(T-t) \quad and \quad B_t = \sigma T \xi_t/(T-t) - \frac{\tilde{\gamma}\kappa + 1}{\kappa}. \tag{2.36}$$

Since the pricing measure depends only on the terminal cash-flow as a consequence of the attainable endowments, changing from P to Q could be interpreted as a different view $\tilde{v}$ of the representative agent on the a-priori-distribution of S_T. More precisely, under Q the cash-flow S_T is exponentially distributed with new parameter $(\tilde{\gamma}\kappa + 1)/\kappa$ also appearing in (2.36), which can be seen by working out the adjusted density

$$\tilde{v}(x) = \frac{e^{-\tilde{\gamma}x}v(x)}{\int e^{-\tilde{\gamma}y}v(y)dy}. \tag{2.37}$$

Formulas (2.35) and (2.36) now follow from [4, Section VII].

2.2.4. Example with dependent market factors

We include an example with one security S depending on two market facors X_1 and X_2, which may be dependent random variables, that is, $S = f(X_1, X_2)$, for some function $f : \mathbb{R}^2 \to \mathbb{R}$ determining the payoff structure. The information processes may now be dependent, though. Within this setup, the agents need to have an a priori estimate of the *joint* distribution of X_1 and X_2, denoted by μ. That is, $P[(X_1, X_2) \in dx \times dy] = \mu(dx \times dy)$. We assume that μ allows for a density ϱ with respect to the Lebesgue measure on $\mathbb{R}^2$ and thus $\mu(dx \times dy) = \varrho(x,y)dxdy$.

Corollary 2.19. *For $t < T$, the equilibrium price process of $S = f(X_1, X_2)$ is given by*

$$S_t = \frac{\int_{\mathbb{R}^2} \exp(-\tilde{\gamma}f(x_1, x_2))f(x_1, x_2)\pi_t(x_1, x_2)dx_1\, dx_2}{\int_{\mathbb{R}^2} \exp(-\tilde{\gamma}f(x_1, x_2))\pi_t(x_1, x_2)dx_1\, dx_2},$$

where the regular conditional density function π_t associated with the dependent market factors is given by

$$\pi_t(x,y) = \frac{\varrho(x,y)\exp\left[\frac{T}{T-t}\left(\sigma_1 x \xi_t^1 - \frac{1}{2}(\sigma_1 x)^2 t\right)\right]\exp\left[\frac{T}{T-t}\left(\sigma_2\, y\, \xi_t^2 - \frac{1}{2}(\sigma_2\, y)^2 t\right)\right]}{\int_{\mathbb{R}^2} \varrho(x,y)\exp\left[\frac{T}{T-t}\left(\sigma_1 x \xi_t^1 - \frac{1}{2}(\sigma_1 x)^2 t\right)\right]\exp\left[\frac{T}{T-t}\left(\sigma_2\, y\, \xi_t^2 - \frac{1}{2}(\sigma_2\, y)^2 t\right)\right]dx\, dy}.$$

A. Proofs and addenda to Section 2

A.1. Regular affine processes

This proposition concerning the characterization of an affine process by its admissible parameters is stated without proof and we refer to [14, Theorem 2.7] or [28, Theorem 2.6 and Equations (2.2a),(2.2b)] for two different approaches to prove it.

Proposition A.1. *Let Y be a regular affine process with state space D. Let F and R be as in Definition 2.1. Then there exists a set of admissible parameters $(A, A^i, b, b^i, c, c^i, m, \mu^i)_{i \in \{1, \cdots, d\}}$ such that F and R are of the Lévy-Khintchine form.*

$$F(u) = \frac{1}{2}\langle u, Au \rangle + \langle b, u \rangle - c + \int\limits_{\mathbb{R}^d \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle h(\xi), u \rangle \right) m(d\xi) \tag{A.1}$$

$$R_i(u) = \frac{1}{2}\langle u, A^i u \rangle + \langle b^i, u \rangle - c^i + \int\limits_{\mathbb{R}^d \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle \chi^i(\xi), u \rangle \right) \mu^i(d\xi), \tag{A.2}$$

where $A, A^1, \cdots, A^d$ are positive semi-definite real $d \times d$-matrices; $b, b^1, \cdots, b^d$ are $\mathbb{R}^d$-valued vectors; $c, c^1, \cdots, c^d$ are positive non-negative numbers; $m, \mu^1, \cdots, \mu^d$ are Lévy measures on $\mathbb{R}^d$ and $h, \chi^1, \cdots, \chi^d$ are suitably chosen truncation functions for the respective Lévy measures. Furthermore, the generator $\mathcal{A}$ of Y is given by

$$
\begin{aligned}
\mathcal{A}\varphi(x) \;=\; & \frac{1}{2} \sum_{k,l=1}^d \left(A_{kl} + \sum_{i \in I} A_{kl}^i x_i \right) \frac{\partial^2 \varphi(x)}{\partial x_k \partial x_l} + \left\langle b + \sum_{i=1}^d b^i x_i, \nabla\varphi(x) \right\rangle - \left(c + \sum_{i \in I} c^i x_i \right) \varphi(x) \\
& + \int\limits_{D \setminus \{0\}} \left(\varphi(c + \xi) - \varphi(x) - \langle h(\xi), \nabla\varphi(x) \rangle \right) m(d\xi) \\
& + \sum_{i \in I} \int\limits_{D \setminus \{0\}} \left(\varphi(c + \xi) - \varphi(x) - \langle \chi^i(\xi), \nabla\varphi(x) \rangle \right) x_i \mu^i(d\xi),
\end{aligned}
\tag{A.3}
$$

and ϕ, ψ satisfy the following system of ODEs

$$\partial_t \phi(t, u) = F(\psi(t, u)) \quad, \quad \phi(0, u) = 0 \tag{A.4}$$

$$\partial_t \psi(t, u) = R(\psi(t, u)) \quad, \quad \psi(0, u) = u. \tag{A.5}$$

A.2. Proofs of Section 2

Proof (Theorem 2.5). By assumption and with $V := (V^1, \cdots, V^{d-1})$, the process $Y := (X, V)$ is analytic affine and hence we know from Section 2.1 that its conditional characteristic function allows for the representation

$$E\left[\exp\left(u \cdot Y_T\right) \mid \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t\right], \tag{A.6}$$

for all $u \in \mathbb{C}^d$ such that $(T, u) \in \mathcal{E}_{\mathbb{C}}$. This holds, since the fact that $\mathcal{D}_t \supseteq \mathcal{D}_T$ whenever $t \leq T$ and $(T, u) \in \mathcal{E}_{\mathbb{C}}$ imply that formula (2.1) holds for t whenever it holds for T, and hence (2.2) as well.

Assume for the moment that (1.5) holds. This will be verified later. We then know from (1.6) that the equilibrium pricing measure Q is given by its Radon-Nikodym-density

$$\frac{dQ}{dP} = \frac{\exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)S_T^k\right)}{E\left[\exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)S_T^k\right)\right]} = \frac{\exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)f^k(X_T)\right)}{E\left[\exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)f^k(X_T)\right)\right]}. \tag{A.7}$$

Hence, by applying Bayes formula, we identify

$$S_t^k = E_Q\left[S_T^k \mid \mathcal{F}_t\right] = \frac{E\left[f^k(X_T)\exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)f^k(X_T)\right) \mid \mathcal{F}_t\right]}{E\left[\exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)f^k(X_T)\right) \mid \mathcal{F}_t\right]} \tag{A.8}$$

as the equilibrium price of the k-th security. Recall that the functions g^k and h were defined by

$$g^k(x) := \exp\left(\alpha^k x\right) f^k(x) \exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)f^k(x)\right),$$

$$h(x) := \exp\left(\beta x\right) \exp\left(-\gamma \sum_{k=1}^{K}(n^k + \eta^k)f^k(x)\right).$$

Since we assumed these to be integrable, the Fourier transforms $\hat{g}^k$ and $\hat{h}$ from (2.5) and (2.6), respectively, exist and are by assumption integrable. Hence we apply the Fourier inversion formula[23] to obtain

$$g^k(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{isx}\hat{g}^k(s)ds \quad \text{and} \quad h(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{isx}\hat{h}(s)ds,$$

dx-almost surely. With this at hand, (A.8) transforms to

$$S_t^k = \frac{E\left[\exp\left(-\alpha^k X_T\right)g^k(X_T) \mid \mathcal{F}_t\right]}{E\left[\exp\left(-\beta X_T\right)h(X_T) \mid \mathcal{F}_t\right]} = \frac{E\left[\int_{\mathbb{R}}\exp\left[(-\alpha^k + is)X_T\right]\hat{g}^k(s)ds \mid \mathcal{F}_t\right]}{E\left[\int_{\mathbb{R}}\exp\left[(-\beta + is)X_T\right]\hat{h}(s)ds \mid \mathcal{F}_t\right]}. \tag{A.9}$$

Now we observe that

$$E\left[\left|\int_{\mathbb{R}}\exp\left[(-\alpha^k + is)X_T\right]\hat{g}^k(s)ds\right| \mid \mathcal{F}_t\right] < E\left[\exp\left(-\alpha^k X_T\right)\int_{\mathbb{R}}\left|\hat{g}^k(s)\right|ds \mid \mathcal{F}_t\right] < \infty,$$

since $\left(T, (-\alpha^k, 0_{d-1})\right) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$ and $\hat{g}^k$ is integrable. The same holds analogously for the denominator in (A.9). In particular, we can assure the existence of ε, strictly positive, such that

$$E\left[\exp\left(-\gamma(1 + \varepsilon)\sum_{k=1}^{K}(n^k + \eta^k)f^k(X_T)\right) \mid \mathcal{F}_t\right] < \infty \quad \text{for all } t \in [0, T],$$

since the set $\mathcal{E}_{\mathbb{C}}$ is open[24], and since we require that $\left(T, (-\beta, 0_{d-1})\right) \in \mathcal{E}$. Hence, (1.5) is satisfied. We may now apply Fubini's Theorem to exchange the order of integration, and we get that

$$E\left[\int_{\mathbb{R}}\exp\left[(-\alpha^k + is)X_T\right]\hat{g}^k(s)ds \mid \mathcal{F}_t\right] = \int_{\mathbb{R}} E\left[\exp\left[(-\alpha^k + is)X_T\right] \mid \mathcal{F}_t\right]\hat{g}^k(s)ds$$

$$= \int_{\mathbb{R}}\exp\left[\phi\left(\tau, (-\alpha^k, 0_{d-1})\right) + \psi\left(\tau, (-\alpha^k, 0_{d-1})\right) \cdot Y_t\right]\hat{g}^k(s)ds. \tag{A.10}$$

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[23] See [11, Theorem 9.5.4].
[24] Compare [28, Lemmata 3.12 and 3.19].

The affine transformation formula (A.6) holds, since $(T, (-\alpha^k, 0)) \in \mathcal{E}$. Applying the same arguments to the denominator in (A.9) combined with (A.10) yields the desired form of S_t^k in (2.8). $\qquad\square$

Proof (Proposition 2.8). The proposition is an immediate consequence of Theorem 2.5. We outline the details for $K = 1$, the rest follows by repeating the arguments for the partial derivative with respect to each y^k, evaluated at $\tilde{\gamma}$. So we assume we only have one security S with corresponding $\tilde{\gamma} \in \mathbb{R}$ affecting the density of the pricing measure Q. It follows that

$$\frac{dQ}{dP} = \frac{\exp(-\tilde{\gamma}S_T)}{E\left[\exp(-\tilde{\gamma}S_T)\right]} = \frac{\exp(-\tilde{\gamma}f(X_T))}{E\left[\exp(-\tilde{\gamma}f(X_T))\right]}$$

and the equilibrium price of S at time t can be obtained again by computing

$$S_t = \frac{E\left[f(X_T)\exp(-\tilde{\gamma}f(X_T))\,|\,\mathcal{F}_t\right]}{E\left[\exp(-\tilde{\gamma}f(X_T))\,|\,\mathcal{F}_t\right]}\,. \tag{A.11}$$

We first want to calculate the denominator in (A.11). Since we assumed that $(T, (-\beta, 0_{d-1})) \in \mathcal{E}$ and following the arguments in the proof of Theorem 2.5, the affine transformation formula applies to the denominator and we can express it in terms of the Fourier transforms. In particular,

$$E\left[\exp(-\tilde{\gamma}f(X_T))\,|\,\mathcal{F}_t\right] < \infty\,. \tag{A.12}$$

Now we exchange expectation and differentiation in (A.12). First, [28, Lemma 3.17] ensures that the derivatives $\partial_{u_1}\phi(\tau, u)$ and $\partial_{u_1}\psi(\tau, u)$ exist for $(\tau, (u_1, \cdots, u_d)) \in \mathcal{E}_{\mathbb{C}}$. Next, we observe that by the definition of $\mathcal{E}_{\mathbb{C}}$ and due to the assumptions on f,

$$E\left[\exp\left(yf(X_T)\right)\,|\,\mathcal{F}_t\right] < \infty \tag{A.13}$$

whenever β, which of course depends on y, is chosen such that $(T, (-\beta(-y), 0_{d-1})) \in \mathcal{E}$, as was shown in the proof of Theorem 2.5. Furthermore, since the set $\mathcal{E}_{\mathbb{C}}$ is open, there exists some $\varepsilon > 0$ such that (A.13) holds for $(y + \delta, 0_{d-1})$, whenever $\delta < \varepsilon$. Now define the following functions:

$$w(y) := \exp\left(yf(X_T)\right) \quad, \quad w_\delta(y) := \frac{1}{\delta}(w(y + \delta) - w(y))\,.$$

Hence it holds that $w'(y) = \lim_{\delta \to 0} w_\delta(y)$. Due to the positivity of the exponential function, we have that $w_\delta(y) \leq \frac{1}{\delta}w(y + \delta)$, where $(1/\delta)E\left[w(y + \delta)\,|\,\mathcal{F}_t\right] < \infty$, as long as $\delta < \varepsilon$. Thus, by dominated convergence, we obtain

$$E\left[f(X_T)\exp\left(yf(X_T)\right)\,|\,\mathcal{F}_t\right] = \frac{\partial}{\partial y}E\left[\exp\left(yf(X_T)\right)\,|\,\mathcal{F}_t\right]\,. \tag{A.14}$$

On the other hand we know that from Theorem 2.5 that

$$E\left[\exp\left(yf(X_T)\right)\,|\,\mathcal{F}_t\right]$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[\phi\left(\tau, (-\beta(-y) + is, 0_{d-1})\right) + \psi\left(\tau, (-\beta(-y) + is, 0_{d-1})\right) \cdot Y_t\right] \hat{h}(-y, s)\, ds\,. \tag{A.15}$$

Hence applying (A.14) and (A.15) yields

$$E\left[f(X_T)\exp\left(-\tilde{\gamma}f(X_T)\right)\,|\,\mathcal{F}_t\right] = \frac{\partial}{\partial y}\left(\frac{1}{2\pi}\int_{\mathbb{R}} \exp\left[\phi\left(\tau, (-\beta(-y) + is, 0_{d-1})\right)\right.\right.$$
$$\left.\left. +\psi\left(\tau, (-\beta(-y) + is, 0_{d-1})\right) \cdot Y_t\right] \hat{h}(-y, s)\, ds\right)\bigg|_{y=-\tilde{\gamma}},$$

and we are done. □

Proof (Corollary 2.9). This is an immediate consequence of Proposition 2.8 with $\beta(\gamma) = 0$, $f(x) = x$ and the fact that there is no need of Fourier methods to compute the denominator in the analogue to (A.11)

$$S_t^1 = \frac{E\left[X_T \exp(-\tilde{\gamma}^1 X_T) \,|\, \mathcal{F}_t\right]}{E\left[\exp(-\tilde{\gamma}^1 X_T) \,|\, \mathcal{F}_t\right]}, \tag{A.16}$$

since the affine transformation formula directly applies to the denominator in (A.16). Remember that we had assumed that $\left(T, (-\tilde{\gamma}^1, 0_{d-1})\right) \in \mathcal{E}$. Now we only need to compute $\frac{\partial}{\partial y} E\left[e^{yX_T} \,|\, \mathcal{F}_t\right]$, the actual derivative in formula (2.9). But from (2.2) it follows that

$$\frac{\partial}{\partial y} E\left[\exp\left(y X_T\right) \,|\, \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t\right] \left[\partial_{u_1}\phi(\tau, u) + \partial_{u_1}\psi(\tau, u) \cdot Y_t\right]\big|_{u=(y, 0_{d-1})}.$$

Combining the above with (A.16) yields $S_t^1 = \left[\partial_{u_1}\phi(\tau, u) + \partial_{u_1}\psi(\tau, u) \cdot Y_t\right]\big|_{u=(-\tilde{\gamma}^1, 0_{d-1})}$. □

Proof (Theorem 2.11). An application of Theorem 2.5 with $\alpha^k = 0$, for all $k = 1, \cdots, N$, in addition to the observation that the Fourier transforms are all integrable functions yields the desired result. As to the second claim of integrability, some straightforward calculations show that there exist constants $\hat{M}, \hat{z} > 0$, just depending on the model parameters, which give

$$\max_{f \in \{\hat{g}, \hat{h}, (\hat{g}^k)_{k=1}^K\}} \int_{\mathbb{R}} |f(s)| ds < \hat{M} \int_{\mathbb{R}} \frac{1}{s^2 + \hat{z}} ds < \infty. \tag{□}$$

Proof (Theorem 2.13). The process $Y = (X, V)$ belongs to a subclass of affine processes, namely to the affine diffusions.[25] That is, Y is a solution to the stochastic differential equation $dY_t = \mu(Y_t)dt + \rho(Y_t)dW_t$, with $Y_0 = y$, for a continuous function $b : D \to \mathbb{R}^d$ and a measurable function $\rho : D \to \mathbb{R}^{d \times d}$ such that $y \mapsto \rho(y)\rho(y)^T$ is continuous. In particular, Y is analytic, since the set $\mathcal{D}$ from (2.3) is non-empty. See for instance the discussion on explosion times of the Heston model in Friz and Keller-Ressel [19]. Moreover, the process Y is conservative and, hence, so is the stopped process Y^T. Combining (A.3) with the fact that the generator of (X, V) is determined by its diffusion matrix $\rho\rho^T$ and its drift vector b, we identify the admissible parameters in (A.1), (A.2) and (A.3), where the parts connected with jumps do not play a role here. Hence we conclude that the conditional characteristic function of Y allows a representation as follows

$$E\left[\exp\left(u \cdot Y_T\right) \,|\, \mathcal{F}_t\right] = \exp\left[\phi(\tau, u) + \psi(\tau, u) \cdot Y_t\right], \tag{A.17}$$

whenever $(T, u) = \left(T, (u_1, u_2)\right) \in \mathcal{E}_{\mathbb{C}}$, so in particular for $\left(T, (u_1, u_2)\right) \in \mathcal{E}$. The functions ϕ and ψ satisfy the following system of Riccati equations

$$\begin{aligned}
\partial_t \phi(t, u) &= \kappa\, \psi_2(t, u) + \mu\, \psi_1(t, u) & &, \quad \phi(0, u) = 0 \\
\partial_t \psi_1(t, u) &= 0 & &, \quad \psi_1(0, u) = u_1 \qquad \text{(R)} \\
\partial_t \psi_2(t, u) &= \frac{1}{2}\sigma^2 \psi_2(t, u)^2 - \lambda \psi_2(t, u) + \frac{1}{2}\psi_1(t, u)^2 & &, \quad \psi_2(0, u) = u_2.
\end{aligned}$$

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[25] We emphasize that we would not have needed the complete theory on general affine processes including various possible behavior of jumps, had we only considered pure diffusion processes, since it was shown in [16, Theorem 10.1] that every diffusion Markov process with continuous diffusion matrix is affine if and only if the functions b and $\rho\rho^T$ are affine in the state variable and the solutions ϕ and ψ of the Riccati equations satisfy $\mathrm{Re}(\phi(t, u) + \psi(t, u)x) \leq 0$, for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times i\mathbb{R}^d$. Our equilibrium approach can cover more sophisticated models than pure diffusions though.

A solution to the above system (R), evaluated at the vector $u = (u_1, 0)$, is given by[26]

$$\phi\big(t, (u_1, 0)\big) = \frac{2\kappa}{\sigma^2} \log\left(\frac{2\theta(u_1)e^{\frac{\theta(u_1)+\lambda}{2}t}}{\theta(u_1)(e^{\theta(u_1)t}+1) + \lambda(e^{\theta(u_1)t}-1)}\right) + \mu u_1 t, \qquad (A.18)$$

$$\psi_1\big(t, (u_1, 0)\big) = u_1,$$

$$\psi_2\big(t, (u_1, 0)\big) = \frac{u_1^2(e^{\theta(u_1)t}-1)}{\theta(u_1)(e^{\theta(u_1)t}+1) + \lambda(e^{\theta(u_1)t}-1)}, \qquad (A.19)$$

where

$$\theta(u_1) = \begin{cases} \sqrt{\lambda^2 - \sigma^2 u_1^2} & \text{if} \quad |u_1| < \frac{\lambda}{\sigma} \\ i\sqrt{\sigma^2 u_1^2 - \lambda^2} & \text{if} \quad |u_1| > \frac{\lambda}{\sigma} \end{cases}.$$

Following Friz and Keller-Ressel [19] and recalling that $\lambda > 0$, we distinguish two different cases

$$t^+(u_1) = \begin{cases} +\infty & |u_1| < \frac{\lambda}{\sigma} \\ \frac{2}{|\theta(u_1)|}\left(\arctan\frac{|\theta(u_1)|}{-\lambda} + \pi\right) & |u_1| > \frac{\lambda}{\sigma} \end{cases}$$

such that $\big(T, (u_1, 0)\big) \in \mathcal{E} \subseteq \mathcal{E}_{\mathbb{C}}$ for all $T \leq t^+(u_1)$.[27] Hence, as long as $T < t^+(u_1)$, formula (A.17) holds for all $u = (u_1, 0)$, where $u_1 \in \mathbb{R}$. Now all assumptions of Proposition 2.8 are satisfied and it follows from (2.10) that

$$S_t = \big[\partial_{u_1}\phi(\tau, u) + \partial_{u_1}\psi_1(\tau, u)X_t + \partial_{u_1}\psi_2(\tau, u)V_t\big]\big|_{u=(-\gamma, 0)}, \quad \text{for all } t \in [0, T]. \qquad (A.20)$$

It remains to compute the derivatives of $\phi(t, u)$ and $\psi(t, u)$ with respect to u_1. Of course $\partial_{u_1}\psi_1(\tau, u) \equiv 1$ and a straightforward calculation yields, with $\theta := \theta(-\gamma)$ and $\theta' := [\frac{\partial}{\partial u_1}\theta](-\gamma)$,

$$\partial_{u_1}\phi(\tau, (-\gamma, 0)) = T(\tau, \gamma) \quad \text{and} \quad \partial_{u_1}\psi_2(\tau, (-\gamma, 0)) = -\gamma\frac{N(\tau, \gamma)}{D(\tau, \gamma)},$$

This, together with (A.20), is (2.15), the proof is complete. $\qquad\square$

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[26]Compare [17, Section 6]. For $u_1 = \frac{\lambda}{\sigma}$ we set $\psi_2(t, (\frac{\lambda}{\sigma}, 0)) = \frac{t}{2+\lambda t}$, resembling the limit and still satisfying $\psi_2(0, (\frac{\lambda}{\sigma}, 0)) = 0$.

[27]Friz and Keller-Ressel [19] refer to Andersen and Piterbarg [1] for the explicit calculations. Basically this is exactly the time interval on which the solutions of the Riccati equations do not explode.

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