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S I G N U M P I I G R A T I Q U E A N I M I

DEDICAT

A U C T O R.

§ 1.

I n t r o d u c t i o .

In sequentes disquisitiones incidi considerando problema notum ad theoriam superficierum secundi gradus pertinens:

Axes principales sectionis conicae determinare, qua data superficies secundi gradus per datum planum secatur. Quod problema geometricum ad algebraicum revocatur hoc:

Substitutiones

$$x = aX + a'Y + a''Z$$

$$y = bX + b'Y + b''Z$$

$$z = cX + c'Y + c''Z$$

determinare, quae aequationes

$$x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2 + 2X(\mu'Y + \mu''Z)$$

$$\varphi(xyz) = \lambda_0X^2 + \lambda_1Y^2 + \lambda_2Z^2 + 2X(\nu'Y + \nu''Z)$$

identicas reddant, ea conditione, ut a, b, c sint quantitates datae aequationi $a^2 + b^2 + c^2$ satisfacientes. — Posito enim $X = o$ fit $x^2 + y^2 + z^2 = Y^2 + Z^2$, $\varphi(xyz) = \lambda_1Y^2 + \lambda_2Z^2$.

Quaestionem mihi proposui, quomodo hoc problema extendi posset ad functiones quotlibet variabilium, et primo hoc sese praebuit problema:

Datis duabus formis quadraticis φ et ψ et lineari una X variabilium $x_1x_2\dots x_n$, alias $X_1\dots X_{n-1}$ ita determinare, ut habeatur

$$\varphi = aX^2 + a_1X_1^2 + \dots + a_{n-1}X_{n-1}^2 + 2X(c_1X_1 + \dots + c_{n-1}X_{n-1})$$

$$\psi = bX^2 + b_1X_1^2 + \dots + b_{n-1}X_{n-1}^2 + 2X(d_1X_1 + \dots + d_{n-1}X_{n-1}).$$

Sed mox hoc non semper solvi posse intellexi, quum e theorematis ill. Weierstrass (Monatsber. 1868), quorum infra accuratior fiet mentio, non binas quaslibet functiones quadraticas n variabilium per eadem n quadrata exhiberi posse scirem. Postulavi igitur, ut functiones $(n-1)$ variabilium, quae ex φ et ψ ponendo $X=0$ prodeunt, in eam formam redigantur, quam l. c. docuit ill. Weierstrass. Nec non disquisitiones meas ad functiones, quas bilineares dicunt, extendi.

Quo facilius autem sequentia intelligantur, brevi problemata et theorematum quedam ad transformationem duarum functionum secundi gradus pertinentia memorabo. Inter quae primum locum obtinet hoc:

Datis duabus formis quadraticis φ et ψ variabilium $x_1 x_2 \dots x_n$, scilicet

$$\varphi = \sum_{\alpha\beta} a_{\alpha\beta} x_\alpha x_\beta, \quad \psi = \sum_{\alpha\beta} b_{\alpha\beta} x_\alpha x_\beta,$$

functiones $x'_1 x'_2 \dots x'_n$ lineares earundem variabilium et constantes $c_1 c_2 \dots c_n$ ita determinare, ut sit

$$\varphi = \sum_a x'^a, \quad \psi = \sum_a c_a x'^a.$$

Saepenumero evenit, ut, coefficientibus $a_{\alpha\beta}$ et $b_{\alpha\beta}$ realibus, forma φ , si variabilibus $x_1 x_2 \dots x_n$ reales tantum valores tribuas, evanescere nequeat nisi omnibus illis valoribus simul evanescentibus. Et conservata et omissa hac conditione problema saepe tractatum est, praesertim ab ill. Cauchy (Exercitationum mathematicarum Vol. IV, p. 140 sqq.) et Jacobi (Diarii Crelleani V. XII, p. 1 sqq.). Totius problematis resolutionem pendere constat e determinatione quantitatum $c_1 c_2 \dots c_n$, quae inveniuntur ut radices unius aequationis n^{ti} gradus. Cujus aequationis in proprietatibus versati sunt praeter eos, quos modo laudavi, imprimis viri ill. Kummer, Borchardt, Sylvester, omnes tamen ratione non habita immutationum, quae adhibendae sunt, si radices illae non omnes inter se differunt. Hunc casum primus perscrutatus est ill. Weierstrass, partim anno 1858 (Monatsbericht 1858, p. 207 sqq.), partim anno 1868 (ibidem 1868, p. 310 sqq.). In hac posteriori commentatione generaliter in formis, quas dicunt, bilinearibus versatur disquisitio, cuius summam indicare licet, quatenus ad sequentia melius intelligenda necessarium videtur.

Sint, ut semper in sequentibus, α et β numeri indefiniti serici 1, 2, ..., n , $x_1 x_2 \dots x_n$ et $y_1 y_2 \dots y_n$ 2n variables, et proponantur duas formae harum variabilium:

$$P = \sum_{\alpha\beta} A_{\alpha\beta} x_\alpha y_\beta, \quad Q = \sum_{\alpha\beta} B_{\alpha\beta} x_\alpha y_\beta.$$

Introducamus duas variabiles sive indeterminatas p , q et consideremus systema nn quantitatum

$$\left\{ \begin{array}{l} pA_{11} + qB_{11}, \quad pA_{12} + qB_{12}, \quad \dots, \quad pA_{1n} + qB_{1n} \\ pA_{21} + qB_{21}, \quad pA_{22} + qB_{22}, \quad \dots, \quad pA_{2n} + qB_{2n} \\ \dots \dots \dots \dots \dots \dots \dots \\ pA_{n1} + qB_{n1}, \quad pA_{n2} + qB_{n2}, \quad \dots, \quad pA_{nn} + qB_{nn} \end{array} \right\},$$

cujus determinans per $[P, Q]$ significemus, neque identice, i. e. pro valoribus arbitrariis ipsarum p et q , evanescere supponamus. Ejus determinantis quilibet factor linearis sit $(ap + bq)$, cuius potestates l^a , l'^a , l''^a , ..., neve his altiores deinceps metiantur 1. ipsum determinans $[P, Q]$, 2. omnia ejus determinantia partialia $(n-1)^a$, 3. determinantia partialia $(n-2)^a$.. gradus, et sit $l^{(r)}$ primus numerorum l , l' , l'' ., cuius valor evanescat. Quibus positis erit

$$l^{(r)} = 0, \quad l^{(r+1)} = 0, \dots, \quad l^{(n)} = 0$$

$$l > l' > l'' \dots > l^{(r-1)} > 0.$$

Jam positio

$$l - l' = e, \quad l' - l'' = e', \dots, \quad l^{(r-1)} = e^{(r-1)},$$

erunt $(ap + bq)^e$, $(ap + bq)^{e'}$, ..., $(ap + bq)^{e^{(r-1)}}$ divisores elementarii determinantis

$[P, Q]$ ad divisorem primi gradus ($ap + bq$) pertinentes. Eodem modo reliquos factores lineares tractantes habebimus systema

$$(a_1 p + b_1 q)^{e_1}, (a_2 p + b_2 q)^{e_2}, \dots, (a_\varrho p + b_\varrho q)^{e_\varrho} \quad (e_1 + e_2 + \dots + e_\varrho = n)$$

omnium divisorum elementariorum ipsius $[P, Q]$, et valebit duplex theorema:

Ut formae $P(x_1 x_2 \dots x_n | y_1 y_2 \dots y_n)$, $Q(x_1 x_2 \dots x_n | y_1 y_2 \dots y_n)$ per easdem substitutiones formae

$$x_\alpha = \sum_\gamma h_{\alpha\gamma} x'_\gamma, \quad y_\beta = \sum_\delta k_{\beta\delta} y'_\delta$$

(designantibus etiam ipsis γ, δ numeros indeterminatos seriei 1, 2, ..., n) transformari possint in $P'(x'_1 x'_2 \dots x'_n | y'_1 y'_2 \dots y'_n)$, $Q'(x'_1 x'_2 \dots x'_n | y'_1 y'_2 \dots y'_n)$, neque determinanti $[P, Q]$ neque $[P', Q']$ identice evanescere, necesse est et sufficit, determinantia illa eosdem habere divisores elementarios.

Primae ejus theorematis partis demonstratio satis obvia est; ad secundam probandum singulari quadam opus est transformatione, quam in §° 2. commentationis laudatae proposuit ill. Weierstrass, et quae hoc theoremate continetur:

Delectis numeris g, h talibus, ut determinans formae $gP + hQ$ non evanescat, et quantitatibus $a_1, b_1; a_2, b_2; \dots a_\varrho, b_\varrho$ ita determinatis, ut sit $ga_1 + hb_1 = 1, ga_2 + hb_2 = 1, \dots, ga_\varrho + hb_\varrho = 1$, n functiones lineares variabilium $x_1 x_2 \dots x_n$, scilicet

$$X_{10} X_{11} \dots X_{1(e_1-1)}$$

$$\vdots \quad \vdots \quad \vdots$$

$$X_{\varrho 0} X_{\varrho 1} \dots X_{\varrho(e_\varrho-1)},$$

et totidem functiones

$$Y_{10} Y_{11} \dots Y_{1(e_1-1)}$$

$$\vdots \quad \vdots \quad \vdots$$

$$Y_{\varrho 0} Y_{\varrho 1} \dots Y_{\varrho(e_\varrho-1)}$$

variabilium $y_1 y_2 \dots y_n$ inveniuntur tales, ut, posito generaliter

$$(X_k Y_\lambda)_k = X_{k0} Y_{\lambda(k-1)} + X_{k1} Y_{\lambda(k-2)} + \dots + X_{k(e_k-1)} Y_{\lambda 0}$$

$$(X_k Y_\lambda)_0 = 0,$$

functiones P, Q in has redeant formas:

$$P = \sum_1^\varrho [a_\lambda (X_\lambda Y_\lambda)_{e_\lambda} - h (X_\lambda Y_\lambda)_{e_\lambda - 1}]$$

$$Q = \sum_1^\varrho [b_\lambda (X_\lambda Y_\lambda)_{e_\lambda} + g (X_\lambda Y_\lambda)_{e_\lambda - 1}].$$

Adnoto, quod l. c. § 4 demonstratur, quoties P et Q simul in formam

$$P = a_1 X_1 Y_1 + a_2 X_2 Y_2 + \dots + a_n X_n Y_n$$

$$Q = b_1 X_1 Y_1 + b_2 X_2 Y_2 + \dots + b_n X_n Y_n$$

redigi possint, hanc transformationem per methodum §° 2 obtineri posse.

Ex his prorsus similia derivantur theorematata de formis P et Q secundi gradus variabilium $x_1 x_2 \dots x_n$.

Quoties igitur de transformatione simultanea duarum formarum quadraticarum aut bilinearum agitur, ante omnia determinantis ipsi $[P, Q]$ respondentis divisores elementarii investigandi sunt. Quod sequentibus fiet, ubi transformandae proponuntur forme bilineares, quae e P, Q nascuntur eliminando singulas variabiles x et y ope aequationum linearium

$$\begin{aligned} a_1x_1 + a_nx_n + \dots + a_nx_n &= 0 \\ b_1y_1 + b_ny_n + \dots + b_ny_n &= 0. \end{aligned}$$

Proponimus ergo

Problem a I.

Datae sint functiones bilineares variabilium $x_1x_2\dots x_n; y_1y_2\dots y_n$

$$P = \sum_{\alpha\beta} A_{\alpha\beta} x_\alpha y_\beta, \quad Q = \sum_{\alpha\beta} B_{\alpha\beta} x_\alpha y_\beta$$

una cum linearibus

$$X = \sum_\alpha a_\alpha x_\alpha, \quad Y = \sum_\beta b_\beta y_\beta;$$

jungantur functioni X aliae $x'_1x'_2\dots x'_{n-1}$ ipsarum $x_1x_2\dots x_n$ et functioni Y aliae $y'_1y'_2\dots y'_{n-1}$ ipsarum $y_1y_2\dots y_n$ tales, ut $Xx'_1\dots x'_{n-1}$ et $Yy'_1\dots y'_{n-1}$ sint inter se independentes; exhibeantur P et Q ut functiones P' ($Xx'_1\dots x'_{n-1} | Yy'_1\dots y'_{n-1}$) et Q' ($Xx'_1\dots x'_{n-1} | Yy'_1\dots y'_{n-1}$), e quibus ponendo $X = 0, Y = 0$

$$P''(x'_1\dots x'_{n-1} | y'_1\dots y'_{n-1}) \text{ et } Q''(x'_1\dots x'_{n-1} | y'_1\dots y'_{n-1})$$

orientur: quaeritur, ut divisores elementarii determinantis $[P'', Q'']$ indagentur.

Quos divisores elementarios ab electione variabilium $x'_1\dots x'_{n-1}; y'_1\dots y'_{n-1}$ minime pendere, quum ex ipsa divisorum determinatione, tum e theoremate primo, quod memoravi, hunc in modum sequitur. Designantibus enim $Xx'_1\dots x'_{n-1}$ nec minus $Xx''_1\dots x''_{n-1}$ functiones lineares inter se independentes variabilium $x_1x_2\dots x_n$, priores illae per posteriores has (et vice versa) exhiberi possunt; unde posito $X = 0$ $x'_1\dots x'_{n-1}$ functiones lineares ipsarum $x'_1\dots x'_{n-1}$ erunt, et simili modo $y'_1\dots y'_{n-1}$ ipsarum $y''_1\dots y''_{n-1}$. Quodsi $P''(x'_1\dots x'_{n-1} | y'_1\dots y'_{n-1})$ et $Q''(x'_1\dots x'_{n-1} | y'_1\dots y'_{n-1})$ per $x''_1\dots x''_{n-1}; y'_1\dots y'_{n-1}$ exhibere volumus, valores illi variabilium x' et y' substituendi sunt. Per ejusmodi substitutionem divisores elementarii mutationem non subeunt. Quibus inventis functionibus P'', Q'' transformatio illa singularis, quam exposuimus, adhibenda erit. Sed propter licentiam in deligendis functionibus $x'_1x'_2\dots x'_{n-1}; y'_1y'_2\dots y'_{n-1}$ haec nova transformatio cum anteriori illa, qua P, Q in P', Q' mutabantur, in unam reddit et habemus

Problem a II.

Variabiles $x'_1\dots x'_{n-1}; y'_1\dots y'_{n-1}$ ita determinare, ut functiones P'', Q'' sponte forma illa normali gaudeant, nec non ipsas formas P, Q per X, Y et novas illas variabiles exhibere (omnibus signis cum iis, quae in problemate primo adhibitae sunt, convenientibus).

Resolutis his duobus problematis, quae quodammodo unum constituant, similes quaestiones de formis quadraticis variabilium $x_1x_2\dots x_n$ una solutae erunt. Jam illa aggrediamur.

§ 2.

De lemmatis quibusdam ad transformationem simultaneam functionum

$$R = \sum_{\alpha\beta} C_{\alpha\beta} x_\alpha y_\beta \quad 1)$$

$$X = \sum_\alpha a_\alpha x_\alpha \quad 2)$$

$$Y = \sum_\beta b_\beta y_\beta \quad 3)$$

spectantibus.

Si functiones propositae R, X, Y per substitutiones

$$x_\alpha = \sum_\gamma h_{\alpha\gamma} x'_\gamma \quad 4)$$

$$y_\beta = \sum_\delta k_{\beta\delta} y'_\delta \quad 5)$$

in quibus indices γ, δ ut α, β valores $1, 2, \dots, n$ induunt, transeunt in

$$R' = \sum_{\gamma\delta} C'_{\gamma\delta} x'_\gamma y'_\delta \quad 6)$$

$$X' = \sum_\gamma a'_\gamma x'_\gamma \quad 7)$$

$$Y' = \sum_\delta b'_\delta y'_\delta, \quad 8)$$

tum sistema $(n+1)^2$ quantitatum

$$\left\{ \begin{array}{c} C'_{11} C'_{12} \dots C'_{1n} a'_1 \\ C'_{21} C'_{22} \dots C'_{2n} a'_2 \\ \dots \dots \dots \dots \\ C'_{n1} C'_{n2} \dots C'_{nn} a'_n \\ b'_1 b'_2 \dots b'_n 0 \end{array} \right\} \quad 9)$$

ex his tribus:

$$\left\{ \begin{array}{c} h_{11} h_{21} \dots h_{n1} 0 \\ h_{12} h_{22} \dots h_{n2} 0 \\ \dots \dots \dots \\ h_{1n} h_{2n} \dots h_{nn} 0 \\ 0 0 \dots 0 1 \end{array} \right\} \quad 10)$$

$$\left\{ \begin{array}{c} C_{11} C_{12} \dots C_{1n} a_1 \\ C_{21} C_{22} \dots C_{2n} a_2 \\ \dots \dots \dots \\ C_{n1} C_{n2} \dots C_{nn} a_n \\ b_1 b_2 \dots b_n 0 \end{array} \right\} \quad 11)$$

$$\left\{ \begin{array}{c} k_{11} k_{12} \dots k_{1n} 0 \\ k_{21} k_{22} \dots k_{2n} 0 \\ \dots \dots \dots \\ k_{n1} k_{n2} \dots k_{nn} 0 \\ 0 0 \dots 0 1 \end{array} \right\} \quad 12)$$

eodem modo compositum est, quo generaliter systemata coëfficientium in substitutionibus linearibus componuntur.

Cujus theorematis brevissima haec nobis videtur demonstratio. Designando per z novam quandam variabilem, derivatae functionis $P' + zX'$ secundum $x'_1 x'_2 \dots x'_n$ sumptae una cum Y' , ut functiones variabilium $y'_1 y'_2 \dots y'_n z$ consideratae, sistema illud coëfficientum 9) suppeditant. Habemus vero

$$\frac{\partial P'}{\partial x'_\gamma} = \sum_a h_{a\gamma} \frac{\partial P}{\partial x_a}$$

$$\frac{\partial X'}{\partial x'_\gamma} = \sum_a h_{a\gamma} \frac{\partial X}{\partial x_a},$$

et proinde

$$\left\{ \begin{array}{l} \frac{\partial(P' + zX')}{\partial x'_\gamma} = \sum_a h_{a\gamma} \frac{\partial(P + zX)}{\partial x_a} + 0 \cdot Y \quad (\gamma = 1, 2, \dots, n) \\ Y' = \sum_a 0 \cdot \frac{\partial(P + zX)}{\partial x_a} + 1 \cdot Y. \end{array} \right.$$

Quibus aequationibus si adjungimus sequentes ($n + 1$):

$$\left\{ \begin{array}{l} \frac{\partial(P + zX)}{\partial x_a} = \sum_\beta C_{a\beta} y_\beta + a_a z \\ Y = \sum_\beta b_\beta y_\beta + 0 \cdot z, \end{array} \right.$$

deinde has:

$$\left\{ \begin{array}{l} y_\beta = \sum_\delta k_{\beta\delta} y'_\delta + 0 \cdot z \\ z = \sum_\delta 0 \cdot y'_\delta + 1 \cdot z, \end{array} \right.$$

veritas theorematis enunciati eluet.

Per notissima theorematia de determinantibus ex iis, quae exposuimus, sequitur, quodvis determinans partiale ($m + 1$)st gradus ex elementis 9) formatum aequale esse summae productorum ternorum factorum, scilicet determinantium partialium ejusdem gradus e systematis 10), 11), 12) ea ratione formatarum, ut

series horizontales ex 10) sumptae respondeant serieb. horiz. ex 9) sumptis,

" " " 11) " " " vertic. " 10) " ,

" " " 12) " " " vertic. " 11) " ,

denique " verticales " 12) " " " " " 9) " .

Sed ob singularem systematum 10), 12) indolem evenit, ut permulta determinantia partialia ex his formata identice (sine respectu valorum h et k) evanescant, alia in determinantia inferioris gradus ex elementis h et k formata redeant. Cujus rei quo commodius rationem habeam, omnia determinantia partialia ejusdem ($m + 1$)st gradus ex elementis 9), 11) in quatuor genera distribuo, primo generi attribuens ea, quae cum $a_1 a_2 \dots a_n$, tum $b_1 b_2 \dots b_n$ continent, linearia quidem utrinque systematis, secundo generi ea, quae a quantitatibus a_α , non ab ipsis b_β pendent, tertio ea, quae non ab ipsis a_α , sed a b_β pendent, quarto ea, quae neque quantitas a_α neque b_β continent. His positis facile est intellectu, quodvis determinans partiale ($m + 1$)st gradus et primi generis aequari summae producto-

rum ex terminis factoribus, quorum primus et tertius sint determinantia partialia m^{a} gradus e systematis

$$\left\{ \begin{array}{l} h_{11} h_{12} \dots h_{1n} \\ h_{21} h_{22} \dots h_{2n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ h_{n1} h_{n2} \dots h_{nn} \end{array} \right\} \quad (13)$$

$$\left\{ \begin{array}{l} k_{11} k_{12} \dots k_{1n} \\ k_{21} k_{22} \dots k_{2n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ k_{n1} k_{n2} \dots k_{nn} \end{array} \right\}, \quad (14)$$

et

secundus vero factor determinans $(m+1)^{\text{a}}$ gradus et primi generis ex 11), et quidem *omnia* haec determinantia systematis 11) ducta esse in bina determinantia systematum 13), 14) non identice evanescunt. Similia valent de determinantibus secundi, tertii, quarti generis, ea solum mutatione, ut gradus determinantium ex 13) et 14) vel alteruter vel uterque unitate maiores sint.

Hae observationes, praesertim quatenus ad determinantia primi generis spectant, postea utilitatem nobis afferent.

§ 3.

Lemmatum praecedentium usus ad solvendum problema I.

Jam in formulis §¹ praecedentis ponamus

$$C_{\alpha\beta} = pA_{\alpha\beta} + qB_{\alpha\beta} \quad (15)$$

$$R = pP + qQ, \quad (16)$$

designantibus P et Q easdem functiones atque in §¹. Omnia determinantia partialia systematis 11) functiones integrae homogeneae variabilium p, q evident. Ex iis, quae modo exposuimus sequitur, quemvis divisorem communem omnium horum determinantium gradus generisque ad libitum sumpti etiam *omnia* determinantia ejusdem gradus generisque ex elementis 9) formata metiri. Quodsi determinantia $H = \Sigma \pm h_{11} h_{22} \dots h_{nn}$ et $K = \Sigma \pm k_{11} k_{22} \dots k_{nn}$ non evanescunt, etiam vice versa variabiles x', y' ut functiones ipsarum x, y per formulas ipsis 4) et 5) similes exhibere licet, et similes valent consequentiae. Hinc habemus

Theorema.

Maximus divisor communis omnium determinantium certi cujusdam gradus generisque, quae ex systemate $(n+1)^2$ elementorum

$$\left\{ \begin{array}{l} pA_{11} + qB_{11}, \dots, pA_{1n} + qB_{1n} a_1 \\ \cdot \quad ; \\ pA_{n1} + qB_{n1}, \dots, pA_{nn} + qB_{nn} a_n \\ b_1 \quad , \dots, \quad b_n \quad 0 \end{array} \right\}$$

formari possunt, non mutatur, si formae $P = \sum_{\alpha\beta} A_{\alpha\beta} x_\alpha y_\beta$, $Q = \sum_{\alpha\beta} B_{\alpha\beta} x_\alpha y_\beta$, $X = \sum_\alpha a_\alpha x_\alpha$, $Y = \sum_\beta b_\beta y_\beta$ per easdem substitutiones 4), 5), quarum determinantia non evanescunt, transformantur.

Jam ut problema primum solvamus, substitutiones illas ita determinemus, ut fiat

$$X' = x'_n \text{ i. e. } a'_1 = 0, \dots, a'_{n-1} = 0, a'_n = 1$$

$$Y' = y'_n \text{ i. e. } b'_1 = 0, \dots, b'_{n-1} = 0, b'_n = 1,$$

quod infinite multis modis diversis fieri potest salva conditione, ut substitutionum determinantia non evanescant; ceterum notaciones in problemate primo adhibitas servemus, adiecta hac pro forma, quam adjunctam formae $(pP + qQ)$ dicunt:

$$17) \quad T = \begin{vmatrix} pA_{11} + qB_{11}, & \dots, & pA_{1n} + qB_{1n} & a_1 \\ \vdots & & \vdots & \vdots \\ pA_{n1} + qB_{n1}, & \dots, & pA_{nn} + qB_{nn} & a_n \\ b_1 & & b_n & 0 \end{vmatrix}.$$

Determinans ipsum et determinantia partialia primi generis e schemate 9) formanda, quae non identice evanescunt, in determinantia partialia systematis

$$\left\{ \begin{array}{c} pA'_{11} + qB'_{11}, & \dots, & pA'_{1(n-1)} + qB'_{1(n-1)} \\ \vdots & & \vdots \\ pA'_{(n-1)1} + qB'_{(n-1)1}, & \dots, & pA'_{(n-1)(n-1)} + qB'_{(n-1)(n-1)} \end{array} \right\},$$

enius determinans per $[P'', Q'']$ significandum est, redeunt. Praesertim invenimus formulam:

$$18) \quad [P', Q'] = -HKT,$$

in qua etiamnunc H, K determinantia systematum 13), 14) denotant. Solutio igitur problematis I haec est.

Theorem a.

Determinans $[P'', Q'']$ formae $(pP'' + qQ'')$ praeter factorem constantem convenit cum determinanti T (17). Si $(ap + bq)$ divisorem aliquem linearem functionis T denotat, quam non identice evanescere in sequentibus supponemus, si ejus divisoris $l^{ta}, l'^{ta}, l''^{ta} \dots$ potestates ex ordine metiuntur ipsum T et ejus determinantia partialia primi generis et n^t, n'^t, \dots gradus, si porro $l^{(r)}$ primus numerorum $l, l', l'' \dots$ evanescit, denique si ponitur

$$e = l - l', e' = l' - l'', \dots, e^{(r-1)} = l^{(r-1)},$$

divisores elementarii ipsius $[P'', Q'']$ ad $(ap + bq)$ pertinentes erunt:

$$(ap + bq)^e, (ap + bq)^{e'}, \dots, (ap + bq)^{e^{(r-1)}}.$$

§ 4.

Varia lemmata ad solvendum problema II.

Quamvis divisores elementarii functionis [P'', Q''] per sola determinantia partialia primi generis ex elementis 11) determinati sint, tamen in solvenda quaestione secunda etiam determinantibus reliquorum generum utemur, de quibus hoc viget theorema:

Per quem divisorem maximum omnia determinantia partialia ($m + 1$)ⁿ gradus generisque primi dividi possunt, idem divisor etiam cetera determinantia ejusdem gradus metietur.

Hujus theorematis demonstrationem tantummodo pro $m = (n - 1)$ proponam, et hoc quidem, ne nimium signorum numerum introducere cogar; principium vero demonstrationis idem manet pro reliquis numeri m valoribus. Praeterea ad tempus quantitates a_α, b_β omnes a cifra diversas supponam. Ponatur

$$T_{\alpha\beta} = \frac{1}{p} \frac{\partial T}{\partial A_{\alpha\beta}} = \frac{1}{q} \frac{\partial T}{\partial B_{\alpha\beta}} \quad 19)$$

$$V_\alpha = \frac{\partial T}{\partial a_\alpha} \quad 20)$$

$$U_\beta = \frac{\partial T}{\partial b_\beta} \quad 21)$$

$$S = \begin{vmatrix} pA_{11} + qB_{11}, \dots, pA_{1n} + qB_{1n} \\ \vdots & \ddots \\ pA_{n1} + qB_{n1}, \dots, pA_{nn} + qB_{nn} \end{vmatrix}, \quad 22)$$

ubi in differentiando quantitates $A_{\alpha\beta}, B_{\alpha\beta}, a_\alpha, b_\beta$ tamquam variables a se invicem independentes spectentur. Omnes hae quantitates erunt determinantia n^{th} gradus, et quidem deinceps primi, tertii, secundi, quarti generis. Si determinans T secundum elementa seriei horizontalis α^{tae} ordinatur, coëfficientes singulorum elementorum erunt $T_{\alpha 1}, T_{\alpha 2}, \dots, T_{\alpha n}, V_\alpha$ et habebimus aequationes notissimas:

$$\begin{aligned} T &= \sum_\beta (pA_{\alpha\beta} + qB_{\alpha\beta}) T_{\alpha\beta} + a_\alpha V_\alpha \\ 0 &= \sum_\beta (pA_{\alpha'\beta} + qB_{\alpha'\beta}) T_{\alpha\beta} + a_{\alpha'} V_\alpha (\alpha' \geq \alpha). \end{aligned}$$

Quarum aequationum posterior docet, etiam functionem V_α per quemvis divisorem omnium $T_{\alpha\beta}$ dividi posse, deinde prior idem de functione T valere probat. Simili modo theorema, quatenus ad functiones U_β respicit, comprobatur. Nec non inde perspicitur, differentiando functionem T secundum variables p et q , necessario $l > l'$ esse, quod supra memoratum est. Ut intelligamus denique, etiam S per illum divisorem dividi posse, respicimus aequationem ex eodem fonte haustam

$$\sum_\beta U_\beta (pA_{\alpha\beta} + qB_{\alpha\beta}) + a_\alpha S = 0.$$

Restat, ut hypothesin illam, quantitatum a_α, b_β nullam evanescere, tollamus. Quod facillime fit introducendo loco variabilium $x_\alpha y_\beta$, alias $x'_\alpha y'_\beta$ per substitutiones formae 4,5. Coefficients barum substitutionum ita eligere licet, ut in formis transformatis X' , Y' conditioni illi satisfiat. Divisores communes determinantium $T_{\alpha\beta}, V_\alpha, U_\beta, S$ (scilicet singulorum generum) per hujusmodi transformationes mutationem non subire scimus; quod igitur de formis transformatis demonstratum est, idem etiam de primitivis viget. Ceterum facile perspicitur, per substitutiones multo minus generales idem impetrari posse, velut, si a_1 non evanescat, ponendo $x_1 = x'_1 + h_1 x'_2 + \dots + h_n x'_n$, $x_2 = x'_2, \dots, x_n = x'_n$.

Per similem transformationem etiam effici potest, ut pro $r = 1, 2, \dots (n-1)$ determinantia primi generis $T^{(r)}$, quae e T obtinentur primis r seriebus et horizontalibus et verticalibus omittendis, cum ipso T divisores communes non habeant, qui non omnia determinantia respective eorundem gradum metiantur. Quod statim sequitur ex iis, quae sub finem § 2 diximus, co-efficients $h_{\alpha\gamma}$ et $k_{\beta\delta}$ primo indeterminatos fingendo et postea ita determinando, ut functiones quaedam non identice evanescentes etiam pro valoribus determinatis quantitatum $h_{\alpha\gamma}$ et $k_{\beta\delta}$ non evanescant. Quum in hac, tum in antecedenti transformatione, si $A_{\alpha\beta} = A_{\beta\alpha}, B_{\alpha\beta} = B_{\beta\alpha}, a_\alpha = b_\alpha$ est, etiam $h_{\alpha\gamma} = k_{\alpha\gamma}$ ponere licet, ita ut inter co-efficients functionum transformatum eadem intercedant relationes. Generaliter hac transformatione non opus erit; sin secus, hoc quoque loco particulares adhiberi possunt substitutiones, omnino earum similes, quibus ill. Weierstrass sub finem § 2 commentationis citatae usus est.

§ 5.

Problema II solvitur.

Sint g, h numeri quicunque (integri, si vis) ejusmodi, ut loco p, q positi functioni T valorem non evanescentem tribuant, g', h' alii aequationi

$$gh' - hg' = 1$$

satisfacientes. Ope substitutionis

$$23) \quad p = sg - g', \quad q = sh - h'$$

fit:

$$24) \quad pA_{\alpha\beta} + qB_{\alpha\beta} = s(gA_{\alpha\beta} + hB_{\alpha\beta}) - (g'A_{\alpha\beta} + h'B_{\alpha\beta}) = sa_{\alpha\beta} - b_{\alpha\beta}$$

$$25) \quad pP + qQ = sp - \psi$$

$$26) \quad a_\lambda p + b_\lambda q = s - e_\lambda,$$

siquidem, quod licet, $ga_\lambda + hb_\lambda = 1$ supponimus. $T, T_{\alpha\beta}, V_\alpha, U_\beta, S$ in functiones integras variabilis s abeunt gradum $n-1, n-2, n-1, n-1, n$, pro quibus nova signa introducamus non necesse est. In functione T terminus in s^{n-1} ductus re vera aderit, reliquae functiones inferioris quam assignati gradus esse possunt. Divisores vero elementarii ipsius T abeunt in

$$(s - e_1)^{e_1}, (s - e_2)^{e_2}, \dots, (s - e_q)^{e_q},$$

ubi summa

$$e_1 + e_2 + \dots + e_{q-1} = n - 1.$$

Ut functiones φ, ψ in formam ulterioribus transformationibus commodam redigamus, resolvimus $(n+1)$ aequationes:

$$\begin{cases} \Sigma_\beta (sa_{\alpha\beta} - b_{\alpha\beta}) y_\beta = s \frac{\partial \varphi}{\partial x_\alpha} - \frac{\partial \psi}{\partial x_\alpha} & (\alpha = 1, 2, \dots, n) \\ \Sigma_\beta b_\beta \quad . \quad y_\beta = Y \end{cases},$$

et quidem multiplicando per $T_{\alpha\beta}$, U_β et addendo, unde fit:

$$y_\beta = \Sigma_\alpha \frac{T_{\alpha\beta}}{T} \left(s \frac{\partial \varphi}{\partial x_\alpha} - \frac{\partial \psi}{\partial x_\alpha} \right) + \frac{U_\beta}{T} Y; \quad 27)$$

ex iisdem aequationibus omnes variabiles y_β eliminando prodit:

$$0 = \Sigma_\alpha V_\alpha \left(s \frac{\partial \varphi}{\partial x_\alpha} - \frac{\partial \psi}{\partial x_\alpha} \right) + SY. \quad 28)$$

Simili ratiocinio obtinentur aequationes:

$$x_\alpha = \Sigma_\beta \frac{T_{\alpha\beta}}{T} \left(s \frac{\partial \varphi}{\partial y_\beta} - \frac{\partial \psi}{\partial y_\beta} \right) + \frac{V_\alpha}{T} X \quad 29)$$

$$0 = \Sigma_\beta U_\beta \left(s \frac{\partial \varphi}{\partial y_\beta} - \frac{\partial \psi}{\partial y_\beta} \right) + SX. \quad 30)$$

Ex his aequationibus colligitur:

$$\varphi = \Sigma_\beta y_\beta \frac{\partial \varphi}{\partial y_\beta} = \Sigma_{\alpha\beta} \frac{T_{\alpha\beta}}{T} \left(s \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial y_\beta} - \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \varphi}{\partial y_\beta} \right) + Y \Sigma_\beta \frac{U_\beta}{T} \frac{\partial \varphi}{\partial y_\beta}$$

$$\psi = \Sigma_\alpha x_\alpha \frac{\partial \psi}{\partial x_\alpha} = \Sigma_{\alpha\beta} \frac{T_{\alpha\beta}}{T} \left(s \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \varphi}{\partial y_\beta} - \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \psi}{\partial y_\beta} \right) + X \Sigma_\alpha \frac{V_\alpha}{T} \frac{\partial \psi}{\partial x_\alpha}$$

$$s\varphi + \psi = \Sigma_{\alpha\beta} \frac{T_{\alpha\beta}}{T} \left(s \frac{\partial \varphi}{\partial x_\alpha} \frac{\partial \varphi}{\partial y_\beta} - \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \varphi}{\partial y_\beta} \right) + Y \Sigma_\beta \frac{sU_\beta}{T} \frac{\partial \varphi}{\partial y_\beta} + X \Sigma_\alpha \frac{V_\alpha}{T} \frac{\partial \psi}{\partial x_\alpha}. \quad 31)$$

Ope aequationum 28), 30) dextra hujus aequationis pars dupli modo transformari potest, scilicet:

$$\begin{aligned} Y \Sigma_\beta \frac{sU_\beta}{T} \frac{\partial \varphi}{\partial y_\beta} + X \Sigma_\alpha \frac{V_\alpha}{T} \frac{\partial \psi}{\partial x_\alpha} &= Y \Sigma_\beta \frac{U_\beta}{T} \frac{\partial \psi}{\partial y_\beta} + X \Sigma_\alpha \frac{V_\alpha}{T} \frac{\partial \psi}{\partial x_\alpha} - \frac{S}{T} XY \\ &= Y \Sigma_\beta \frac{sU_\beta}{T} \frac{\partial \varphi}{\partial y_\beta} + X \Sigma_\alpha \frac{sV_\alpha}{T} \frac{\partial \varphi}{\partial x_\alpha} + \frac{S}{T} XY. \end{aligned} \quad \left. \right\} 32)$$

Ut formulam 31) secundum potestates descendentes variabilis evolvamus, sit

$$\Sigma_{\alpha, \beta} \frac{T_{\alpha\beta}}{T} \eta_\alpha \xi_\beta = F(\xi_1 \dots \xi_n | \eta_1 \dots \eta_n) = s^{-1} F_1 + s^{-2} F_2 + \dots \quad 33)$$

$$34) \quad \begin{cases} \sum_a \frac{V_\alpha}{T} \eta_a = H(\eta_1 \dots \eta_n) = H_o(\eta_1 \dots \eta_n) + s^{-1} H_1(\eta_1 \dots \eta_n) + \dots \\ \sum_\beta \frac{U_\beta}{T} \xi_\beta = G(\xi_1 \dots \xi_n) = G_o(\xi_1 \dots \xi_n) + s^{-1} G_1(\xi_1 \dots \xi_n) + \dots \end{cases}$$

$$35) \quad \frac{S}{T} = -Cs + C' + \dots$$

Hic signis adhibitis fit ex 31) et 32)

$$36) \quad \begin{cases} \varphi = F_1 \left(\frac{\partial \varphi}{\partial y_1} \dots \frac{\partial \varphi}{\partial y_n} \Big| \frac{\partial \varphi}{\partial x_1} \dots \frac{\partial \varphi}{\partial x_n} \right) + CXY \\ \psi = F_2 \left(\frac{\partial \varphi}{\partial y_1} \dots \frac{\partial \varphi}{\partial y_n} \Big| \frac{\partial \varphi}{\partial x_1} \dots \frac{\partial \varphi}{\partial x_n} \right) + YG_1 \left(\frac{\partial \varphi}{\partial y_1} \dots \frac{\partial \varphi}{\partial y_n} \right) + XH_1 \left(\frac{\partial \varphi}{\partial x_1} \dots \frac{\partial \varphi}{\partial x_n} \right) + CXY \end{cases}$$

Ad evolutionem vero functionum F, G, H secundum potestates descendentes ipsius s nos perducet disceptio harum functionum fractarum in fractiones simplices, quae hic sequenti modo peragi potest.

Sit, ut supra, $T^{(r-1)}$ determinans id, quod e T oritur sublatis $(r-1)$ primis series horizontalibus et verticalibus, $(-1)^{\alpha+\beta} T_{\alpha\beta}^{(r-1)}$ id, in quo praeterea series horizontalis α^{ta} , verticalis β^{ta} omissae sunt, simili modo determinentur $V_\alpha^{(r-1)}, U_\beta^{(r-1)}$. Si $\alpha < r$, $T_{\alpha\beta}^{(r-1)}$ et $V_\alpha^{(r-1)} = 0$, sin $\beta < r$, $T_{\alpha\beta}^{(r-1)}$ et $U_\beta^{(r-1)} = 0$ ponantur. Inter has functiones praeter alias hae intercedunt relationes e determinantium theoria nota:

$$T_{11} T_{\alpha\beta} - T_{\alpha 1} T_{1\beta} = T T'_{\alpha\beta}$$

$$T'_{22} T'_{\alpha\beta} - T'_{\alpha 2} T'_{2\beta} = T T''_{\alpha\beta}$$

• • • • • • • •

$$T_{11} V_\alpha - T_{\alpha 1} V_1 = T V'_\alpha$$

$$T'_{22} V'_\alpha - T'_{\alpha 2} V'_2 = T' V''_\alpha$$

• • • • • • • •

$$T_{11} U_\beta - T_{1\beta} U_1 = T U'_\beta$$

$$T'_{22} U'_\beta - T'_{1\beta} U'_1 = T' U''_\beta$$

• • • • • • • •

e quibus sequuntur hae:

$$37) \quad \begin{cases} \frac{T_{\alpha\beta}}{T} = \frac{T_{\alpha 1} T_{1\beta}}{T T'} + \frac{T'_{\alpha 2} T'_{2\beta}}{T' T''} + \dots \\ \frac{V_\alpha}{T} = \frac{T_{\alpha 1} V_1}{T T'} + \frac{T'_{\alpha 2} V'_2}{T' T''} + \dots \\ \frac{U_\beta}{T} = \frac{T_{1\beta} U_1}{T T'} + \frac{T'_{2\beta} U'_2}{T' T''} + \dots \end{cases}$$

Hinc posito

$$\left. \begin{aligned} X' &= T_{11} \xi_1 + T_{12} \xi_2 + \dots + T_{1n} \xi_n \\ X'' &= T'_{22} \xi_2 + \dots + T'_{2n} \xi_n \\ \cdot &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ Y' &= T_{11} \eta_1 + T_{21} \eta_2 + \dots + T_{n1} \eta_n \\ Y'' &= T'_{22} \eta_2 + \dots + T'_{n2} \eta_n \\ \cdot &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \right\} 38)$$

nanciscimur

$$\left. \begin{aligned} F &= \frac{X' Y'}{T T'} + \frac{X'' Y''}{T'' T''} + \frac{X''' Y'''}{T''' T''' \dots} \\ H &= \frac{V_1' Y'}{T T'} + \frac{V_2' Y''}{T'' T''} + \frac{V_3' Y'''}{T''' T''' \dots} \\ G &= \frac{U_1' X'}{T T'} + \frac{U_2' X''}{T'' T''} + \frac{U_3' X'''}{T''' T''' \dots} \end{aligned} \right\} 39)$$

Jam sit $(s - c)$ quilibet factor linearis functionis T et $(s - c_\lambda)^{\epsilon_\lambda}$ in serie divisorum elementariorum ad $(s - c)$ pertinentium. Si functiones $T^{(x-1)}$, $T^{(x)}$ per altiores ipsius $(s - c)$ potestates quam resp. $t^{(x-1)_{\text{tam}}}$, $t^{(x)_{\text{tam}}}$ dividi non possunt, ponere licet

$$\left. \begin{aligned} \frac{X^{(x)}}{\sqrt{T^{(x-1)} T^{(x)}}} &= (s - c_\lambda)^{-\frac{1}{2} \epsilon_\lambda} \sum_0^\infty X_{\lambda\mu} (s - c_\lambda)^\mu \\ \frac{Y^{(x)}}{\sqrt{T^{(x-1)} T^{(x)}}} &= (s - c_\lambda)^{-\frac{1}{2} \epsilon_\lambda} \sum_0^\infty Y_{\lambda\nu} (s - c_\lambda)^\nu \\ \frac{V_x^{(x-1)}}{\sqrt{T^{(x-1)} T^{(x)}}} &= (s - c_\lambda)^{-\frac{1}{2} \epsilon_\lambda} \sum_0^\infty D_{\lambda\mu} (s - c_\lambda)^\mu \\ \frac{U_x^{(x-1)}}{\sqrt{T^{(x-1)} T^{(x)}}} &= (s - c_\lambda)^{-\frac{1}{2} \epsilon_\lambda} \sum_0^\infty C_{\lambda\nu} (s - c_\lambda)^\nu, \end{aligned} \right\} 40)$$

designantibus $X_{\lambda\mu}$, $Y_{\lambda\nu}$ functiones lineares homogeneas variabilium ξ_β , η_α , quarum coëfficientes praeter factorem $\frac{1}{\sqrt{C_\lambda}}$ rationaliter e datis coëfficientibus $a_{\alpha\beta}$, $b_{\alpha\beta}$, a_α , b_β et radice c_λ formantur; idem de constantibus $D_{\lambda\mu}$, $C_{\lambda\nu}$ valet; denique ipsum C_λ rationaliter per easdem quantitates exprimitur. Multiplicando fit e formulis 40)

$$\left. \begin{aligned} \frac{X^{(x)} Y^{(x)}}{T^{(x-1)} T^{(x)}} &= \sum_{\mu\nu} X_{\lambda\mu} Y_{\lambda\nu} (s - c_\lambda)^{\mu + \nu - \epsilon_\lambda} \\ \frac{U_x^{(x-1)} X^{(x)}}{T^{(x-1)} T^{(x)}} &= \sum_{\mu\nu} C_{\lambda\nu} X_{\lambda\mu} (s - c_\lambda)^{\mu + \nu - \epsilon_\lambda} \\ \frac{V_x^{(x-1)} Y^{(x)}}{T^{(x-1)} T^{(x)}} &= \sum_{\mu\nu} D_{\lambda\mu} Y_{\lambda\nu} (s - c_\lambda)^{\mu + \nu - \epsilon_\lambda} \end{aligned} \right\} (\mu, \nu = 0, 1, 2, \dots \infty).$$

Harum evolutionum retinemus potestates negativas formulae ($s - e_\lambda$), et eodem modo pro omnibus numeri λ valoribus (1, 2, ..., ϱ) procedimus; invenimus secundum 34), 39):

$$\left. \begin{aligned} F &= \Sigma_{\lambda} \left(\Sigma X_{\lambda\mu} Y_{\lambda\nu} (s - e_\lambda)^{\mu + \nu - e_\lambda} \right) \\ H &= \Sigma_{\lambda} \left(\Sigma D_{\lambda\mu} Y_{\lambda\nu} (s - e_\lambda)^{\mu + \nu - e_\lambda} \right) + H_o (\eta_1 \dots \eta_n) \\ G &= \Sigma_{\lambda} \left(\Sigma C_{\lambda\mu} X_{\lambda\nu} (s - e_\lambda)^{\mu + \nu - e_\lambda} \right) + G_o (\xi_1 \dots \xi_n) \end{aligned} \right\} (\mu + \nu < e_\lambda),$$

sive ponendo

$$41) \quad \left. \begin{aligned} (X_\lambda Y_\lambda)_k &= \Sigma X_{\lambda\mu} Y_{\lambda\nu} \\ (D_\lambda Y_\lambda)_k &= \Sigma D_{\lambda\mu} Y_{\lambda\nu} \\ (C_\lambda X_\lambda)_k &= \Sigma C_{\lambda\mu} X_{\lambda\nu} \end{aligned} \right\} (\mu + \nu = k - 1)$$

$$42) \quad \left. \begin{aligned} F &= \Sigma_{\lambda} \left(\frac{(X_\lambda Y_\lambda)_{e_\lambda}}{(s - e_\lambda)} + \frac{(X_\lambda Y_\lambda)_{e_\lambda - 1}}{(s - e_\lambda)^2} + \dots + \frac{(X_\lambda Y_\lambda)_1}{(s - e_\lambda)^{e_\lambda}} \right) \\ H &= H_o + \Sigma_{\lambda} \left(\frac{(D_\lambda Y_\lambda)_{e_\lambda}}{(s - e_\lambda)} + \frac{(D_\lambda Y_\lambda)_{e_\lambda - 1}}{(s - e_\lambda)^2} + \dots + \frac{(D_\lambda Y_\lambda)_1}{(s - e_\lambda)^{e_\lambda}} \right) \\ G &= G_o + \Sigma_{\lambda} \left(\frac{(C_\lambda X_\lambda)_{e_\lambda}}{(s - e_\lambda)} + \frac{(C_\lambda X_\lambda)_{e_\lambda - 1}}{(s - e_\lambda)^2} + \dots + \frac{(C_\lambda X_\lambda)_1}{(s - e_\lambda)^{e_\lambda}} \right) \end{aligned} \right)$$

Ceterum ex formulis 28), 30) facile sequitur

$$H_o \left(\frac{\partial \varphi}{\partial x_1} \dots \frac{\partial \varphi}{\partial x_n} \right) = CY$$

$$G_o \left(\frac{\partial \varphi}{\partial y_1} \dots \frac{\partial \varphi}{\partial y_n} \right) = CX,$$

constantii C eadem manente atque in 35).

Ex formulis 36) nunc sequitur:

$$43) \quad \left. \begin{aligned} \varphi &= \Sigma_{\lambda} (X_\lambda Y_\lambda)_{e_\lambda} + CXY \\ \psi &= \Sigma_{\lambda} c_{\lambda} (X_\lambda Y_\lambda)_{e_\lambda} + \Sigma_{\lambda} \left((X_\lambda Y_\lambda)_{e_\lambda - 1} + X(D_\lambda Y_\lambda)_{e_\lambda} + Y(C_\lambda X_\lambda)_{e_\lambda} \right) + C'XY. \end{aligned} \right.$$

et:

$$44) \quad \left. \begin{aligned} P &= \Sigma_{\lambda} a_{\lambda} (X_\lambda Y_\lambda)_{e_\lambda} - h \Sigma_{\lambda} \left((X_\lambda Y_\lambda)_{e_\lambda - 1} + X(D_\lambda Y_\lambda)_{e_\lambda} + Y(C_\lambda X_\lambda)_{e_\lambda} \right) + (hC - hC')XY \\ Q &= \Sigma_{\lambda} b_{\lambda} (X_\lambda Y_\lambda)_{e_\lambda} + g \Sigma_{\lambda} \left((X_\lambda Y_\lambda)_{e_\lambda - 1} + X(D_\lambda Y_\lambda)_{e_\lambda} + Y(C_\lambda X_\lambda)_{e_\lambda} \right) + (-g'C + gC')XY \end{aligned} \right.$$

ubi in functionibus $X_{\lambda\mu}$, $Y_{\lambda\nu}$ substituendum est

$$\xi_\beta = \frac{\partial \varphi}{\partial y_\beta}, \quad \eta_\alpha = \frac{\partial \varphi}{\partial x_\alpha}.$$

Formulae 44) problema nostrum resolvunt; nam posito $X = 0$, $Y = 0$ fit

$$P = \Sigma_k a_k (X_k Y_k)_{e_1} - h \Sigma_k (X_k Y_k)_{e_1} - 1$$

$$Q = \Sigma_k b_k (X_k Y_k)_{\varepsilon_i} + g \Sigma_k (X_k Y_k)_{\varepsilon_i} = 1.$$

unde $2(n-1)$ variables $X_{10} \dots X_{1(e_1-1)}$ $Y_{10} \dots Y_{1(e_1-1)}$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100

$$X_{\rho_0} \dots X_{\rho(e_n - 1)} \quad Y_{\rho_0} \dots Y_{\rho(e_n - 1)}$$

z'z — *z'z* — sumi possunt

pro quaesitis $x'_1 \dots x'_{n-1}$; $y'_1 \dots y'_{n-1}$ sumi possunt.

Si ei conditioni, cujus in formulis 40) usus est factus, ut functiones T' , T'' ,.. cum T divisorem communem non habeant, qui non idem omnia determinantia partialia respective n^{μ} , $(n-1)^{\mu}$,.. gradus metiatur, non satisfit, ex iis, quae in fine §¹ 4. diximus, scimus, functiones transformatas inveniri posse, quae postulato satisfaciant; quae quum in formam 44) redactae erunt, nihil nisi expressiones functionum $X_{\lambda\mu}$, $Y_{\lambda\nu}$ mutandum erit, ut ipsae P , Q eodem modo exhibitae sint.

Fractiones simplices ad ejusmodi divisorem ($s - c$) pertinentes, qui functionem S non metitur, etiam alia via determinari possunt. Per ejusmodi divisorem non omnia $T_{\alpha\beta}$ dividendi poterunt, ut ex theoremate §¹ 4 sequitur. Hinc *unus* divisor elementarius $(s - c_1)^{\epsilon_1}$ invenietur ad radicem c pertinens. Formula demonstratu facilis

$$ST_{\alpha\beta} - V_\alpha U_\beta = TS_{\alpha\beta}$$

give

$$\frac{T_{\alpha\beta}}{T} = \frac{V_\alpha U_\beta}{TS} + \frac{S_{\alpha\beta}}{S}$$

perducit ad

$$F = \frac{\Sigma_\alpha V_\alpha \eta_\alpha \cdot \Sigma_\beta U_\beta \xi_\beta}{TS} + \Sigma_{\alpha\beta} \frac{S_{\alpha\beta}}{S} \eta_\alpha \xi_\beta.$$

Ponamus igitur

$$\frac{\sum_\alpha V_\alpha \eta_\alpha}{\sqrt{VTS}} = (s - c_\lambda)^{-\frac{1}{2}c_\lambda} \sum_0^\infty Y_{\lambda\nu} (s - c_\lambda)^\nu$$

$$\frac{\Sigma_\beta U_\beta \xi_\beta}{\sqrt{TS}} = (s - c_\lambda)^{-\frac{1}{2}e_\lambda} \sum_0^\infty X_{\lambda\mu} (s - o_\lambda)^\mu$$

$$V\sqrt{\frac{S}{T}} = (s - c_\lambda)^{-\frac{1}{2}e_\lambda} \sum_{\mu=0}^{\infty} C_{\lambda\mu} (s - c_\lambda)^\mu$$

In vicinia igitur valoris c , evolutiones functionum F, H, G haec sunt:

$$F = \Sigma X_{k\mu} Y_{k\nu} (s - e_k)^{\mu + \nu - e_k}$$

$$H = \sum C_{\lambda\mu} Y_{\lambda\nu} (s - c_\lambda)^{\mu + \nu - e_\lambda}, \quad (\mu, \nu = 0, 1, 2, \dots, \infty)$$

$$G = \Sigma C_{1..} X_{1..} (s - e_1)^{\mu + \nu - e_\lambda}$$

unde proceditur ut supra. Haec methodus magis quam prior symmetrica est respectu variabilium, sed non eadem generalitate gaudet.

Sub finem adjungò transformationem, quae tum adhibenda est, quum T neque cum $\frac{\partial T}{\partial s}$ neque cum S divisorem habet communem. Omnium evolutionum 45) primi tantum termini retinendi sunt. Si igitur $T'_1, S_1, V_{\alpha 1}, U_{\beta 1}$ valores significant functionum $\frac{\partial T}{\partial s}, S, V_\alpha, U_\beta$ pro $s = c_1$, et si statuimus

$$47) \quad \text{fit} \quad X_1 = \Sigma_\beta \frac{U_{\beta 1}}{\sqrt{S_1 T'_1}} \xi_\beta, \quad Y_1 = \Sigma_\alpha \frac{V_{\alpha 1}}{\sqrt{S_1 T'_1}} \eta_\alpha,$$

$$\varphi = \Sigma_1 X_1 Y_1 + CXY \quad \left. \begin{array}{l} \\ \psi = \Sigma_1 c_1 X_1 Y_1 + X \Sigma_1 \sqrt{\frac{S_1}{T'_1}} Y_1 + Y \Sigma_1 \sqrt{\frac{S_1}{T'_1}} X_1 + C' XY \end{array} \right\} (1 = 1, 2, \dots n - 1),$$

$$48) \quad \left. \begin{array}{l} P = \Sigma_1 a_1 X_1 Y_1 - h \Sigma_1 \sqrt{\frac{S_1}{T'_1}} (XY_1 + YX_1) + (h'C - hC') XY \\ Q = \Sigma_1 b_1 X_1 Y_1 + g \Sigma_1 \sqrt{\frac{S_1}{T'_1}} (XY_1 + YX_1) + (-g'C + gC) XY. \end{array} \right.$$

§ 6.

De formis quadraticis.

Ex datis duabus formis \mathfrak{P} et \mathfrak{Q} secundi gradus variabilium $x_1 x_2 \dots x_n$ formentur functiones bilineares

$$49) \quad \left. \begin{array}{l} P = \Sigma_\beta \frac{1}{2} \frac{\partial \mathfrak{P}}{\partial x_\beta} y_\beta = \Sigma_{\alpha\beta} A_{\alpha\beta} x_\alpha y_\beta \\ Q = \Sigma_\beta \frac{1}{2} \frac{\partial \mathfrak{Q}}{\partial x_\beta} y_\beta = \Sigma_{\alpha\beta} B_{\alpha\beta} x_\alpha y_\beta, \end{array} \right.$$

unde sequitur:

$$50) \quad \left. \begin{array}{l} A_{\alpha\beta} = A_{\beta\alpha} = \frac{1}{2} \frac{\partial^2 \mathfrak{P}}{\partial x_\alpha \partial x_\beta} \\ B_{\alpha\beta} = B_{\beta\alpha} = \frac{1}{2} \frac{\partial^2 \mathfrak{Q}}{\partial x_\alpha \partial x_\beta}; \end{array} \right.$$

porro functioni
jungatur

$$X = \Sigma_\alpha a_\alpha x_\alpha \\ Y = \Sigma_\beta a_\beta y_\beta.$$

Propter symmetriam systematis

$$\left. \begin{array}{c} pA_{11} + qB_{11}, \dots, pA_{1n} + qB_{1n} \quad a_1 \\ \vdots \\ pA_{n1} + qB_{n1}, \dots, pA_{nn} + qB_{nn} \quad a_n \\ a_1 \quad \dots, \quad a_n \quad 0 \end{array} \right\},$$

cujus determinans est T , erit etiam

$$\begin{aligned} T_{\alpha\beta} &= T_{\beta\alpha}, \quad T'_{\alpha\beta} = T'_{\beta\alpha}, \dots \\ V_\alpha &= U_\alpha, \quad V'_\alpha = U'_\alpha, \dots \end{aligned}$$

Hinc quantitates $Y_{\lambda\mu}$ caedem functiones variabilium $\eta_\alpha = \frac{\partial(gP + hQ)}{\partial x_\alpha}$ sunt, ac $X_{\lambda\mu}$ functiones variabilium $\xi_\alpha = \frac{\partial(gP + hQ)}{\partial y_\alpha}$, et idem valet, si illas ut functiones ipsarum y_β , has ipsarum x_α spectamus. Praeterea in formulis 40) erit $D_{\lambda\mu} = C_{\lambda\mu}$. Posito igitur $y_\beta = x_\beta$ fit $\mathfrak{P} = \Sigma_\lambda (a_\lambda (X_\lambda X_\lambda)_{e_\lambda} - h(X_\lambda X_\lambda)_{e_\lambda} - 1) - 2hX \Sigma_\lambda (C_\lambda X_\lambda)_{e_\lambda} + (h'C - hC')XX$
 $\mathfrak{Q} = \Sigma_\lambda (b_\lambda (X_\lambda X_\lambda)_{e_\lambda} + g(X_\lambda X_\lambda)_{e_\lambda} - 1) + 2gX \Sigma_\lambda (C_\lambda X_\lambda)_{e_\lambda} + (-g'C + gC')XX$. { 51)

Si coëfficientes formarum \mathfrak{P} , \mathfrak{Q} , X reales sunt, et \mathfrak{P} ea gaudet proprietate, ut, variabilibus realibus $x_1 x_2 \dots x_n$ per aequationem $X = 0$ inter se dependentibus, evanescere non possit, nisi sit $x_1 = 0, x_2 = 0, \dots, x_n = 0$, per eadem ratiocinia, quibus usus est ill. Weierstrass (§ 5, F l. c.) invenitur, omnes divisores elementarios quum reales tum primi gradus esse.

Sub finem observo, similes methodos adhiberi posse ei casui, ubi praeter duas functiones \mathfrak{P} et \mathfrak{Q} secundi gradus dantur m functiones lineares inter se independentes:

$$\begin{aligned} X &= a_1 x_1 + a_2 x_2 + \dots + a_n x_n \\ X' &= a'_1 x_1 + a'_2 x_2 + \dots + a'_n x_n \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \\ X^{(m-1)} &= a_1^{(m-1)} x_1 + a_2^{(m-1)} x_2 + \dots + a_n^{(m-1)} x_n. \end{aligned}$$

Posito enim $X = 0, X' = 0, \dots, X^{(m-1)} = 0$ variables $x_1 \dots x_n$ per $(n-m)$ novas exhiberi possunt, quarum P et Q functiones evadent. Determinatio autem divisorum elementariorum pendebit e determinanti

$$\left| \begin{array}{cccccc} pA_{11} + qB_{11}, \dots, pA_{1n} + qB_{1n} & a_1 \dots a_1^{(m-1)} \\ \vdots & \vdots & \vdots & \vdots \\ pA_{n1} + qB_{n1}, \dots, pA_{nn} + qB_{nn} & a_n \dots a_n^{(m-1)} \\ a_1 & \dots & a_n & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1^{(m-1)} & \dots & a_n^{(m-1)} & 0 & \dots & 0 \end{array} \right|$$

et ejus determinantibus partialibus. Ulteriorem ejus observationis evolutionem alii occasioni reservo.

Quum maximam hujus dissertationis partem scripsisse, obvenit mihi commentatio nova ill^r. Kronecker (Ueber Scharen von quadratischen Formen, Berl. Monatsberichte 1874, p. 1 ff.), qua vir sagacissimus transformationi illi singulare Weierstrassiana, quae in hac dissertatione saepissime commemorata est, novam lucem afferat et similem transformationem extare docet, etiam si determinans $[P, Q]$ identice evanescat. Hujus commentationis in nostra disquisitione rationem haberi nondum potuisse, valde lugemus.

T H E S E S.

I.

Ad theoriam transformationis duarum formarum secundi gradus stabiendum methodi ill^r. Jacobi et Cauchy non sufficiunt.

II.

Optima theorematis fundamentalia algebraici demonstratio existimanda est Gaussiana altera.

III.

Notiones calculi differentialis facililime intelliguntur ab eo, qui theoriam functionum rationalium satis intellixit.

Natus sum, Ludovicus Stickelberger, anno 1850, in vico Buch pagi Scaufensis patre Emmanuel, matre Maria Julia Elisabeth e gente Courvoisier, quos adhuc superstites maxime veneror. Fidei addictus sum reformatae. Postquam primis literarum elementis imbutus sum, per tres annos privatum a patre praesertim lingua latina et matheci doctus sum, et eo provectus, ut, quum anno 1863 gymnasium Scaufense adiisse, anno 1867 maturitatis testimonio munitus inde prodirem. Aestatem ejusdem anni domi consumpsi. Deinde per tria semestria Heidelbergae disserentes audivi viros illustrissimos Bunsen, du Bois-Reymond, F. Eisenlohr, Helmholtz, Hesse, Kirchhoff, Kopp, Leenhard, Lüroth, Weber. Postea per septem semestria Berolini interfui scholis illustrissimorum Harms, Helmholtz, Kronecker, Kummer, Thomé, Weierstrass, nec minus per tres annos seminariorum, quod dicitur, mathematico, cui moderantur celeberrimi Kummer et Weierstrass. Quibus omnibus viris, optime de me meritis, imprimis illustrissimis Kronecker, Kummer, Weierstrass, quorum benevolentiae multum debeo, gratias maximas ago.
