Intertemporal Surplus Management*

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September 1999

Abstract:

This paper presents an intertemporal portfolio selection model for pension funds that maximize the intertemporal expected utility of the surplus of assets net of liabilities. Following Merton (1973) it is assumed that both the asset and the liability return follow Itô processes as functions of a state variable. The optimum occurs for investors holding four funds: the market portfolio, the hedge portfolio for the state variable, the hedge portfolio for the liabilities, and the riskless asset. It is shown that pension funds should purchase hedging for liabilities. In contrast to Merton's result in the assets only case, this hedge depends exclusively on the funding ratio of a specific pension fund and not on preferences. With HARA utility the investments in the state variable hedge portfolios are also preference independent. With log utility the market portfolio investment depends only on the current funding ratio.

Key Words:

Asset, Beta, Funding Ratio, HARA utility function, Hedge Portfolio, Intertemporal Capital Asset Pricing Model, Itô process, *J*-function, Liabilities, Log utility function, Safety First, Shortfall risk, State variable, Surplus management

JEL classification: G23, G11

* Without implicating them, the authors would like to thank the participants in a workshop at the ETH in Zürich, May 1996, the annual conference of the Deutsche Gesellschaft für Finanzwirtschaft (DGF) in Berlin, September 1996, the EURO INFORMS meeting in Barcelona, July 1997, and the 8th International Conference on Stochastic Programming, Vancouver (August 1998) for their comments. We are especially grateful to Christian Hipp, Stanley Pliska, Bernd Schips, and Heinz Zimmermann for detailed comments and suggestions.

1. Introduction

Intertemporal Asset Allocation Models date to Merton (1969,1971,1973) (see also the summary in Merton 1990), and Samuelson (1969). Merton presents a continuous time intertemporal model whereas Samuelson discusses the discrete time case. Both models formulate a lifetime portfolio selection problem. Merton's (1973) intertemporal capital asset pricing model derives equilibrium asset premia. In contrast to Sharpe's (1964) CAPM, Merton's CAPM is based on a three fund theorem. Each rational investor holds the riskless asset, the market portfolio, and a hedge portfolio for a so called state variable in order to maximize his lifetime expected utility. The state variable is a stochastic term which affects the asset price processes. The hedge portfolio provides maximum correlation to the state variable, i.e. it provides the best possible hedge against the state variable variance. Continuous time models can be applied to surplus optimization problems, since surplus optimizers, such as pension funds, are legally obliged to cover their liabilities with their assets in each point of time during their lifetime. Pension funds frequently are broadly internationally diversified. Exchange rate movements are considered as state variables.

Surplus Management means investing in assets considering the dynamics of the liabilities. Leibowitz and Henriksson (1988) consider shortfall risk minimization for American pension funds. Further papers co-authored by Leibowitz are summarized in Leibowitz, Bader and Kogelman (1996). Shortfall risk is due to Roy (1952) where the „safety first" principle was introduced. Safety first is more general than shortfall risk because in contrast to Leibowitz and Henriksson (1988), Roy uses the Tschebyscheff inequality which enables him to work with non-normally distributed asset returns. Surplus management is related to stochastic wage growth rates. See Ezra (1991) and Black (1989) who investigate different asset allocation mixes for pension funds due to different assumptions of inflation rate volatility. Inflation rates are highly correlated to nominal wage growth rates. Other aspects of asset-liability management are surveyed in Ziemba and Mulvey (1998).

A problem similar to surplus optimization is addressed by Merton (1993 reprinted in Ziemba and Mulvey 1998: see also Constantinides' 1993 comments on Merton's paper). He advocates the view that University endowment funds can be managed by using Merton's (1969) intertemporal portfolio selection model. His objective function is to maximize the lifetime expected utility of University activities with specific costs. The University activity portfolio includes things like education, training, research, storage of knowledge, etc. Since the activities of a University have to be optimized with respect to the their costs, the activity costs are like "liabilities" for universities. However, Merton (1993) specifies the University's non-endowment cash flows as Itô processes dependent on the activity costs. This is the classical setting of Merton's (1973) intertemporal CAPM, where the state variables are given by the activity costs. The result is that each utility maximizing University holds three funds, the market portfolio, the riskless portfolio and a hedge portfolio against fluctuations of the activity costs. The composition across those three funds depends on the risk preferences towards market and activity cost volatility.

This paper provides a synthesis of the surplus management and the continuous time finance literature. In contrast to Merton's (1993) setting, the liability returns are modeled as a part of the surplus return and not as a state variable. Furthermore, the surplus return, which implies the asset- and the liability return, is assumed to depend on one or more state variables. This is a reasonable setting if the state variables are for instance imagined as currency returns. Three important results developed are: First, each pension fund maximizes lifetime expected utility by investing in four distinct mutual funds: the market portfolio, the state variable hedge portfolio, the liabilities hedge portfolio, and the riskless asset. Hence, the result of the model developed here extents Merton's (1993) approach by an additional hedge portfolio. Secondly, the investment in the liability hedge portfolio depends only on the current funding ratio of a pension fund and is independent of the utility function. This result differs from Merton's (1993) findings. The preference independence of the liability hedge portfolio has major implications for pension fund monitoring. Thirdly, the pension funds state variable hedging policy is preference-independent if HARA (hyperbolic absolute risk aversion) utility functions are assumed. With log utility, the market portfolio investment is independent of preferences, and the investment policy depends only on the current funding.

The paper has six sections. In section two, the intertemporal surplus management model and a four fund theorem is derived. In section three we show that with HARA utility, the risk aversion coefficients of the model are related to the funding ratio and to the currency betas of the portfolio. A k state variable case is derived in section 4. Section five describes a case study for a surplus optimizer who is diversified across the stock, the bond markets, and cash equivalents of four countries. Section six concludes the paper.

2. An intertemporal surplus management model - a four fund theorem

The surplus return $\tilde{R}_{St}$ in period *t* (see e.g. Sharpe and Tint 1990) is

(1) $$\tilde{R}_{St} \equiv \frac{\tilde{S}_{t+1} - S_t}{A_t} = \tilde{R}_{At} - \frac{1}{F_t}\tilde{R}_{Lt} ,$$

where $\tilde{R}_{At}$ represents the asset return and $\tilde{R}_{Lt}$ the liability return. Using *A* instead of *S* as the denominator avoids undefined results for *S*=0. The surplus in period *t* equals the difference between the assets and the liabilities, namely $S_t \equiv A_t - L_t$. The funding ratio is $F_t = A_t / L_t$. Let *Y* be a normally distributed state variable such as an exchange rate. Adler and Dumas (1983) take purchasing power parities as state variables. An extension of the setting to *k* state variables is possible without difficulties (see e.g. Richard 1979 and Adler and Dumas 1983). Let E_A, E_L, E_Y be the drift terms of the asset, liabilities and state variables processes, respectively. The volatilities of the asset, liabilities and state variables processes are s_A, s_L, s_Y, respectively. Then

(2) $$\begin{aligned} \tilde{R}_A &= \frac{d\tilde{A}}{A} \equiv E_A(Y,t)dt + s_A(Y,t)\cdot d\tilde{z}_A \\ \tilde{R}_L &= \frac{d\tilde{L}}{L} \equiv E_L(Y,t)dt + s_L(Y,t)\cdot d\tilde{z}_L \\ \tilde{R}_Y &= \frac{d\tilde{Y}}{Y} \equiv E_Y dt + s_Y d\tilde{z}_Y , \end{aligned}$$

where $d\tilde{z}_A \equiv \tilde{z}_A\sqrt{dt}$, $d\tilde{z}_L \equiv \tilde{z}_L\sqrt{dt}$ and $d\tilde{z}_Y \equiv \tilde{z}_Y\sqrt{dt}$ are standard Wiener processes for assets, liabilities, and the state variable, respectively. Substituting (2) into (1) yields the surplus return process:

(3) $$\begin{aligned} \tilde{R}_S &= \frac{d\tilde{S}}{A} \equiv \frac{d\tilde{A}}{A} - \frac{1}{F}\cdot\frac{d\tilde{L}}{L} \\ &= \left(E_A(Y,t) - \frac{1}{F}E_L(Y,t)\right)dt + \left(s_A(Y,t)\tilde{z}_A - \frac{1}{F}s_L(Y,t)\tilde{z}_L\right)\sqrt{dt} . \end{aligned}$$

Then (for proof see the appendix)

$$E\left(d\tilde{S}\right) = \left(E_A - \frac{1}{F}E_L\right) \cdot A \cdot dt$$

(4)
$$E\left(d\tilde{S}^2\right) = \left(s_A^2 + \frac{1}{F^2}s_L^2 - 2\frac{1}{F}s_{AL}\right) \cdot A^2 \cdot dt$$

$$E\left(d\tilde{S} \cdot d\tilde{Y}\right) = \left(s_{AY} - \frac{1}{F}s_{LY}\right) \cdot A \cdot Y \cdot dt, \text{ and}$$

$$E\left(d\tilde{Y}^2\right) = s_Y^2 \cdot Y^2 \cdot dt,$$

where functional dependencies on Y are omitted for simplicity and s_{AY} and s_{LY} are the covariances of the state variable with assets and liabilities, respectively.

The objective is to maximize the expected lifetime utility of surplus which implies identifying an optimum surplus strategy. The expected utility is positively related to the surplus in each period. This is due to the fact that positive surpluses improve the wealth position of the insurants of a pension fund, even if the yearly retirement benefits are over-covered by the surplus. Hence, in this interpretation, the insurants are like shareholders of the pension fund. Applying the Bellman principle (see e.g. Dixit and Pindyck 1994, chapter 4) with the utility function U assumed to be additively separable where t represents today's period and T the end of the pension fund's existence yields:

(5)
$$J(S,Y,t) \equiv \max_w E_t\left(\int_t^T U(S,Y,t)dt\right).$$

The J-function is the maximum of the pension fund's expected lifetime utility. According to Merton (1969)

(6)
$$\begin{aligned} J(S,Y,t) &= \max_w E_t\left(\int_t^{t+dt} U(S,Y,t)dt + \int_{t+dt}^T U(S,Y,t)dt\right) \\ &= \max_w E_t\left(\int_t^{t+dt} U(S,Y,t)dt\right) + J\left(S + d\tilde{S}, Y + d\tilde{Y}, t + dt\right) \\ &= U(S,Y,t)\,dt + J\left(S + d\tilde{S}, Y + d\tilde{Y}, t + dt\right). \end{aligned}$$

Applying Itô's lemma, where J_S is the first partial derivative of J with respect to S, J_{SS} is the second partial derivative of J with respect to S, J_Y and J_{YY}, respectively, are the first and second partial derivatives with respect to Y, and J_{SY} is the derivative of J with respect to S and Y (see the appendix for proof) yields

(7)
$$0 = \max_w \Big[J_S E_t\big(d\tilde{S}\big) + J_Y E_t\big(d\tilde{Y}\big) + J_t dt + \frac{1}{2} J_{SS} E_t\big(d\tilde{S}^2\big) + \frac{1}{2} J_{YY} E_t\big(d\tilde{Y}^2\big) + J_{SY} E_t\big(d\tilde{S} d\tilde{Y}\big) + U dt \Big].$$

Let n be the number of risky assets in the portfolio (in addition to the riskless asset), w is the vector of portfolio fractions of the risky assets, and m_A is the vector of expected asset returns of the risky assets, all of dimension n, e is a n-dimensional vector of ones, V is the covariance matrix of dimension n x n, $V_{AL}' \equiv (s_{1L}, \ldots, s_{nL})$ and $V_{AY}' \equiv (s_{1Y}, \ldots, s_{nY})$ are the vectors of covariances between the n assets and the liabilities, respectively, with the state variable. Rearranging (7) yields the following partial differential equation (see the appendix for proof):

(8)
$$\begin{aligned} 0 = \max_w \Big[& J_S \Big(w'(m_A - re) + r - \frac{1}{F} E_L \Big) \cdot A + J_Y E_Y + J_t \\ & + \frac{1}{2} J_{SS} \Big(w'Vw + \frac{1}{F^2} s_L^2 - 2\frac{1}{F} w'V_{AL} \Big) \cdot A^2 \\ & + \frac{1}{2} J_{YY} s_Y^2 \cdot Y^2 + J_{SY} \Big(w'V_{AY} - \frac{1}{F} s_{LY} \Big) \cdot A \cdot Y + U \Big]. \end{aligned}$$

Differentiating (8) with respect to w yields[1]

(9)
$$\begin{aligned} w &= -\frac{J_S}{AJ_{SS}} V^{-1}(m_A - re) - \frac{YJ_{SY}}{AJ_{SS}} V^{-1} V_{AY} + \frac{1}{F} V^{-1} V_{AL} \\ &= -a \frac{J_S}{AJ_{SS}} w_M - b \frac{YJ_{SY}}{AJ_{SS}} w_Y + c \frac{1}{F} w_L \end{aligned}$$

where $w_M \equiv \frac{V^{-1}(m_A - re)}{e'V^{-1}(m_A - re)}$, $w_Y \equiv \frac{V^{-1}V_{AY}}{e'V^{-1}V_{AY}}$, $w_L \equiv \frac{V^{-1}V_{AL}}{e'V^{-1}V_{AL}}$, $a \equiv e'V^{-1}(m_A - re)$, $b \equiv e'V^{-1}V_{AY}$, and $c \equiv e'V^{-1}V_{AL}$. The vectors w_M, w_Y, and w_L are of dimension n with elements that sum to 1; a, b, and c are real constants.

The optimum portfolio consists of four single portfolios: the market portfolio w_M, the hedge portfolio for the state variable w_Y, which is Merton's (1973) state variable hedge portfolio, the hedge portfolio for the liabilities w_L, and the riskless asset. The state variable hedge port-

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[1] The derivative of (8) with respect to w is: $0 = J_S(m_A - re) + AJ_{SS}Vw - \frac{A}{F} J_{SS} V_{AL} + YJ_{SY} V_{AY}$,

folio w_Y reveals the maximum correlation with the state variable Y (see the appendix). A perfect hedge for the state variable could be achieved if the universe of n risky assets contains forward contracts on the state variable. Then the state variable hedge portfolio would consist of a single asset which is the forward contract. The third portfolio w_L is interesting. For the liability hedge portfolio (i.e. a portfolio which hedges wage increases or inflation rates), no hedging opportunities are offered by the financial markets. Equation (9) shows how a liability hedge can be constructed. This is related to the problem addressed by Ezra (1991) and Black (1989) and solves it intertemporally. In the four fund theorem, pension funds invest in the following four funds:

First, the market portfolio w_M with level $-a\frac{J_S}{AJ_{SS}}$, secondly, the state variable hedge portfolio w_Y with level $-b\frac{YJ_{SY}}{AJ_{SS}}$, thirdly the riskless asset with level $1+a\frac{J_S}{AJ_{SS}}+b\frac{YJ_{SY}}{AJ_{SS}}-\frac{c}{F}$, and finally, the liability hedge portfolio w_L with level $c\frac{1}{F}$. Thus, the holdings of the liability hedge portfolio are independent of preferences. The most interesting result in the portfolio selection equation (9) is that the liability hedge portfolio holdings depend only on the current funding ratio and not on the form of the utility function. Each pension fund in order to maximize its lifetime expected utility should hedge the liabilities according to the financial endowment.

The percentages of each of the three other funds differ according to the risk preferences of the investors. For example, $-a\frac{J_S}{AJ_{SS}}$ is the percentage invested in the market portfolio. Since J is a "derived" utility function, this ratio is a times the Arrow/Pratt relative risk tolerance with respect to changes in the surplus. That is, the higher the risk tolerance towards market risk is, the higher the fraction of the market portfolio holdings will be. The percentage of the state variable hedge portfolio is $-b\frac{J_{SY}}{AJ_{SS}}$. Merton (1973) showed that this ratio is b times the Arrow/Pratt relative risk tolerance with respect to changes in the state variable. The percentage of the liability hedge portfolio is $c\frac{1}{F}$. Surprisingly, and in contrast to Merton’s results, this portfolio does not depend on preferences nor on a specific utility function, but only on the funding ratio of the specific pension fund. The lower the funding ratio, the higher will be the percentage of the liability hedge portfolio.

This allows for a simple technique to monitor pension funds, which extents Merton's (1993) approach. In most pension fund systems, pension funds are legally obliged to invest subject to a deterministic threshold return. Since payments of pension funds depend on the growth and the volatility of wage rates, this is not appropriate. For instance, if the threshold return is 4% p.a. and the wages grow by more than this, the liabilities cannot be covered by the assets. Our model suggests instead that a portfolio manager of a pension fund should invest in a portfolio which smoothes the fluctuation of the surplus returns caused by wage volatility, i.e. in a liability hedge portfolio. Since the liability hedge portfolio depends only on the funding ratio, preferences of the insurants have not to be specified.

3. Risk Preference, Funding Ratio, and Currency Betas

Assume that the utility function is from the HARA (hyperbolic absolute risk aversion) class. This is the class of linear risk tolerance utility functions. Let a be the risk aversion coefficient. Then

(10) $$U(S,Y,t) \equiv \frac{S^a}{a} \quad \Rightarrow \quad J(S,Y,t) = \max_w E_t\left(\int_t^T \frac{S(t)^a}{a} dt \right)$$

where $S = S(t) = S[A(Y,t)]$ according to (3).

Log utility is that member of HARA when a approaches 0, since by applying l'Hôpital's rule

(11) $$\lim_{a \to 0} U(S,Y,t) = \lim_{a \to 0} \frac{S^a}{a} = \lim_{a \to 0} \left(S^a \cdot \ln S \right) = \ln S \, .$$

Under the HARA utility function assumption

(12) $$J_S = S^{a-1} \quad J_{SS} = (a-1) \cdot S^{a-2}$$
$$J_{SY} = \frac{dS[A(Y)]^{a-1}}{dY} = J_{SS} \cdot \frac{dS}{dA} \cdot \frac{dA}{dY} = J_{SS} \cdot \frac{dA}{dY}.$$

The percentage holdings of the market portfolio may then be re-expressed as:

(13) $$-\frac{J_S}{A\cdot J_{SS}}=-\frac{S}{A(a-1)}=\frac{1}{1-a}\cdot\left(1-\frac{1}{F}\right).$$

The market portfolio holdings thus depend on the funding ratio (the higher *F*, the higher the investment in the market portfolio will be) and on the risk aversion (the higher a, the lower the market portfolio investment will be). If a approaches 0 (log utility case), the coefficient for the market portfolio investment becomes 1-1/*F*. Thus, for log utility pension funds, risk aversion does not matter. Indeed only the funding ratio matters to determine the market portfolio investment. For either case of utility functions, if the funding ratio is 100%, there will be no investment in the risky market portfolio. Thus, the funding ratio of a pension fund does not only determine the capability to bear risk but also the willingness to take risk.

We now consider the percentage holding of the state variable hedge portfolio. Suppose the state variable *Y* is an exchange rate fluctuation which affects the surplus of a pension fund. Given (12) it follows that:

(14) $$-\frac{Y\cdot J_{SY}}{A\cdot J_{SS}}=-\frac{dA/A}{dY/Y}=-\frac{\tilde{R}_A}{\tilde{R}_Y}.$$

Assume the regression model $\tilde{R}_A=b\left(\tilde{R}_A,\tilde{R}_Y\right)\cdot\tilde{R}_Y$ (asset returns are linear functions of currency returns). Hence, $-\tilde{R}_A/\tilde{R}_Y$ is the negative beta of the portfolio with respect to the state variable. Using (14) it follows that:

(15) $$-\frac{Y\cdot J_{SY}}{A\cdot J_{SS}}=-b\left(\tilde{R}_A,\tilde{R}_Y\right).$$

If *Y* is an exchange rate, then -*b* equals the minimum variance hedge ratio for the foreign currency position. The holdings of the state variable hedge portfolio are independent of preferences; they only depend on the foreign currency exposure of the portfolio. The higher the exchange rate risk in the portfolio is, the higher the currency hedging will be.

Hence, for the HARA case, only the investment in the market portfolio depends on the risk aversion a. The investments in all other funds are preference independent. They depend only on the funding ratio and on the exposure of the asset portfolio with the state variable.

4. The multiple state variable case

Let k be the number of foreign currencies contained in a pension fund's portfolio, $Y_1,\ldots, Y_k$ be the exchange rates in terms of the domestic currency, and $\tilde{R}_{Y_1},\ldots,\tilde{R}_{Y_k}$ the exchange rate returns of the k currencies. Then $-\frac{Y_1 J_{SY_1}}{AJ_{SS}} = -b\left(\tilde{R}_A,\tilde{R}_{Y_1}\right),\ldots,-\frac{Y_k J_{SY_k}}{AJ_{SS}} = -b\left(\tilde{R}_A,\tilde{R}_{Y_k}\right)$, which are the Arrow/Pratt risk tolerances with respect to changes in the exchange rates for the HARA case. Let $V_{AY_1},\ldots,V_{AY_k}$ be the covariance vectors of the returns of the asset portfolio with the k exchange rate returns, $w_{Y_1} \equiv \frac{V^{-1}V_{AY_1}}{e'V^{-1}V_{AY_1}},\ldots,w_{Y_k} \equiv \frac{V^{-1}V_{AY_k}}{e'V^{-1}V_{AY_k}}$ which are the state variable hedge portfolios 1 to k, and $b_1 \equiv e'V^{-1}V_{AY_1},\ldots,b_k \equiv e'V^{-1}V_{AY_k}$ are the coefficients of the state variable hedge portfolios. A pension fund with HARA utility function facing k state variables has the following investment strategy:

(16)
$$w = \frac{a}{1-a}\left(1-\frac{1}{F}\right)w_M - \sum_{i=1}^{k} b_i b\left(R_A, R_{Y_i}\right)w_{Y_i} + c\frac{1}{F}w_L$$

where $b\left(R_A, R_{Y_i}\right) = \sum_{l=1}^{n} w_l b\left(R_{A_l}, R_{Y_i}\right)$.

Since the right hand side of equation (16) depends on the portfolio allocation w, it is not possible to solve (16) analytically for w, therefore numerical solutions must be used. For k state variables a k+3 fund theorem thus follows.

5. Case Study

The following case study is based on an USD based surplus optimizer investing in the stock and bond markets of the US, UK, Japan, and Germany. Monthly MSCI data between January 1986 and March 1997 (135 observations) are used for the stock markets. The monthly JP Morgan indices are used for the bond markets in this period. The stochastic benchmark for a surplus optimizer is the growth of wages and salaries, using the Datastream index for US - wages and salaries. The average growth rate of wages and salaries in the US between January

1986 and March 1997 was 5.4% p.a. (see **Table 1**) with annualized volatility of 4%. **Table 1** also contains the stock and bond market descriptive statistics in terms of USD, and the currency betas of the indices. All foreign currencies, i.e. GBP, JPY, and DEM, have volatility about 12% p.a., and all currencies appreciated against the USD by 1% to 4.5% per year. From a USD viewpoint, the GBP-beta is especially high for the UK bond market, the Japanese stock and bond market reveal a JPY-beta of 1.2 and 1.12, respectively, and the German bond market has a DEM-beta of 1.01. All other countries have substantially lower currency betas. Since the betas are close to zero, the wages and salaries obviously do not depend on currency movements.

The investor faces an exposure against three foreign currencies (GBP, JPY, DEM), and has to invest into six funds. Three of them are hedge portfolios for the state variables which are assumed to be currency returns. The next step is therefore to calculate the compositions of the six funds. The results are shown in **Table 2**. The major holdings in the market portfolio are investments in the US stock market and the UK bond market. Substantial short positions for the tangency portfolio are obtained for the UK stock and the US bond market. The US bond portfolio is a major part of a portfolio providing the best hedge against fluctuations in wages and salaries. More than 126% of the liability hedge portfolio are invested in US bonds market. The currency hedge portfolios are dominated by the bond markets of the respective currencies.

As derived in section three, the holdings of the six funds depend only on the funding ratio and on the currency betas of the distinct markets. This is shown in **Table 3**. For a funding ratio of one, there is no investment in the market portfolio and only diminishing investments in the currency hedge portfolios. The portfolio betas against the three currencies are zero. The higher the funding ratio is, the higher is the investment in the market portfolio, the lower are the investments in the liability and the state variable hedge portfolios, and the lower is the investment in the riskless fund. For funding ratios exceeding one, the currency hedge portfolios are shortened which implies an increase of the currency exposure instead of a hedge against it. As a consequence, the portfolio beta against the currency returns increases to 0.29 for a funding ratio of 150%. Hence, a significant currency risk is accepted, when the liabilities are broadly covered by the assets. The increase of the market portfolio holdings and the reduction of the hedge portfolio holdings and the riskless fund for increasing funding ratios shows that the funding ratio is directly related to the ability to bear risk. Rather than risk aversion coefficients, the funding ratio provides an objective measure to quantify attitudes towards risk.

6. Conclusions

This model derives a four fund theorem for intertemporal surplus optimizers. In addition to the three funds identified by Merton (1973), the expected utility maximizing portfolio contains a liability hedge portfolio which is preference independent. Its holdings depend only on the funding ratio of a pension fund. The higher the funding ratio is, the lower the necessity for liabilities hedging will be. As a practical consequence, the hedging policy of pension funds could very easily be monitored by authorities. This is due to the fact that in the optimum, only the funding ratio is decisive for the liabilities hedge, and not the utility function, which hardly can be determined by law. Today's pension fund laws do not contain any rules for the treatment of stochastic wage growths. Although Merton (1993) addresses a similar problem of the optimal investment strategy for University endowment funds, his setting is different. He describes the "liabilities" of universities, i.e. the costs for their activities, as state variables. As a results, he obtains a preference dependent hedge portfolio for the activity costs. In contrast, here the liability returns are specified as a part of the surplus return. In contrast to Merton's results, the hedge portfolio for the liabilities is exclusively dependent on the funding ratio (and not on risk preferences) of a pension fund. The funding ratio is an objective measure whereas risk preferences are hard to determine.

The model provides an intertemporal portfolio selection approach for surplus optimizers. The intertemporal surplus management approach holds for investors who have to cover liabilities by their assets in each moment of time. Since the investment strategy of all investors is consumption orientated, it is reasonable to assume that all investors invest in order to cover their liabilities. If the growth rate of individual consumption is related to the growth of wages and salaries, the relevant benchmark for surplus optimizers refers to the growth rate and the volatility of wages and salaries. Finally, this model suggests a new type of product for investment banks. Pension funds need to protect themselves against unanticipated changes in the growth rates of wages and salaries. Investment bank could offer liability protection and hedge those positions by purchasing a liability hedge portfolio as is given by equation (9).

Tables

Table 1:
Descriptive statistics

The stock data is based on MSC indices and the bond data on JP Morgan indices. The wage growth rates are from Datastream. Monthly data between January 1986 and March 1997 (135 observations) is used. All coefficients are in USD. The average returns and volatilities are in percent per annum.

		Mean return	Volatility	Beta GBP	Beta JPY	Beta DEM
Stocks	USA	11.83	14.35	-0.13	-0.20	-0.21
	UK	11.10	19.57	0.66	0.37	0.32
	Japan	7.62	26.95	0.81	1.2	0.65
	Germany	8.45	20.76	0.48	0.28	0.54
Bonds	USA	8.11	4.97	0.06	0.06	0.10
	UK	11.59	14.38	1.03	0.61	0.67
	Japan	11.03	14.30	0.63	1.12	0.76
	Germany	10.37	12.74	0.74	0.73	1.01
Exchange rates	GBP	1.01	11.93	1.0	0.54	0.75
	JPY	4.49	12.05	0.55	1.0	0.68
	DEM	3.27	11.94	0.75	0.67	1.00
	Wages and salaries	5.41	3.97	0.01	-0.01	0.00

Table 2:
Optimum portfolios of an internationally diversified pension fund

The portfolio holdings are based on equation (9). All portfolio fractions are percentages. A riskless rate of interest of 7% per annum is assumed.

		Market portfolio	Liability hedge portfolio	Hedge portfolio GBP	Hedge portfolio JPY	Hedge portfolio DEM
Stocks	USA	92.0	-13.3	-1.9	-19.8	-3.4
	UK	-41.0	17.1	-10.2	11.7	3.2
	Japan	-15.5	14.0	6.6	5.3	0.4
	Germany	-11.3	-2.8	1.6	2.0	-5.2
Bonds	USA	-30.5	126.2	-124.2	-36.4	-46.9
	UK	55.8	-17.6	165.8	-15.5	-4.7
	Japan	31.1	-15.4	-30.4	135.8	-9.1
	Germany	19.5	-8.3	92.7	16.8	165.7

Table 3
Weightings of the funds due to different funding ratios

The weightings of the portfolios according to equation (16), where a=0, i.e. log utility, is assumed. The weightings of the six funds are in percent.

Funding ratio	0.9	1	1.1	1.2	1.3	1.5
Market portfolio	-70.5	0.0	57.7	105.7	146.4	211.4
Liability hedge portfolio	12.7	11.4	10.4	9.5	8.8	7.6
Hedge portfolio GBP	3.7	-0.1	-3.1	-5.7	-7.8	-11.3
Hedge portfolio JPY	2.4	0.0	-2.0	-3.7	-5.1	-7.3
Hedge portfolio DEM	2.6	-0.1	-2.3	-4.1	-5.7	-8.2
Riskless assets	149.0	88.7	39.3	-1.8	-36.6	-92.3
Portfolio beta against GBP	-0.10	0.00	0.08	0.15	0.20	0.29
Portfolio beta against JPY	-0.06	0.00	0.05	0.10	0.13	0.19
Portfolio beta against DEM	-0.07	0.00	0.06	0.11	0.15	0.21

Appendix

<u>Derivation of equation (4):</u>

Since $E(\tilde{z}_A) = E(\tilde{z}_L) = 0$ the first expression is straightforward:

$$E\left(d\tilde{S}^2\right) = E\left[\left(\left(E_A - \frac{1}{F}E_L\right)\cdot A \cdot dt + \left(s_A \tilde{z}_A - \frac{1}{F} s_L \tilde{z}_L\right) A\sqrt{dt}\right)^2\right]$$

$$= E\left[\left(s_A \tilde{z}_A - \frac{1}{F} s_L \tilde{z}_L\right)^2\right]\cdot A^2 \cdot dt \text{, because } dt^2 = dt^{3/2} = 0$$

$$= \left(s_A^2 \cdot E\left(\tilde{z}_A^2\right) + \frac{1}{F^2} s_L^2 \cdot E\left(\tilde{z}_L^2\right) - \frac{1}{F}\cdot 2 \cdot s_A \cdot s_L \cdot E\left(\tilde{z}_A \cdot \tilde{z}_L\right)\right)\cdot A^2 \cdot dt\,.$$

Since $\tilde{z}_A$ and $\tilde{z}_L$ are distributed standardnormal $E\left(\tilde{z}_A^2\right) = E\left(\tilde{z}_L^2\right) = 1$. Furthermore:

$$s_A \cdot s_L \cdot E\left(\tilde{z}_A \cdot \tilde{z}_L\right) = E\left(\tilde{R}_A \cdot \tilde{R}_L\right) = E\left[\left(\tilde{R}_A - E_A\right)\left(\tilde{R}_L - E_L\right)\right] = s_{AL}\,.$$

This yields $E\left(d\tilde{S}^2\right)$ in (4). Moreover:

$$E\left(d\tilde{S}\cdot d\tilde{Y}\right) = AYs_A s_Y E\left(\tilde{z}_A \cdot \tilde{z}_Y\right)dt - \frac{A}{F} Ys_L s_Y E\left(\tilde{z}_L \cdot \tilde{z}_Y\right)dt$$

$$= \left(s_{AY} - \frac{1}{F} s_{LY}\right)\cdot A \cdot Y \cdot dt.$$

This provides $E\left(d\tilde{S}\cdot d\tilde{Y}\right)$ in (4).

<u>Derivation of (7):</u>

The arguments of the *J-function* are dropped for simplicity. *o(dt)*summarizes all terms of higher order than 1 of *dt*. For *o(dt)*: $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$. Applying Itô’s lemma yields:

$$J(S,Y,t) = U(S,Y,t)\,dt + \max_{w} E_t\left[J\left(S + d\tilde{S}, Y + d\tilde{Y}, t + dt\right)\right]$$

$$= U\,dt + \max_{w} E_t\left[J + J_S d\tilde{S} + J_Y d\tilde{Y} + J_t dt + \frac{1}{2} J_{SS} d\tilde{S}^2 + \frac{1}{2} J_{YY} d\tilde{Y}^2 + J_{SY} d\tilde{S} d\tilde{Y} + o(dt)\right].$$

This yields the fundamental partial differential equation:

$$0 = U\,dt + \max_{w} E_t\left[J_S d\tilde{S} + d\tilde{Y} J_Y + J_t dt + \frac{1}{2} J_{SS} d\tilde{S}^2 + \frac{1}{2} J_{YY} d\tilde{Y}^2 + J_{SY} d\tilde{S} d\tilde{Y} + o(dt)\right].$$

<u>Derivation of (8):</u>

Substituting (4) into (7) yields:

$$\begin{aligned} 0 = \max_{w} \Bigg[& J_S\left(E_A - \frac{1}{F} E_L\right) \cdot A + J_Y E_Y + J_t \\ & + \frac{1}{2} J_{SS}\left(s_A^2 + \frac{1}{F^2} s_L^2 - 2\frac{1}{F} s_{AL}\right) \cdot A^2 \\ & + \frac{1}{2} J_{YY}^2 s_Y^2 \cdot Y^2 + J_{SY}\left(s_{AY} - \frac{1}{F} s_{LY}\right) \cdot A \cdot Y + U \Bigg]. \end{aligned}$$

Furthermore

$$\begin{aligned} E_A &= w'(m_A - re) + r \\ s_A^2 &= w'Vw \\ s_{AL} &= w'V_{AL} \\ s_{AY} &= w'V_{AY}\,, \end{aligned}$$

which yields (8).

<u>Discussing equation (9):</u>

Maximizing the covariance between the asset portfolio and the state variable:

$$Cov\left(w'\begin{pmatrix} \tilde{R}_1 \\ \vdots \\ \tilde{R}_n \end{pmatrix}, \frac{d\tilde{Y}}{Y}\right) = w'V_{AY} \underset{w}{\rightarrow} max \quad s.t. \quad w'Vw = s_A^2$$

where $\tilde{R}_1, \ldots, \tilde{R}_n$ refer to the *n* asset returns. Let λ be a Lagrange multiplier:

$$L = w' V_{AY} - \lambda \left(w' V w - \sigma_A^2 \right) \Rightarrow \frac{\partial L}{\partial w} = V_{AY} - 2\lambda V w = 0 \Leftrightarrow w = \frac{1}{2\lambda} V^{-1} V_{AY}$$

$$\frac{\partial^2 L}{(\partial w)^2} = -2\lambda V < 0 \quad \text{if and only if } V \text{ is positive definite.}$$

Hence, w_Y (the state variable hedge portfolio) maximizes the correlation between the asset portfolio and the state variable. Furthermore:

$$2\lambda V w = V_{AY} \Leftrightarrow 2\lambda w' V w = w' V_{AY} \Leftrightarrow 2\lambda = \frac{\sigma_{AY}}{\sigma_A^2} = \beta_{AY} \Rightarrow V^{-1} V_{AY} = \beta_{AY} w$$

Multiplying the fractions of the asset portfolio by the regression coefficient β_{AY} provides the fractions of the hedge portfolio.

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