A Multi-stage Stochastic Integer Programming Approach for Capacity Expansion under Uncertainty

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November 15, 2000

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Abstract

This paper addresses a multi-period investment model for capacity expansion in an uncertain environment. Using a scenario tree approach to model the evolution of uncertain demand and cost parameters, and fixed-charge cost functions to model the economies of scale in expansion costs, we develop a multi-stage stochastic integer programming formulation for the problem. A reformulation of the problem is proposed using variable disaggregation to exploit the lot-sizing substructure of the problem. The reformulation significantly reduces the LP relaxation gap of this large scale integer program. A heuristic scheme is presented to perturb the LP relaxation solutions to produce good quality integer solutions. Finally, we outline a branch and bound algorithm that makes use of the reformulation strategy as a lower bounding scheme, and the heuristic as an upper bounding scheme, to solve the problem to global optimality. Our preliminary computational results indicate that the proposed strategy has significant advantages over straightforward use of commercial solvers.

1 Introduction

Planning for capacity expansion forms a crucial part of the strategic level decision making in many applications. Examples can be found in heavy process industries [26, 35], communication networks [10, 36, 22], electric utilities [30, 31], automobile industries [14], service industries [5, 4], and more recently, in electronic goods and semiconductor industries [34, 6, 37]. In all of these applications, the expansion of production capacity requires the commitment of substantial capital resources over long periods of time. Furthermore, the economies-of-scale in the expansion costs, as well as the uncertainties in the long range forecasts for costs and demands, make these decision problems very complex. Consequently, quantitative models for economic capacity expansion planning has been the subject of intense research since the early 1960s.

Early approaches for solving stochastic capacity expansion problems were based on stochastic control theory [25, 17, 12, 3]. In these models, the demands are assumed to be simple stochastic processes to render analytical tractability. With the advent of stochastic programming and increased computational power, the use of scenarios to model uncertainties in planning models has become increasingly popular [20, 7]. These models allow inclusion of greater logistical details in the form of constraints than conventional dynamic programming approaches. In capacity expansion problems however, fixed-charge expansion cost functions prevent the use of standard stochastic programming decomposition approaches. To overcome this difficulty, existing stochastic programming approaches for capacity planning, either, assume linear expansion costs [15, 5, 11], or are restricted to two decision stages [14, 24, 37]. In two stage stochastic capacity expansion models, the first decision stage constitutes determining the capacity expansion schedule for the entire planning horizon, while scenario dependent second stage decision constitutes taking recourse actions in order to correct any infeasibilities. These recourse actions can be interpreted as outsourcing additional capacity. Multi-stage models extend the two-stage stochastic programming models by allowing revised decisions in each time stage based upon the uncertainty realized so far. The uncertainty information in a multi-stage stochastic program is modeled as a multi-layered scenario tree, and the optimization problem consists of determining an expansion schedule that hedges against this scenario tree. A notable exception to the existing literature in this area is that of Rajagopalan *et al.* [34], where the authors addressed a multi-stage stochastic capacity planning model with concave expansion costs. The authors assumed a single product family with non-decreasing *deterministic* demand, with the uncertainties in the timing of capacity availability. For this model, the authors exploited the problem structure to design an efficient dynamic programming algorithm.

This paper addresses a multi-stage capacity expansion problem with uncertainties in demand and cost parameters, and economies of scale in expansion costs. Using a scenario tree approach to model the evolution of uncertain parameters, and fixed-charge cost functions to model the economies of scale in expansion costs, we develop a multi-stage stochastic integer programming formulation for the problem. A reformulation of the problem is proposed using variable disaggregation to exploit the lot-sizing substructure of the problem. We show that the proposed reformulation significantly reduces the LP relaxation gap of the original large-scale integer program. We describe a heuristic scheme to perturb the LP relaxation solutions to produce good quality integer solutions. Finally, we outline a branch and bound algorithm that makes use of the reformulation strategy as a lower bounding scheme, and the heuristic as an upper bounding scheme, to solve the problem to global optimality.

The remainder of this paper is organized as follows. The next section presents a multi-stage stochastic integer programming formulation for the problem under study. A reformulation strategy is developed in Section 3. In Section 4, we discuss a heuristic for constructing feasible solutions to the problem. A branch and bound algorithm is discussed in Section 5. Finally, some computational results are presented in Section 6.

2 Formulation

In this section, we present a multi-stage stochastic integer programming formulation for the multi-facility capacity expansion problem.

Let us first address the deterministic problem. Consider a planning horizon of T time periods, over which the capacity investment costs, and demands are assumed to be known. The objective is to determine a schedule of timing and level of capacity acquisitions of a set of $\mathcal{I}$ resources or technology types to satisfy the demand of a product family while minimizing the total discounted cost over the entire planning horizon. Fixed-charge cost models are assumed for the economies of scale in the investment costs. Without loss of generality, we assume zero initial capacities. Using x_{it} to denote the capacity expansion of resource type $i \in \mathcal{I}$ in period t and y_{it} to denote the boolean variable for the corresponding capacity expansion decision, the problem can be stated as follows:

$$\text{(CAP)}: \quad \min \quad \sum_{t=1}^{T} \sum_{i \in \mathcal{I}} (\alpha_{it} x_{it} + \beta_{it} y_{it}) \tag{1}$$

$$\text{s.t.} \quad 0 \le x_{it} \le M_{it} y_{it} \qquad t = 1, \ldots, T; i \in \mathcal{I} \tag{2}$$

$$\sum_{\tau=1}^{t} \sum_{i \in \mathcal{I}} x_{i\tau} \ge d_t \qquad t = 1, \ldots, T \tag{3}$$

$$y_{it} \in \{0, 1\} \qquad t = 1, \ldots, T; i \in \mathcal{I} \tag{4}$$

where α_{it} and β_{it} are the discounted variable and fixed investment cost components, and

d_t are the demand parameters, respectively. M_{it} are the variable upper bounds on the capacity additions. Constraint (2) enforces that capacity acquisition levels are bounded by the expansion bounds M_{it}. For the purposes of this paper, it is assumed that M_{it} is sufficiently large. Constraint (3) ensures that total capacity installed is sufficient to satisfy the demand. Finally, the objective (1) is to minimize the total discounted expansion cost.

To extend the formulation (CAP) to a stochastic setting, we assume that the uncertain problem parameters $(\alpha_{it}, \beta_{it}, d_t)$ evolve as discrete time stochastic processes with a finite probability space and generate a filtration. This information structure can be interpreted as a scenario tree where the nodes n in stage (or level) t of the tree constitute the states of the world that can be distinguished by information available up to time stage t. Each node n of the scenario tree, except the root ($n = 0$), has a unique parent $a(n)$, and each non-terminal node n is the root of a sub-tree $\mathcal{T}(n)$. Thus, $\mathcal{T}(0)$ denotes the entire tree. The probability associated with the state of the world in node n is p_n. $\mathcal{S}_t$ denotes the set of nodes corresponding to time stage t, and t_n is the time stage corresponding to node n. The path from the root node to a node n will be denoted by $\mathcal{P}(n)$. If n is a terminal (leaf) node then $\mathcal{P}(n)$ corresponds to a *scenario*, and represents a joint realization of the problems parameters over all periods $1, \dots, T$. There are S leaf nodes corresponding to S scenarios. The notation just described is illustrated in Figure 1. Note that, unlike simple stochastic process models, scenario tree representation of uncertainty can approximate a wide variety of distributions and correlations. Accurately approximating complex stochastic processes or probability distributions by scenario trees is an active research area in itself [29, 18, 13]. With the scenario tree specified, and considering a risk-neutral objective of minimizing expected total cost, the stochastic capacity expansion problem can be written as:

$$\text{(SCAP)}: \quad \min \quad \sum_{n \in \mathcal{T}(0)} p_n \left\{ \sum_{i \in \mathcal{I}} (\alpha_{in} x_{in} + \beta_{in} y_{in}) \right\} \tag{5}$$

$$\text{s.t.} \quad 0 \le x_{in} \le M_{in} y_{in} \qquad n \in \mathcal{T}(0); i \in \mathcal{I} \tag{6}$$

$$\sum_{m \in \mathcal{P}(n)} \sum_{i \in \mathcal{I}} x_{im} \ge d_n \qquad n \in \mathcal{T}(0) \tag{7}$$

$$y_{in} \in \{0, 1\} \qquad n \in \mathcal{T}(0); i \in \mathcal{I} \tag{8}$$

Formulations (CAP) and (SCAP) represent the basic structure of multi-resource capacity expansion problems. These can be extended and generalized in a number of ways. For example, a deterministic expansion lead time of L can be modeled in (CAP) by changing the summation in constraint (3) to $\sum_{\tau=1}^{(t-L)^+}$. Similarly, for (SCAP), the summation in constraint (7) can be changed to $\sum_{m \in \cup_{t=t_n-L}^{t_n} \mathcal{P}(n) \setminus \mathcal{S}_t}$. Versions of (SCAP) that consider operating decisions (*i.e.*, how much of existing capacity should be committed for production), resources with unequal yield rates, and multiple demand families have been addressed in [1]. The inclusion of inventory balances is also a straightforward extension, as is the consideration of multiple product families. Much of the subsequent developments in this paper are applicable to these model extensions without any added conceptual difficulty.

(SCAP) is a multi-stage stochastic integer program for which no practical general purpose solution methodology exists. In principle, with the scenario tree specified, the problem

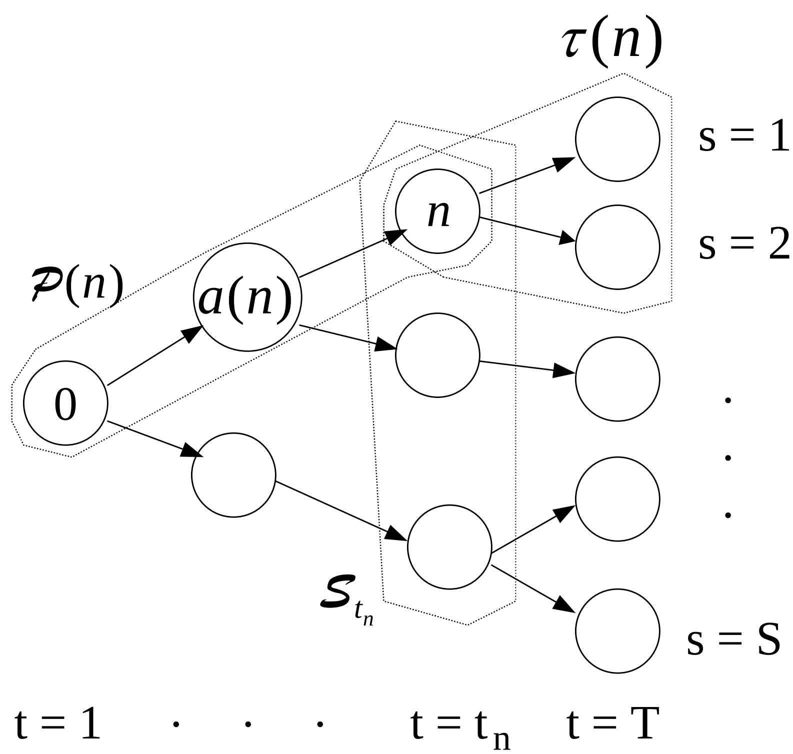

Figure 1: The Scenario Tree Notation

is a large scale deterministic mixed integer program and can be solved by standard IP techniques. However, such a scheme will computationally very expensive. In the following sections, we develop a specialized solution strategy to take advantage of the problem structure.

3 Problem Reformulation

This section explores lot sizing substructures in the stochastic capacity expansion problem that can be exploited to obtain "tight" reformulations, *i.e.*, reformulations with small LP relaxation gaps. Similar substructures for the deterministic capacity expansion problem have been investigated in [35]. Tighter problem reformulations can help rounding heuristics to produce approximate integer feasible solutions of good quality. Furthermore, the reformulations can provide better lower bounds and help expedite convergence in exact branch and bound algorithms.

We begin by drawing the equivalence between the stochastic uncapacitated lot-sizing problem and single resource ($|\mathcal{I}| = 1$) instances of (SCAP). Next, we extend a well known reformulation scheme for the deterministic uncapacitated lot-sizing problem to the stochastic case. This scheme is then used to obtain reformulations of (SCAP) with tight LP relaxation gaps.

3.1 The Stochastic Lot-Sizing Problem

The deterministic uncapacitated lot sizing problem is stated as [32]:

$$
\begin{aligned}
(\text{LSP}):\quad \min \quad & \sum_{t=1}^{T}(\alpha_t X_t + \beta_t Y_t + h_t I_t) \\
\text{s.t.} \quad & I_{t-1} + X_t = \overline{d}_t + I_t \quad t = 1, \ldots, T \\
& X_t \leq M_t Y_t \qquad t = 1, \ldots, T \\
& I_o = 0 \\
& X_t, I_t \geq 0, Y_t \in \{0,1\} \quad t = 1, \ldots, T,
\end{aligned}
$$

where X_t, I_t represents the production and inventory level in period t, and Y_t indicates whether a production set-up is carried out in period t. Problem parameters α_t, β_t, h_t, and $\overline{d}_t$ represent the production cost, set-up cost, holding cost, and the demand in period t. M_t are sufficiently large upper bounds on X_t. Since there is no backlogging, these bounds can be set as $M_t = \sum_{\tau=t}^{T} \overline{d}_\tau$.

Using the notation described for the stochastic capacity expansion problem, the stochastic lot sizing problem can be formulated as:

$$
\begin{aligned}
(\text{SLSP}):\quad \min \quad & \sum_{n \in \mathcal{T}(0)} p_n(\alpha_n X_n + \beta_n Y_n + h_n I_n) \\
\text{s.t.} \quad & I_{a(n)} + X_n = \overline{d}_n + I_n \quad n \in \mathcal{T}(0) \\
& X_n \leq M_n Y_n \qquad n \in \mathcal{T}(0) \\
& X_n, I_n \geq 0, Y_n \in \{0,1\} \quad n \in \mathcal{T}(0).
\end{aligned}
$$

Note that it is important to have a tight upper bound on X_n. It is easy to see that a valid upper bound on X_n is given by

$$M_n = \max_{m \in \mathcal{S}_T \cap \mathcal{T}(n)} \left\{ \sum_{k \in \mathcal{P}(m) \cap \mathcal{T}(n)} \overline{d}_m \right\}.$$

Proposition 3.1 *There is a one-to-one correspondence between the set of feasible solution* (x_n, y_n) *of a single resource instance of* (SCAP) *with parameters* (α_n, β_n, d_n)*, and the set of feasible solutions* (X_n, Y_n, I_n) *of an instance of* (SLSP) *with parameters* $(\alpha_n, \beta_n, \overline{d}_n)$*, where* $\overline{d}_n = (d_n - \max_{m \in \mathcal{P}(n) \setminus n} \{d_m\})^+$.

Proof: See Appendix □

By the above result, we can solve single facility instances of (SCAP) by solving equivalent instances of (SLSP) with similar cost coefficients. For the deterministic lot-sizing problem (LSP), there are well known reformulations for which the LP relaxations yields integral solutions. These results are mainly based on the Wagner-Whitin conditions on the structure of the optimal solution. However, when parameter uncertainties are present, the extension of these results is not obvious. Next we investigate the reformulation scheme of Krarup and Bilde [21] in the context of the stochastic lot sizing problem. We discover that, although the relaxation of the reformulated problem does not yield integral solutions, the scheme serves to significantly tighten the relaxation gap.

3.2 The Krarup-Bilde reformulation

Krarup and Bilde [21] presented a formulation of (LSP) by defining $Q_{t\tau}$ as the quantity produced in period t to satisfy the demand in period $\tau = t, \ldots, T$. Then:

$$X_t = \sum_{\tau=t}^{T} Q_{t\tau} \quad t = 1, \ldots, T. \tag{9}$$

Using these variables and eliminating the inventory variables, the K-B reformulation of the (LSP) is as follows.

$$\begin{aligned}
\text{(RLSP)}: \quad \min \quad & \sum_{t=1}^{T} \sum_{\tau=t}^{T} (\alpha_t + h_t + h_{t+1} + \ldots + h_{\tau-1}) Q_{t\tau} + \sum_{t=1}^{T} \beta_t Y_t \\
\text{s.t.} \quad & \sum_{\tau=1}^{t} Q_{\tau t} = \overline{d}_t \quad t = 1, \ldots, T \\
& Q_{t\tau} \leq \overline{d}_\tau Y_t \quad t = 1, \ldots, T; \tau = t, \ldots, T \\
& Q_{t\tau} \geq 0, Y_t \in \{0, 1\}.
\end{aligned}$$

Proposition 3.2 (cf. [32]) *The solution to the LP relaxation of* (RLSP) *yields* $0-1$ *values for the* $Y-$*variables. In addition, the image in the* (X, I, Y) *space under the transformation (9) of all points* (Q, Y) *feasible in the LP relaxation of* (RLSP) *produces the convex hull of* (LSP).

It thus follows that one only needs to solve the LP relaxation of (RLSP) and obtain a solution to (LSP). A number of other reformulation of (LSP) exist for which the above result also hold [2, 27, 33].

To extend the K-B reformulation strategy to (SLSP), let us introduce variables Q_{nk} for all $k \in \mathcal{T}(n)$ to indicate the *part* of the production X_n in node n that is used to satisfy the demand in node k. However, in the stochastic case, the production at a node n can be used to satisfy various demand scenarios corresponding to a particular time period. The main observation here is that the amount of production required at a node n is the *maximum* total amount carried over from node n to the successive periods. Thus we modify the K-B transformation in eq. (9) as follows:

$$X_n = \max_{m \in \mathcal{S}_T \cap \mathcal{T}(n)} \left\{ \sum_{k \in \mathcal{P}(m) \cap \mathcal{T}(n)} Q_{nk} \right\}.$$

We can now reformulate (SLSP) as follows:

$$\begin{aligned}
\text{(RSLSP)}: \quad \min \quad & \sum_{n \in \mathcal{T}(0)} p_n \left[\alpha_n X_n + h_n I_n + \beta_n Y_n \right] \\
\text{s.t.} \quad & X_n \geq \sum_{k \in \mathcal{P}(m) \cap \mathcal{T}(n)} Q_{nk} && m \in \mathcal{S}_T \cap \mathcal{T}(n), n \in \mathcal{T}(0) \\
& \sum_{k \in \mathcal{P}_n} Q_{kn} = \overline{d}_n && n \in \mathcal{T}(0) \\
& Q_{nk} \leq \overline{d}_k Y_n && k \in \mathcal{T}(n), n \in \mathcal{T}(0) \\
& I_{a(n)} + X_n = \overline{d}_n + I_n && n \in \mathcal{T}(0) \\
& Q_{nk}, I_n \geq 0, Y_n \in \{0, 1\}.
\end{aligned}$$

Proposition 3.3 *The optimal objective value of the LP relaxation of* (RSLSP) *is no smaller than that of* (SLSP), *and it may be strictly greater.*

Proof: Given a feasible solution (Q, X, I, Y) to the LP relaxation of (RSLSP), we need to show that (X, I, Y) is a feasible solution to the LP relaxation of (SLSP) with the same objective function value.

We only need to show that the solution to (RSLSP) satisfies the constraints: $X_n \leq M_n Y_n$, since all other constraints are implied. Notice that

$$\begin{aligned}
X_n &= \max_{m \in \mathcal{S}_T \cap \mathcal{T}(n)} \left\{ \sum_{k \in \mathcal{P}(m) \cap \mathcal{T}(n)} Q_{nk} \right\} \\
&= \max_{m \in \mathcal{S}_T \cap \mathcal{T}(n)} \left\{ \sum_{k \in \mathcal{P}(m) \cap \mathcal{T}(n)} \overline{d}_k . \frac{Q_{nk}}{\overline{d}_k} \right\} \\
&\leq \max_{m \in \mathcal{S}_T \cap \mathcal{T}(n)} \left\{ \sum_{k \in \mathcal{P}(m) \cap \mathcal{T}(n)} \overline{d}_k Y_n \right\} \\
&= M_n Y_n
\end{aligned}$$

where the last two steps follow from the fact that $Q_{nk} \leq \overline{d}_k Y_n$, and the definition of M_n. Also note that we only consider those $k \in \mathcal{P}(m) \cap \mathcal{T}(n)$ for which $\overline{d}_k > 0$, since otherwise $Q_{nk} = 0$.

Thus the solution (X, Y, I) is feasible to (SLSP), and also has the same objective function value. It then follows that the optimal value of the LP relaxation of (RSLSP) is no smaller than that of (SLSP). The numerical example below shows that the value can indeed be strictly greater. □

Example 1

Consider an instance of (SLSP) with zero holding costs. The uncertain parameters evolve over the scenario tree depicted in Figure 2. The corresponding problem data are provided in Table 1. The optimal IP and LP objective values of formulations (SLSP) and (RSLSP) are compared in Table 2. We observe that the reformulation has very small LP relaxation gap (0.79%) in comparison to the original formulation (26.04%).

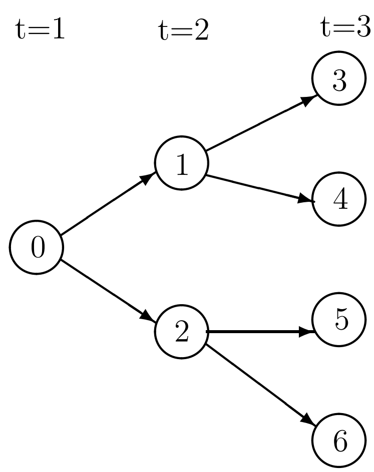

Figure 2: The Scenario Tree

n	α_n	β_n	d_n	p_n
1	5	20	5	1
2	3	59	5	0.3
3	1	21	15	0.7
4	1	10	5	0.1
5	2	16	10	0.2
6	1	10	10	0.3
7	2	10	20	0.4

Table 1: Problem Parameters

Formulation	IP obj. val.	LP obj. val.	% Gap
(SLSP)	114.4	84.6	26.04
(RSLSP)	114.4	113.5	0.70

Table 2: Comparison of LP relaxation gaps

Unfortunately, unlike that of its deterministic counterpart, the LP relaxation of the reformulation of (SLSP) does not yield integral solutions. This is because of the structural properties enjoyed by an optimal solution of (LSP) break down for the stochastic case. Table 3 displays the optimal solution for the numerical example. We can observe that although inventory is carried in to node 3, there is still production in this node, thus $I_{a(n)}X_n \neq 0$. Thus, the Wagner-Whitin conditions are *not* satisfied by an optimal solution to the stochastic problem.

n	X_n	Y_n	I_n
1	10	1	5
2	0	0	0
3	30	1	20
4	5	1	0
5	10	1	0
6	0	0	10
7	0	0	0

Table 3: The Optimal Solution

3.3 Reformulation of (SCAP)

Let us now apply the above reformulation scheme to the multi-facility capacity expansion problem (SCAP). To see the lot-sizing substructure in this case, we introduce non-negative variables $X_n = \sum_{i\in\mathcal{I}} x_{in}$ and binary variables z_n, to denote the total capacity addition in node n, and the decision to add capacity to any facility in node n, respectively. (SCAP) can then be written as:

$$\begin{aligned}
\min \quad & \sum_{n\in\mathcal{T}(0)} p_n \left\{ \sum_{i\in\mathcal{I}} (\alpha_{in}x_{in} + \beta_{in}y_{in}) \right\} \\
\text{s.t.} \quad & 0 \leq x_{in} \leq M_{in}y_{in} && n \in \mathcal{T}(0); i \in \mathcal{I} \\
& y_{in} \in \{0,1\} && n \in \mathcal{T}(0); i \in \mathcal{I} \\
& X_n = \sum_{i\in\mathcal{I}} x_{in} && n \in \mathcal{T}(0) \\
& 0 \leq X_n \leq (\sum_{i\in\mathcal{I}} M_{in})z_n && n \in \mathcal{T}(0)
\end{aligned}$$

$$\sum_{m\in\mathcal{P}(n)} X_m \geq d_n \qquad n\in\mathcal{T}(0)$$
$$z_n \in \{0,1\} \qquad n\in\mathcal{T}(0)$$

Note that the last three constraints of the above problem are identical to the constraints of a single facility instance of (SCAP). Based on the variable disaggregation scheme for stochastic lot-sizing problems, the above problem can then be reformulated as:

$$\begin{aligned}
\min \quad & \sum_{n\in\mathcal{T}(1)} p_n \left[\sum_{i\in\mathcal{I}}(\alpha_{in}x_{in}+\beta_{in}y_{in})\right] \\
\text{s.t.} \quad & 0\leq x_{in}\leq M_{in}y_{in} && n\in\mathcal{T}(0); i\in\mathcal{I} \\
& y_{in}\in\{0,1\} && n\in\mathcal{T}(0); i\in\mathcal{I} \\
& X_n=\sum_{i\in\mathcal{I}} x_{in} && n\in\mathcal{T}(0) \\
& X_n \geq \sum_{k\in\mathcal{T}(n)\cap\mathcal{P}(m)} Q_{nk} && m\in\mathcal{S}_T\cap\mathcal{T}(n); n\in\mathcal{T}(0) \\
& \sum_{k\in\mathcal{P}(n)} Q_{kn}=\overline{d}_n && n\in\mathcal{T}(0) \\
& 0\leq Q_{nk}\leq \overline{d}_k z_n && k\in\mathcal{T}(n); n\in\mathcal{T}(0) \\
& z_n\in\{0,1\} && n\in\mathcal{T}(0); i\in\mathcal{I}
\end{aligned}$$

where $\overline{d}_n = (d_n - \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\})^+$.

Owing to the binary restrictions on z_n and the constraints on Q_{kn}, it is easily verified that we can equivalently reformulate the above problem by substituting $\sum_{i\in\mathcal{I}} x_{in}$ for X_n, and $\sum_{i\in\mathcal{I}} y_{in}$ for z_n. Thus the reformulation of multi-facility (SCAP) is as follows:

$$\begin{aligned}
(\text{RSCAP}): \quad \min \quad & \sum_{n\in\mathcal{T}(1)} p_n \left[\sum_{i\in\mathcal{I}}(\alpha_{in}x_{in}+\beta_{in}y_{in})\right] \\
\text{s.t.} \quad & 0\leq x_{in}\leq M_{in}y_{in} && n\in\mathcal{T}(0); i\in\mathcal{I} \\
& y_{in}\in\{0,1\} && n\in\mathcal{T}(0); i\in\mathcal{I} \\
& \sum_{i\in\mathcal{I}} x_{in} \geq \sum_{k\in\mathcal{T}(n)\cap\mathcal{P}(m)} Q_{nk} && m\in\mathcal{S}_T\cap\mathcal{T}(n); n\in\mathcal{T}(0) \\
& \sum_{k\in\mathcal{P}(n)} Q_{kn}=\overline{d}_n && n\in\mathcal{T}(0) \\
& 0\leq Q_{nk}\leq \overline{d}_k \left(\sum_{i\in\mathcal{I}} y_{in}\right) && k\in\mathcal{T}(n); n\in\mathcal{T}(0)
\end{aligned}$$

The following example demonstrates the tightened LP relaxations obtained by the proposed reformulation. Further computational experiments are reported in Section 6.

Example 2

Consider an instance of (SCAP) with three facilities. The uncertain parameters evolve over the scenario tree depicted in Figure 2. The corresponding problem data are provided in Table 4. The optimal IP and LP objective values of formulations (SCAP) and (RSCAP) are compared in Table 5. We observe that the reformulation has significantly smaller LP relaxation gap (29.0%) in comparison to the original formulation (50.22%).

n	α_{1n}	α_{2n}	α_{3n}	β_{1n}	β_{2n}	β_{3n}	d_n	p_n
0	2	1	2	10	15	5	5	1
1	1	1	1	10	30	20	15	0.3
2	3	1	2	11	5	10	10	0.7
3	1	2	1	5	10	3	5	0.1
4	2	1	1	10	3	5	10	0.2
5	2	1	3	3	10	5	10	0.3
6	1	3	2	10	5	3	20	0.4

Table 4: Problem Parameters

Formulation	IP obj. val.	LP obj. val.	% Gap
(SCAP)	34.0	16.925	50.22
(RSCAP)	34.0	24.140	29.00

Table 5: Comparison of LP relaxation gaps

4 A Heuristic Strategy

In this section, we describe a heuristic strategy to construct feasible integer solutions to the stochastic capacity expansion problem (SCAP). Note that simply rounding up the fractional values of the boolean variables (y_{in}) in the LP relaxation solution provides a feasible integer solution. However, such a naive strategy might result in very poor solutions, possibly requiring capacity additions to be carried out in all periods. Recently, Ahmed and Sahinidis [1] a rounding heuristic for an alternative formulation of (SCAP). We briefly describe this heuristic strategy and adapt it to the formulation presented in this paper.

Let us consider an alternative formulation for (SCAP). Instead of defining the problem variables over the nodes of the scenario tree, we define these over each individual scenario path $s = 1, \ldots, S$. A joint realization of the problem parameters corresponding to scenario s will be denoted by $\omega^s := (\omega_1^s, \ldots, \omega_n^s)$ where $\omega_t^s := (\alpha_{it}^s, \beta_{it}^s, d_t^s)$, with corresponding probability p^s. The technological constraints (2)-(4) in the deterministic problem (CAP) with the parameters ω^s corresponding to scenario s will be concisely denoted by $\mathcal{X}(\omega^s)$. The decision variables corresponding to scenario s will be denoted by $X^s := (X_1^s, \ldots, X_n^s)$ with

$X_t^s := (x_{it}^s, y_{it}^s)$. The objective function (1) corresponding to scenario s for an $n-$period problem will be denoted by $f_n^s(\cdot)$. The decision maker cannot distinguish between the scenarios passing through the same node at any time stage. Consequently, the feasible solutions X_t^s must satisfy:

$$X_t^{s_1} = X_t^{s_2} \qquad \forall (s_1, s_2) \in n, \forall n \in \mathcal{S}_t, \forall t = 1, \ldots, T.$$

These conditions are known as the *non-anticipativity* constraints, and we shall collectively denote them by $\mathcal{N}$. Using this notation, we can formulate the stochastic capacity expansion problem as follows:

$$\begin{aligned} (\text{SCAP}'): \quad \min \quad & \sum_{s=1}^{S} p^s f_n^s(X^s) \\ \text{s.t.} \quad & X^s \in \mathcal{X}(\omega^s) \cap \mathcal{N} \quad \forall s = 1, \ldots, S \end{aligned}$$

Observe that, in the absence of the non-anticipativity constraint $\mathcal{N}$, the stochastic problem (SCAP′) decomposes into S instances of the deterministic problem (CAP). Ahmed and Sahinidis [1] used this observation to decompose the problem across scenarios, then to construct integer solutions for each scenario subproblem, and finally to re-enforce the non-anticipativity constraints to construct a feasible integer solution to (SCAP′). The key steps of this heuristic are as follows. The details of the method can be found in [1].

Heuristic A:

1. Relax the integrality requirements in $\mathcal{X}(\omega^s)$ and solve the multi-stage stochastic *linear* program using standard solvers. Let $\overline{X}^s$ be the LP relaxation solution. Note that $\overline{X}^s \in \mathcal{N}$. If $\overline{X}^s \in \mathcal{X}(\omega^s)$ stop, else go to Step 2.

2. For each scenario s, construct an integral solution from $\overline{X}^s$ by shifting capacity additions from latter to earlier periods (see [1] for details). Let $\underline{X}^s$ be this solution. Note that $\underline{X}^s \in \mathcal{X}(\omega^s)$. If $\underline{X}^s \in \mathcal{N}$ stop, else go to Step 3.

3. Construct a solution $\widehat{X}^s$ from $\underline{X}^s$, such that $\widehat{x}^s \in \mathcal{X}(\omega^s) \cap \mathcal{N}$. Note that the capacity shifting step might destroy the non-anticipativity structure of the capacity expansion variables (X_t^s). We recover this by *capacity bundling* where we set $\widehat{x}_{it}^s = \max_{s \in n} \{\underline{x}_{it}^s\}$ for all $s \in n$ for all $n \in \mathcal{S}_t$. This guarantees that the capacity acquired in any period is the same in all scenarios of a scenario bundle. Finally, the binary variables are rounded up accordingly.

Figure 3 illustrates the above heuristic strategy for a simple 3-period, 4-scenario example. The solutions obtained in each of the three phases of the heuristic are plotted. The height of the rectangular blocks represent the capacity expansion bounds, and the height of filling in the block represent the amount of capacity added in the corresponding solution. Note that the LP relaxation solution satisfies the non-anticipativity constraints. For example, the capacity additions in scenarios 2 and 3, in time period 2 are the same since these scenarios belong to the same bundle. However, after capacity shifting (step 2), the non-anticipativity

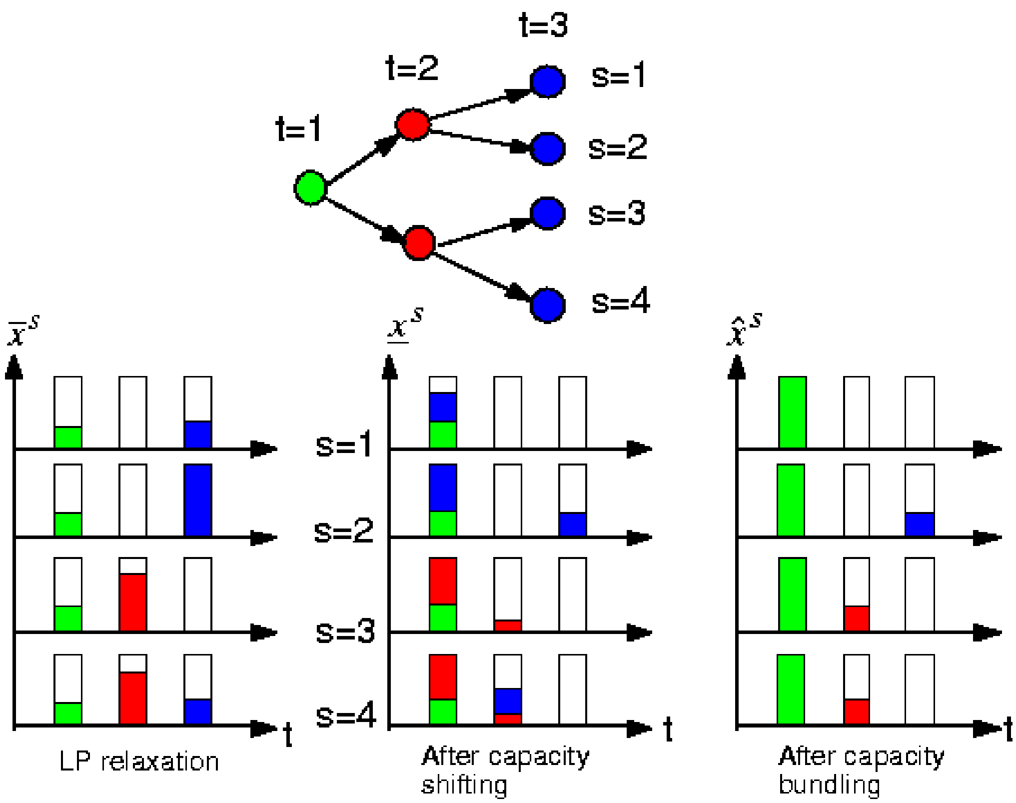

Figure 3: The Heuristic Strategy

structure is destroyed. The capacity bundling phase (step 3) restores the non-anticipativity structure.

It is easily verified that the *scenario* formulation (SCAP′) is entirely equivalent to the *tree* formulation (SCAP) presented in Section 2. Thus, given any solution to one of the formulations, we can convert it to the other. Since the reformulation (RSCAP) provides a tighter LP relaxation than (SCAP), we can apply the above heuristic to this solution to construct an integer feasible solution to (SCAP). The scheme is summarized as follows:

<u>Heuristic B:</u>

1. Solve the LP relaxation of (RSCAP). Let $(\overline{x}_{in}, \overline{y}_{in})$ denote this solution. Convert this solution to the scenario formulation as follows. Let n_s denote the leaf node in the scenario tree corresponding to scenario s. Then set $\overline{x}_{it}^s := \overline{x}_{in}$ and $\overline{y}_{it}^s := \overline{y}_{in}$ if $n \in \mathcal{P}(n_s)$ and $n \in \mathcal{S}_t$. The LP relaxation scenario solution is then $\overline{X}^s := (\overline{x}_{it}^s, \overline{y}_{it}^s)$.

2. Apply steps 2 and 3 in Heuristic A to construct a heuristic solution to the scenario formulation $\widehat{X}^s = (\widehat{x}_{it}^s, \widehat{y}_{it}^s)$ from the LP relaxation solution $\overline{X}^s$.

3. Construct a heuristic solution to the tree formulation (SCAP) as follows: set $\widehat{x}_{in} := \widehat{x}_{it}^s$ and $\widehat{y}_{in} := \overline{y}_{it}^s$ if $n \in \mathcal{P}(n_s)$ and $n \in \mathcal{S}_t$.

Ahmed and Sahinidis [1] proved that Heuristic A produces a feasible solution to (SCAP′). From the equivalence of the two formulations, the following result then follows immediately.

Proposition 4.1 *Heuristic B produces a feasible solution to* (SCAP).

The proposed heuristic can be easily improved by only shifting to periods that offer a cost benefit. Furthermore, the strategy can potentially be integrated with other heuristic methods such as those proposed by Fong and Srinivasan [16] and Li and Tirupati [23]. Such improvements will only produce better quality solutions. Furthermore, with some assumptions on the parameter distributions, Ahmed and Sahinidis [1] also showed that Heuristic A is *asymptotically optimal* in the number of time periods. This result implies that, as the problem size increases, the quality of the heuristic solution also increases and eventually, for sufficiently large problem sizes, the heuristic provides optimal solutions. Intuitively, if the demands are not expected to vary widely (for example, if they have bounded moments), the heuristic can be expected to have carried out enough capacity expansions in the early periods to satisfy the most of the demand. From the equivalence of the two formulations, Heuristic B also posseses this attractive property.

5 A Branch & Bound Algorithm

As mentioned earlier, currently there are no practical general purpose solution algorithms for multi-stage stochastic integer programs. A pioneering effort in this area is that of Caroe and Schultz [9] who proposed a branch and bound scheme coupled with Lagrangian relaxation for scenario formulations of multi-stage stochastic integer programs. In this scheme, the Lagrangian dual obtained by relaxing the non-anticipativity constraints in the scenario formulation is solved to obtain lower bounds. Relaxing the non-anticipativity constraints decomposes the problem and allows each scenario sub-problem to be solved independently. A subgradient approach is then used to improve the dual multipliers. To close the duality gap the authors propose branching on the integer variables. In [8] and [9], the authors present computational results using this algorithm for two-stage problems. Although the method is theoretically applicable to the multi-stage case, the authors acknowledge that owing to the increased dimensionality of the Lagrangian dual and the branching variables, several issues regarding a successful implementation of such an approach for the multi-stage case remain open.

In this paper, we propose to solve (SCAP) by enhancing the standard integer programming branch and bound algorithm with the problem specific reformulation scheme and heuristic strategy described in Sections 3 and 4. In this scheme, we solve the LP relaxation of the reformulated problem (RSCAP) to obtain tight lower bounds. Standard stochastic linear programming decomposition schemes, such as those implemented in solvers such as [19], can be used to solve these relaxations. As an upper bounding routine, Heuristic B (cf. Section 4) is used to obtain good quality feasible integer solutions. Conventional integer programming rules are used for branching.

A potential advantage of the proposed method over that of Caroe and Schultz [9] from an implementation point of view is that the lower bound is obtained by linear programming rather than computationally expensive subgradient methods to solve the Lagrangian dual. Furthermore, the number of branching variables is fewer – the scenario formulation of (SCAP$'$) that would be used in the Caroe and Schultz algorithm requires $|\mathcal{I}| \times T \times S$ binary variables, whereas the tree formulation (SCAP) requires $|\mathcal{I}| \times |\mathcal{T}(0)|$ binary variables, and $|\mathcal{T}(0)| < T \times S$, for example, a binary scenario tree with T periods has $S = 2^{T-1}$ and $n = (2^T - 1) < T \times 2^{T-1}$ for $T > 1$. On the other hand, the Lagrangian bounding scheme of Caroe and Schultz is independent of any specialized problem structure and provides very tight lower bounds for most problems.

6 Computational Results

In this section, we provide some computational experience using the solution strategy described in Section 5 to solve a set of 16 small instances of the multi-stage stochastic capacity expansion problem (SCAP). A ternary scenario tree was assumed with the uncertain parameters jointly realizing as one of three sets of values in each period. Problems were generated by varying the number of time periods from 2 to 5 and the number of facilities from 1 to 4. Data for the problem instances are available from the authors. Table 6 presents the time periods, tree size, number of scenarios, number of facilities, and the number of binary variables, continuous variables, and rows in the original formulation (SCAP) and its reformulation (RSCAP). Note that although the reformulation introduces a large number of additional continuous variables and rows, the number of binary variables is same in both models.

We first investigate the strength of the LP relaxation gap of (RSCAP), and then the performance of the proposed branch and bound algorithm. All computations were carried out on an IBM RS/6000 Model 590 Workstation with 512 Mb RAM and a 66MHz processor. CPLEX 6.6 was used to the solve the linear and integer programs.

					(SCAP)			(RSCAP)		
No.	T	$\|\mathcal{T}(0)\|$	S	$\|\mathcal{I}\|$	Bin.	Cont.	Rows	Bin.	Cont.	Rows
P_2_1	2	4	3	1	4	4	8	4	14	24
P_2_2	2	4	3	2	8	8	12	8	18	28
P_2_3	2	4	3	3	12	12	16	12	22	32
P_2_4	2	4	3	4	16	16	20	16	26	36
P_3_1	3	13	9	1	13	13	26	13	104	141
P_3_2	3	13	9	2	26	26	39	26	117	154
P_3_3	3	13	9	3	39	39	52	39	130	167
P_3_4	3	13	9	4	52	52	65	52	143	180
P_4_1	4	40	27	1	40	40	80	40	860	951
P_4_2	4	40	27	2	80	80	120	80	900	991
P_4_3	4	40	27	3	120	120	160	120	940	1031
P_4_4	4	40	27	4	160	160	200	160	980	1071
P_5_1	5	121	81	1	121	121	242	121	7502	7350
P_5_2	5	121	81	2	242	242	363	242	7623	7471
P_5_3	5	121	81	3	363	363	484	363	7744	7592
P_5_4	5	121	81	4	484	484	605	484	7865	7713

Table 6: Problem Dimensions

6.1 Comparison of LP Relaxation Gaps

Table 7 compares the gap (from the optimal integer solution) and the CPU seconds for the LP relaxation of the original formulation (SCAP) and the reformulation (RSCAP). As expected, the reformulation requires higher CPU time, but provides significantly better LP relaxation bounds.

6.2 Performance of the Proposed Branch and Bound Algorithm

The proposed branch and bound algorithm was implemented by integrating Heuristic B with the CPLEX 6.6 MIP solver, and applying the algorithm to the reformulation (RSCAP). Table 8 compares the performance of the proposed method ((RSCAP) + Heuristic) to a straightforward application of the CPLEX 6.6 MIP solver on the original formulation (SCAP). A node limit of $100,000$ was imposed and the CPLEX default relative tolerance of 0.0001 was used. As can be observed from Table 8, the proposed enhancements offer significant reductions in the number of nodes and CPU seconds. The three largest problems in the set could not be solved within the prescribed resource limits using the straightforward CPLEX implementation.

7 Conclusions and Future Research

The key contributions of this paper are the following:

- We have proposed a multi-stage stochastic integer programming formulation for a

No.	(SCAP) % Gap	(SCAP) CPUs	(RSCAP) % Gap	(RSCAP) CPUs
P_2_1	22.64	0.00	2.63	0.00
P_2_2	28.79	0.01	5.22	0.01
P_2_3	34.19	0.01	13.57	0.00
P_2_4	35.14	0.00	14.10	0.01
P_3_1	21.08	0.00	2.69	0.03
P_3_2	26.43	0.01	4.33	0.03
P_3_3	31.98	0.01	12.39	0.03
P_3_4	33.31	0.00	13.08	0.03
P_4_1	19.29	0.02	2.39	0.11
P_4_2	24.44	0.02	3.01	0.13
P_4_3	30.02	0.03	9.44	0.13
P_4_4	31.49	0.04	10.10	0.14
P_5_1	19.24	0.06	2.75	0.71
P_5_2	24.03	0.11	3.03	0.78
P_5_3	29.45	0.14	8.61	0.88
P_5_4	31.00	0.20	9.13	0.97
Average	27.66	0.04	7.28	0.25

Table 7: LP Relaxation Gaps

general multi-facility capacity expansion problem under uncertainty.

- A reformulation scheme has been developed by exploiting special lot-sizing sub-structure in the problem. The proposed reformulation offers significantly tighter LP relaxation gaps than the original formulation.

- We have modified a recently proposed heuristic strategy for scenario based formulations of capacity expansion problems to be applicable to the formulation presented in this paper.

- We have proposed enhancing standard integer programing branch and bound algorithms by integrating the reformulation scheme and the heuristic strategy to solve the problem to global optimality.

- We have presented computational results demonstrating the effectiveness of the reformulation and the proposed branch and bound algorithm.

The results in this paper pave the way for a number of future research avenues. We assumed that the capacity expansion bounds were large enough, making the problem "unrestricted" and allowing the exploitation of the *uncapacitated* lot-sizing substructure. For the restricted case, recent results on *capacitated* lot-sizing problems [28] can be investigated for possible extensions. The heuristic strategy also has considerable room for improvement. Note that by fixing the solution corresponding to a parent node of the scenario tree after a single pass of the heuristic, we can decouple the problems corresponding to the child

	(SCAP)		(RSCAP)+Heuristic	
No.	Nodes	CPUs	Nodes	CPUs
P_2_1	1	0.02	0	0.02
P_2_2	2	0.05	0	0.02
P_2_3	5	0.04	0	0.03
P_2_4	9	0.07	0	0.02
P_3_1	7	0.04	5	0.05
P_3_2	19	0.14	7	0.09
P_3_3	45	0.39	12	0.13
P_3_4	82	0.74	19	0.22
P_4_1	35	0.3	19	0.32
P_4_2	563	2.67	28	0.51
P_4_3	1822	11.57	41	0.88
P_4_4	4701	41.48	77	1.87
P_5_1	1536	20.43	193	5.9
P_5_2	100000[a]	784.17	492	16.94
P_5_3	100000[b]	1409.26	2142	90.15
P_5_4	100000[c]	2975.70	4728	271.91

[a] Gap = 0.58%, [b] Gap = 1.11%, [c] Gap = 1.74%.

Table 8: Performance of CPLEX 6.6

sub-trees. The heuristic can then be applied recursively to these sub-trees. This multi-pass version of the heuristic can offer significantly better solutions. Furthermore, in the capacity shifting phase of the heuristic, the non-anticipativity constraints are relaxed without any penalties. Incorporating appropriate Lagrange multipliers in the objective can help reduce the non-anticipativity violations and produce better solutions. The generic capacity expansion model addressed in this paper is applicable to a wide variety of industrial settings. We recommend future research efforts to be directed at solving large-scale industry relevant capacity expansion problems.

Appendix: Proof of Proposition 3.1

Given any feasible solution (x_n, y_n) for (SCAP), we can construct a feasible solution (X_n, I_n, Y_n) for (SLSP) by setting $X_n = x_n$, $Y_n = y_n$, and $I_n = \sum_{m\in\mathcal{P}(n)} x_m - \max_{m\in\mathcal{P}(n)}\{d_m\}$. To show that this solution is feasible to (SLSP), we just need to check for the inventory balance constraints. For a given n, the left-hand side (LHS) of this constraint is given by:

$$\begin{aligned} I_{a(n)} + X_n &= \sum_{m\in\mathcal{P}(n)\setminus n} x_m - \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\} + x_n \\ &= \sum_{m\in\mathcal{P}(n)} x_m - \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\}. \end{aligned}$$

The right-hand side of this constraint is:

$$\overline{d}_n + I_n \quad = \quad (d_n - \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\})^+ + \sum_{m\in\mathcal{P}(n)} x_m - \max_{m\in\mathcal{P}(n)}\{d_m\}$$

If $d_n > \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\}$, then $\max_{m\in\mathcal{P}(n)}\{d_m\} = d_n$, and both sides of the inventory balance constraint are equal. Otherwise, $(d_n - \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\})^+ = 0$ and $\max_{m\in\mathcal{P}(n)}\{d_m\} = \max_{m\in\mathcal{P}(n)\setminus n}\{d_m\}$, and once again both sides of the constraint are equal. Thus the constructed solution is feasible to (SLSP).

Given a feasible solution (X_n, I_n, Y_n) to (SLSP), we can construct a solution to (SCAP) by setting $x_n = X_n$, and $y_n = Y_n$. To show that the solution is feasible, observe that by summing up the inventory balance constraints in (SLSP) for all $m \in \mathcal{P}(n)$, we have $\sum_{m\in\mathcal{P}(n)} X_n \geq \sum_{m\in\mathcal{P}(n)} \overline{d}_m$. All we need now is to show that $\sum_{m\in\mathcal{P}(n)} \overline{d}_m \geq d_n$ or

$$\sum_{m\in P(n)} (d_m - \max_{k\in\mathcal{P}(m)\setminus m}\{d_k\})^+ \geq d_n.$$

Since $\max_{m\in\mathcal{P}(n)}\{d_m\} \geq d_n$, it is sufficient to show that

$$\sum_{m\in P(n)} (d_m - \max_{k\in\mathcal{P}(m)\setminus m}\{d_k\})^+ \geq \max_{m\in\mathcal{P}(n)}\{d_m\}.$$

Let us number the nodes on $\mathcal{P}(n)$ as $\{1, 2, \ldots, n\}$. We then prove the following inequality by induction:

$$\sum_{m=1}^{n} (d_m - \max_{k=1,\ldots,m-1}\{d_k\})^+ \geq \max_{m=1,\ldots,n}\{d_m\}. \tag{10}$$

Clearly (10) holds for $n = 1$. Suppose now that it holds for n, we then need to show that

$$\sum_{m=1}^{n+1} (d_m - \max_{k=1,\ldots,m-1}\{d_k\})^+ \geq \max_{m=1,\ldots,n+1}\{d_m\}. \tag{11}$$

Note that the left hand side of inequality (11) is:

$$\begin{aligned} := \quad & \sum_{m=1}^{n} (d_m - \max_{k=1,\ldots,m-1}\{d_k\})^+ + (d_{n+1} - \max_{k=1,\ldots,n}\{d_k\})^+, && (12) \\ \geq \quad & \max_{1,\ldots,n}\{d_m\} + (d_{n+1} - \max_{k=1,\ldots,n}\{d_k\})^+, && (13) \end{aligned}$$

where (13) follows from the induction hypothesis.

If $d_{n+1} < \max_{k=1,\ldots,n}\{d_k\}$, then the first part of expression (13) reduces to $\max_{m=1,\ldots,n}\{d_m\} = \max_{m=1,\ldots,n+1}\{d_m\}$, and the second part reduces to $(d_{n+1} - \max_{k=1,\ldots,n}\{d_k\})^+ = 0$. Thus inequality (11) holds.

Otherwise if $d_{n+1} \geq \max_{k=1,\ldots,n}\{d_k\}$, then (13) reduces to:

$$\begin{aligned} \geq \quad & \max_{1,\ldots,n}\{d_m\} + d_{n+1} - \max_{k=1,\ldots,n}\{d_k\} \\ = \quad & d_{n+1} \\ = \quad & \max_{1,\ldots,n+1}\{d_m\}, \end{aligned}$$

and inequality (11) holds. □

References

[1] S. Ahmed and N. V. Sahinidis. An asymptotically optimal heuristic for a stochastic capacity expansion problem. Working paper, Department of Chemical Engineering, University of Illinois, 2000.

[2] I. Barany, T. Van Roy, and L. A. Wolsey. Uncapacitated lot-sizing: The convex hull of solutions. *Mathematical Programming Study*, 22:32–43, 1984.

[3] J. C. Bean, J. L. Higle, and R. L. Smith. Capacity expansion under stochastic demands. *Operations Research*, 40:S210–S216, 1992.

[4] O. Berman and Z. Ganz. The capacity expansion problem in the service industry. *Computers & Operations Research*, 21:557–572, 1994.

[5] O. Berman, Z. Ganz, and J. M. Wagner. A stochastic optimization model for planning capacity expansion in a service industry under uncertian demand. *Naval Research Logistics*, 41:545–564, 1994.

[6] S. Bermon and S. Hood. Capacity optimization planning system (CAPS). *Interfaces*, 29:31–50, 1999.

[7] J. R. Birge and F. Louveaux. *Introduction to Stochastic Programming.* Springer, New York, NY, 1997.

[8] C. C. Caroe. *Decomposition in stochastic integer programming.* PhD thesis, University of Copenhagen, 1998.

[9] C. C. Caroe and R. Schultz. Dual decomposition in stochastic integer programming. *Operations Research Letters*, 24:37–45, 1999.

[10] S-G. Chang and B. Gavish. Telecommunications network topological design and capacity expansion: Formulations and algorithms. *Telecommunication Systems*, 1:99–131, 1993.

[11] Z-L. Chen, S. Li, and D. Tirupati. A scenario based stochastic programming approach for technology and capacity planning. Working paper, Department of Systems Engineering, University of Pennsylvania, 1999.

[12] M. H. A. David, M. A. H. Dempster, S. P. Sethi, and D. Vermes. Optimal capacity expansion under uncertainty. *Advances in Applied Probability*, 19:156–176, 1987.

[13] J. Dupacova, G. Consigli, and S. W. Wallace. Scenarios for multi-stage stochastic programs. Working paper, URL: `http://www.iot.ntnu.no/iok_html/users/sww/reports.htm`, 2000.

[14] G. D. Eppen, R. K. Martin, and L. Schrage. A scenario approach to capacity planning. *Operations Research*, 37:517–527, 1989.

[15] C. H. Fine and R. M. Freund. Optimal investment in product-flexible manufacturing capacity. *Management Science*, 36:449–466, 1990.

[16] C. O. Fong and V. Srinavasan. The multiregion dynamic capacity expansion problem: Part II. *Operations Research*, 29:800–816, 1981.

[17] J. Freidenfelds. Capacity expansion when demand is a birth-death random process. *Operations Research*, 28:712–721, 1980.

[18] K. Hoyland and S. W. Wallace. Generating scenario trees for multi-stage decision problems. To appear in *Management Science*, 2000.

[19] IBM. IBM Stochastic Solutions. URL: `http://www6.software.ibm.com/es/oslv2/features/stoch.htm`, 1998.

[20] P. Kall and S. W. Wallace. *Stochastic Programming*. John Wiley and Sons, Chichester, England, 1994.

[21] J. Krarup and O. Bilde. Plant location, set covering and economic lot size: An $o(mn)$ algorithm for structured problems. In *Optimierung bei Graphentheoretischen und ganzzahligen Problemen,* Collatz et al. (eds.), International Series of Numerical Mathematics, Birkhauser-Verlag, 36:155–180, 1977.

[22] M. Laguna. Applying robust optimization to capacity expansion of one location in telecommunications with demand uncertainty. *Management Science*, 44:S101–S110, 1998.

[23] S. Li and D. Tirupati. Dynamic capacity expansion problem with multiple products: technology selection and timing of capacity additions. *Operations Research*, 42:958–976, 1994.

[24] M. L. Liu and N. V. Sahinidis. Optimization in process planning under uncertainty. *Industrial & Engineering Chemistry Research*, 35:4154–4165, 1996.

[25] A. S. Manne. Capacity expansion and probabilistic growth. *Econometrica*, 29:632–649, 1961.

[26] A.S. Manne, editor. *Investments for Capacity Expansion*. The MIT press, Cambridge, MA, 1967.

[27] R. K. Martin. Generating alternative mixed-integer programming models using variable redefinition. *Operations Research*, 35:820–831, 1987.

[28] A. J. Miller. *Polyhedral Approaches to Capacitated Lot-Sizing Problems*. PhD thesis, Georgia Institute of Technology, 1999.

[29] J. M. Mulvey. Generating scenario for the towers perrin investment system. *Interfaces*, 26:1–13, 1996.

[30] F. H. Murphy, S. Sen, and A. L. Soyster. Electric utility expansion planning in the presence of existing capacity: A nondifferentiable, convex programming approach. *Computers & Operations Research*, 14:19–31, 1987.

[31] F. H. Murphy and H. J. Weiss. An approach to modeling electric utility capacity expansion planning. *Naval Research Logistics*, 37:827–845, 1990.

[32] G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*. John Wiley & Sons, 1988.

[33] Y. Pochet and L. A. Wolsey. Lot-size models with backlogging: Strong reformulations and cutting planes. *Mathematical Programming*, 40:317–335, 1988.

[34] S. Rajagopalan, M. R. Singh, and T. E. Morton. Capacity expansion and replacement in growing markets with uncertain technological breakthroughs. *Management Science*, 44:12–30, 1998.

[35] N. V. Sahinidis and I. E. Grossmann. Reformulation of the multiperiod milp model for capacity expansion of chemical processes. *Operations Research*, 40, Supp. No. 1:S127–S144, 1992.

[36] I. Saniee. An efficient algorithm for the multiperiod capacity expansion of one location in telecommunications. *Operations Research*, 43:187–190, 1995.

[37] J. M. Swaminathan. Tool capacity planning for semiconductor fabrication facilities under demand uncertainty. *European Journal of Operational Research*, 120:545–558, 2000.