STOCHASTIC PROGRAMMING DUALITY: $\mathcal{L}^\infty$ MULTIPLIERS WITH AN APPLICATION TO MATHEMATICAL FINANCE *

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Abstract. A new duality theory is developed for a class of stochastic programs in which the probability distribution is not necessarily discrete. This provides a new framework for problems which are not necessarily bounded, are not required to have relatively complete recourse, and do not satisfy the typical Slater condition of strict feasibility. These problems instead satisfy a different constraint qualification called 'direction-free feasibility' to deal with possibly unbounded constraint sets, and 'calmness' of a certain finite-dimensional value function to serve as a weaker condition than strict feasibility to obtain the existence of dual multipliers. In this way, strong duality results are established in which the dual variables are finite-dimensional, despite the possible infinite-dimensional character of the second-stage constraints. From this, infinite-dimensional dual problems are obtained in the space of essentially bounded functions. It is then shown how this framework could be used to obtain duality results in the setting of mathematical finance.

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1. Introduction

Stochastic programming addresses the problem of choosing an optimal decision in the midst of an uncertain environment. The approach begins with a mathematical programming model into which a probability distribution is incorporated to describe the uncertain parameters present in the problem. When the probability distribution involved is not concentrated at finitely many points, the problem is akin to a semi-infinite program in that there are infinitely many constraints. Otherwise put, there are cost functions and constraints involved which lie in infinite-dimensional spaces. As a result, these problems can be difficult to solve.

A rich convex duality theory in stochastic programming exists, cf. [3, 9, 11, 10], which can be used to analyze and characterize optimality for a broad class of problems, in which certain functions governing the constraints of the problem are assumed to lie in $\mathcal{L}^\infty$.

While it is often desirable to obtain strong duality results with Lagrange multipliers in infinite-dimensional spaces such as $\mathcal{L}^1$, to achieve this it must usually be assumed either that the constraint sets are bounded, cf. [9], or that the problem satisfies a strict feasibility condition along with a property known as relatively complete recourse, cf. [10]. Due to their restrictiveness, neither of these requirements is very satisfying.

In a development completely separate from stochastic programming, the mathematical finance literature exploits convex duality to obtain the equivalence of "no arbitrage" conditions on a market with the existence of an equivalent martingale measure for the market price process, cf. [14, 2], which can then be used to price financial derivatives. These problems generally possess unbounded constraints, and do not satisfy the property of relatively complete recourse, yet miraculously (to the stochastic programmer) do yield dual variables in $\mathcal{L}^1$ (these turn out to be the Radon-Nikodym derivatives of the aforementioned martingale measures).

It turns out that the assumptions at the heart of this mathematical finance theory, the no arbitrage-type conditions, impose a special property on the problem which allows strong duality results without the usual assumptions of bounded constraint sets or relatively complete recourse present in the stochastic programming literature. Up to now, the ability to frame these duality theorems from mathematical finance, and their variants, in a stochastic programming setting has been hampered due to the absence in the stochastic programming literature of a duality theory on infinite-dimensional spaces general enough to encompass these problems. This paper serves to identify the precise property that allows strong duality results in an infinite-dimensional setting, and to generalize the duality results accordingly for stochastic programs which do not include explicit recourse decisions, though this will be generalized in a subsequent paper that deals with multistage problems. The goal is to add to the existing stochastic programming theory so that it may now encompass these important problems arising in mathematical finance, as well as in other application areas.

The main problems we concern ourselves with have the form

$$\begin{array}{ll} \text{minimize} & f_{10}(x) + E\{f_{20}(\boldsymbol{\xi}, x)\} \\ \text{so that} & f_{1i}(x) \leq 0, \quad i = 1, \ldots, m_1, \quad x \in C \\ & f_{2i}(\xi, x) \leq 0 \quad P\text{-a.s.}, \quad i = 1, \ldots, m_2, \end{array} \tag{$\mathcal{P}$}$$

where $C \subset I\!R^n$ is a closed, convex set, $f_{1i} : I\!R^n \to I\!R$ are convex functions for $i = 0, \ldots, m_1$, $f_{2i} : \Xi \times I\!R^n \to I\!R$ are convex in $x \in I\!R^n$ for fixed $\xi \in \Xi$ for $i = 0, \ldots, m_2$, and $(\Xi, \mathcal{F}, P)$ is a probability space with $\mathcal{F}$ P-complete. For each $x \in I\!R^n$, it's assumed that $f_{2i}(\cdot, x) \in \mathcal{L}^1(\Xi, \mathcal{F}, P; I\!R)$, i.e. $f_{2i}(\cdot, x)$ is P-summable, $i = 1, \ldots, m_2$. The associated dual problem is derived from the Lagrangian $L : I\!R^n \times [I\!R^{m_1} \times \mathcal{L}^\infty(\Xi, \mathcal{F}, P; I\!R^{m_2})] \to \overline{I\!R}$, given by

$$L(x, y, z) := \begin{cases} f_{10}(x) + E\{f_{20}(\boldsymbol{\xi}, x)\} + \sum_{i=1}^{m_1} f_{1i}(x) y_i & \\ + \sum_{i=1}^{m_2} E\{f_{2i}(\boldsymbol{\xi}, x) z_i(\boldsymbol{\xi})\} & \text{if } x \in C,\ y \geq 0,\ z \geq 0\ P\text{-a.s.}, \\ +\infty & \text{if } x \notin C, \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem is then

$$\text{maximize } g(y, z) \text{ so that } y \geq 0,\ z \in \mathcal{L}_+^\infty(\Xi, \mathcal{F}, P; I\!R^{m_2}), \tag{$\mathcal{D}$}$$

where

$$g(y, z) := \inf \{L(x, y, z) \mid x \in C\}.$$

It is always true that for this pair of primal and dual problems, one has the inequality,

$$\inf \mathcal{P} \geq \sup \mathcal{D}.$$

Our goal is to obtain duality results of the form $\inf \mathcal{P} = \sup \mathcal{D}$, and in particular the existence of dual multipliers, i.e. solutions to $\mathcal{D}$; this type of strong duality with existence of dual multipliers we express for convenience as $\inf \mathcal{P} = \max \mathcal{D}$. It is useful to also define the feasibility sets for $\mathcal{P}$. The set of first stage feasible points is given by

$$K_1 := \{x \in C \mid f_{1i}(x) \leq 0,\ i = 1, \ldots, m_1\},$$

and the set of recourse feasible constraints, or *induced constraints*, is given by

$$K_2 := \{x \in I\!R^n \mid f_{2i}(\xi, x) \leq 0\, P\text{-a.s.},\ i = 1, \ldots, m_2\}. \tag{1}$$

The problem $\mathcal{P}$ is said to be *strictly feasible* if there exists $\varepsilon > 0$, $x \in C$ with $f_{1i}(x) \le -\varepsilon$, $i = 1, \dots, m_1$, and $f_{2i}(\xi, x) \le -\varepsilon$, $i = 1, \dots, m_2$. It is said to have *relatively complete recourse* if $K_1 \subset K_2$. In other words, the recourse constraints do not impose any further constraints on the decision than were already present in the first stage. Relatively complete recourse is a type of *constraint qualification* that was used along with the strict feasibility of $\mathcal{P}$ to invoke the duality theorems which currently comprise the theoretical stochastic programming literature, brought about in the latter half of the 1970's by Eisner and Olsen [3] and Rockafellar and Wets [10, 8, 12]. The approach here forgoes both strict feasibility and relatively complete recourse for a constraint qualification of a different nature. The problem $\mathcal{P}$ is said to have *direction-free feasibility* if it is feasible and the recession cone of $K_1 \cap K_2$,

$$[K_1 \cap K_2]^\infty := \{v \in \mathbb{R}^n \mid x + \lambda v \in K_1 \cap K_2 \text{ for all } \lambda \ge 0,\, x \in K_1 \cap K_2\},$$

is a subspace of $\mathbb{R}^n$. Direction-free feasibility is the key property which yields the duality results of the form $\inf \mathcal{P} = \sup \mathcal{D}$ in the sections to follow, and in particular it is the property paralleling no arbitrage type conditions that drives the mathematical finance results in the last section.

A function $f : \mathbb{R}^m \to \overline{\mathbb{R}}$ is said to be *calm from below at* $\bar{u}$ if $f(\bar{u})$ is finite and there is a $\kappa \ge 0$ and a neighborhood V of $\bar{u}$ such that for all $u \in V$,

$$f(u) \ge f(\bar{u}) - \kappa |u - \bar{u}|. \tag{2}$$

Calmness of a certain finite-dimensional optimal value function will yield the existence of solutions to $\mathcal{D}$ as well as the existence of solutions to certain finite-dimensional dual problems. A simplified version of the main theorem may now be stated. Here, $[\cdot]^+ = \max[\cdot, 0]$.

Theorem 1.1. *Suppose the problem $\mathcal{P}$ has direction-free feasibility. Then $\inf \mathcal{P} = \sup \mathcal{D}$. If the finite-dimensional optimal value function $\varphi : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \overline{\mathbb{R}}$ given by*

$$\varphi(u, v) := \inf_x \begin{cases} f_{10}(x) + E\{f_{20}(\boldsymbol{\xi}, x)\} & \text{if } x \in C,\ f_{1i}(x) \le u_{1i},\ i = 1, \dots, m_1, \\ & \int_\Xi [f_{2i}(\xi, x)]^+ P(d\xi) \le u_{2i},\ i = 1, \dots, m_2, \\ +\infty & \text{otherwise,} \end{cases}$$

is calm from below at 0, then $\inf \mathcal{P} = \max \mathcal{D}$.

Compare this with the two duality theorems given below, from Rockafellar and Wets [9, 10]. In each of these theorems, it is assumed that the constraint functions lie in $\mathcal{L}^\infty$ instead of $\mathcal{L}^1$, and the dual variable z is taken in $\mathcal{L}^1$ instead of $\mathcal{L}^\infty$.

Theorem 1.2. *Suppose C is bounded. Then $\min \mathcal{P} = \sup \mathcal{D}$.*

Theorem 1.3. *Suppose the problem $\mathcal{P}$ is strictly feasible and satisfies relatively complete recourse. Then* $\inf \mathcal{P} = \max \mathcal{D}$.

The motivation for moving away from bounded constraint sets and relatively complete recourse towards other forms of constraint qualifications is that many problems have emerged, especially in recent years related to finance applications, that do not possess the property of relatively complete recourse, or even strict feasibility. Sometimes, actually in rare instances, it is possible to determine the induced constraint set K_2 explicitly in the form of deterministic constraints which can then be added to the original problem. But this is not generally the case for applications in finance as will be seen in §7, as well as for many other emergent problems. Also, allowing the constraint functions in $\mathcal{P}$ to lie in $\mathcal{L}^1$ instead of $\mathcal{L}^\infty$ opens up many heretofore unexplored possibilities in theoretical and applied stochastic programming.

Prior to now, in the setting of possibly unbounded constraint sets, the best one was able to achieve without the assumption of relatively complete recourse is a duality theorem in an extended sense, in which the infinite-dimensional dual multipliers lie in the dual of $\mathcal{L}^\infty(\Xi, \mathcal{F}, P; I\!R^{m_2})$ and thus have undesirable esoteric characteristics associated with the singular components of elements of that space, cf. [11].

The theorems presented here are the culmination of an analysis of the theory of mathematical finance, in particular the state-of-the-art papers by Delbaen and Schachermayer [2] and Schachermayer [14]. In those papers, variants of a classical theorem referred to as "the fundamental theorem of asset pricing" equate no arbitrage-type conditions with the existence of what is known in the literature as an "equivalent martingale measure." The no arbitrage-type conditions amount to special types of constraint qualifications which may be generalized to the stochastic programming setting in the form of direction-free feasibility. The equivalence of no arbitrage and the existence of an equivalent martingale measure then translates in the stochastic programming setting to a duality theorem which implies the existence of an $\mathcal{L}^\infty_+(\Xi, \mathcal{F}, P; I\!R^{m_2})$ solution to $\mathcal{D}$. In a precursor to this work, King [5] considers the fundamental theorem of asset pricing in an optimization setting on a finite probability space.

Section 2 presents the equivalent finite-dimensional problems which come into play as an intermediate step. Section 3 contains a review of basic duality theory and optimization, and other necessary foundations for the theory. A new result which exploits direction-free feasibility for deterministic problems is presented in §4. A duality theory in which the dual problem is finite-dimensional is established in §5. The main result 1.1 is restated and proved in Section 6. In Section 7, the results of the preceding sections are applied to derive a variant of the fundamental theorem of asset pricing in a stochastic programming setting. Only a simple one-stage problem is considered. For the full multi-stage arbitrage pricing theory in an $\mathcal{L}^{\infty^*}$ stochastic programming duality setting, consult King and Korf [6]. As already noted, this paper is a precursor to one which will address multistage stochastic programming problems.

2. The deterministic equivalent problems

In this section, we derive a class of finite-dimensional problems equivalent to $\mathcal{P}$. Let $\mathbf{G}$ be the set of all finite partitions of Ξ. A finite partition $\mathcal{S}$ of Ξ is a collection of $\mathcal{F}$-measurable sets, $\mathcal{S} \subset \mathcal{F}$, of positive measure satisfying

$$\bigcup_{A\in\mathcal{S}} A = \Xi, \quad A_1 \cap A_2 = \emptyset \text{ for all } A_1, A_2 \in \mathcal{S}, \quad |\mathcal{S}| \text{ finite.}$$

For an integrable random variable, $X : \Xi \to I\!R$, let $E\{X|A\}$ denote the conditional expectation of X with respect to the σ-field generated by $\mathcal{S}$ evaluated at $A \in \mathcal{S}$, i.e.

$$E\{X|A\} := \frac{1}{P(A)} \int_A X(\xi) P(d\xi).$$

For $\mathcal{S} \in \mathbf{G}$, define the set

$$K_2^{\mathcal{S}} := \{x \in I\!R^n \mid E\{f_{2i}^+(\boldsymbol{\xi}, x)|A\} \leq 0,\ A \in \mathcal{S},\ i = 1, \ldots, m_2\}.$$

We have the following theorem equating K_2 in (1) and $K_2^{\mathcal{S}}$.

Theorem 2.1. *For all $\mathcal{S} \in \mathbf{G}$, $K_2^{\mathcal{S}} = K_2$.*

Proof. Fix an $\mathcal{S} \in \mathbf{G}$. Let $x \in K_2$. Then for all $x \in I\!R^n$, $f_{2i}(\xi, x) \leq 0$ almost surely. Thus $f_{2i}^+(\xi, x) = 0$ almost surely. This implies that for any $A \in \mathcal{S}$, $\int_A f_{2i}^+(\xi, x)P(d\xi) \leq 0$, which yields $K_2 \subset K_2^{\mathcal{S}}$. For the other direction, let $x \in K_2^{\mathcal{S}}$. For all $A \in \mathcal{S}$, $\int_A f_{2i}^+(\xi, x)P(d\xi) \leq 0$, $i = 1, \ldots, m_2$. Thus

$$\int_\Xi f_{2i}^+(\xi, x)P(d\xi) = \sum_{A\in\mathcal{S}} \int_A f_{2i}^+(\xi, x)P(d\xi) \leq 0.$$

Equivalently, $P(f_{2i}(\xi, x) > 0) = 0$, i.e. $f_{2i}(\xi, x) \leq 0$ almost surely. This shows that $K_2^{\mathcal{S}} \subset K_2$, whereby $K_2^{\mathcal{S}} = K_2$. ◻

This leads to the ability to express $\mathcal{P}$ equivalently as a finite-dimensional optimization problem for each choice of $\mathcal{S} \in \mathbf{G}$.

$$\begin{array}{ll} \text{minimize} & f_{10}(x) + F_{20}(x) \\ \text{subject to} & f_{1i}(x) \leq 0, \quad i = 1, \ldots, m_1, \quad x \in C, \\ & F_{2i}^A(x) \leq 0, \quad i = 1, \ldots, m_2,\ A \in \mathcal{S}, \end{array} \qquad (\mathcal{P}_{\mathcal{S}})$$

where

$$F_{20}(x) := E\{f_{20}(\boldsymbol{\xi}, x)\} = \int_{\Xi} f_{20}(\xi, x) P(d\xi) \text{ and } F_{2i}^{A}(x) := E\{f_{2i}^{+}(\boldsymbol{\xi}, x)|A\}.$$

We can analyze these constraints in terms of what it would mean to perturb them slightly. A perturbation of the constraint

$$E\{f_{2i}^{+}(\boldsymbol{\xi}, x)|A\} \leq 0 \longrightarrow E\{f_{2i}^{+}(\boldsymbol{\xi}, x)|A\} \leq \varepsilon \tag{2}$$

means that the *average* value of $f_{2i}(\cdot, x)$ on A is less than or equal to ε, i.e. $f_{2i}(\cdot, x)$ could take on values larger than ε on A but still satisfy this constraint in the average. Thus, this is a more relaxed perturbation than a direct perturbation on the original constraints ($f_{2i}(\xi, x) \leq \varepsilon$ P-a.s.). Note also that if $\varepsilon < 0$ in (2), the problem becomes infeasible. Further remarks about the interpretation of the (second-stage) dual variables as a price system associated with perturbations of the form (2) can be found at the end of §5.

3. Finite-dimensional duality and optimization

This section reviews some of the requisite concepts from finite-dimensional convex duality and optimization. Rather than give all of the proofs of these basic results, the necessary references are provided so that the reader may piece together the proofs if wished. We begin with the finite-dimensional (deterministic) problem

$$\text{minimize } f_0(x) \text{ so that } x \in C,\ f_i(x) \leq 0,\ i = 1, \ldots, m \tag{$\mathcal{P}_d$}$$

for a convex, lsc function $f_0 : I\!R^n \to \overline{I\!R}$, a closed convex set $C \subset I\!R^n$, and finite convex functions $f_i : I\!R^n \to I\!R$, $i = 1, \ldots, m$. Define the feasible set by

$$K_d := \{x \in C \mid f_i(x) \leq 0,\ i = 1, \ldots, m\}$$

The *optimal value function* $\varphi_d : I\!R^m \to \overline{I\!R}$ is given by

$$\varphi_d(u) := \inf\{f_0(x) \mid x \in C,\ f_i(x) \leq u_i,\ i = 1, \ldots, m\}.$$

Theorem 3.1. *The optimal value function φ_d is convex. If K_d is bounded and $\mathcal{P}_d$ is feasible, then $\varphi_d(0)$ is finite and φ_d is lsc.*

Proof. The convexity of φ_d follows from [13, Proposition 2.22(a)]. For each $\bar{u}$, $\varepsilon > 0$, $\alpha \in I\!R$, the set of pairs (x, u) satisfying $|u - \bar{u}| \leq \varepsilon$, $f_0(x) \leq \alpha$, $x \in C$, and $f_i(x) \leq u_i$, $i = 1, \ldots, m$ is bounded whenever K_d is, cf. [13, Theorem 3.5, Exercise 3.24]. Thus applying Theorem 1.19 in [13] yields the lower semi-continuity of φ_d, and since $0 \in \operatorname{dom} \varphi_d$ by the feasibility of $\mathcal{P}_d$, the solution set is compact by [13, Theorem 1.19]. Hence $\varphi_d(0)$ must be finite. □

The Lagrangian $L_d : I\!R^n \times I\!R^m \to \overline{I\!R}$ for $\mathcal{P}_d$ is given by

$$L_d(x,y) := \begin{cases} f_0(x) + \sum_{i=1}^m y_i f_i(x) & \text{if } x \in C,\ y \geq 0, \\ +\infty & \text{if } x \notin C, \\ -\infty & \text{otherwise,} \end{cases}$$

and the dual problem is given by

$$\text{maximize } g(y) \text{ subject to } y \geq 0, \qquad (\mathcal{D}_d)$$

where

$$g(y) = \inf_{x \in I\!R^n} L_d(x,y).$$

Theorem 3.2. *If $\mathcal{P}_d$ is feasible and K_d is bounded then* $\min \mathcal{P}_d = \sup \mathcal{D}_d$.

Proof. This follows from the convexity and lower semicontinuity of φ_d. Observe that

$$\sup_{y \in I\!R^m} g(y) = \liminf_{u \to 0} \varphi_d(u) = \varphi_d(0),$$

where the first equality can be taken from [7, Theorem 7], for example, utilizing the convexity of φ_d. Since $\varphi_d(0) = \inf \mathcal{P}_d$, the result follows. □

In order to obtain the existence of dual multipliers, one can appeal to the constraint qualification of 'calmness' of the optimal value function. We state the definition below.

Definition 3.3 [13]. *A function $f : I\!R^m \to \overline{I\!R}$ is calm at $\bar{u}$ from below with modulus $\kappa \in I\!R_+ = [0, \infty)$ if $f(\bar{u})$ is finite and on some neighborhood V of $\bar{u}$ one has*

$$f(u) \geq f(\bar{u}) - \kappa|u - \bar{u}| \text{ when } u \in V.$$

Theorem 3.4. *Suppose that $\mathcal{P}_d$ is feasible, K_d is bounded, and the value function φ_d for $\mathcal{P}_d$ is calm from below at* 0. *Then* $\min \mathcal{P}_d = \max \mathcal{D}_d$.

Proof. This straightforwardly combines Proposition 10.47 of [13] (calmness as a constraint qualification) with Theorem 11.39(d) of [13], along with Theorem 3.1 above. □

For the convex function, φ_d we define the *subgradient* $\partial\varphi_d : I\!R^m \to I\!R^m$ at $\bar{u}$ by

$$\partial\varphi_d(\bar{u}) := \{v \mid \varphi_d(u) \geq \varphi_d(\bar{u}) + \langle v, u - \bar{u}\rangle\}.$$

The *subderivative* of φ_d at $\bar{u}$ in the direction $u \in I\!R^m$ (equivalent to the directional derivative in this convex case) is given by

$$d\varphi_d(\bar{u})(u) := \lim_{\tau \searrow 0} \frac{\varphi_d(\bar{u} + \tau u) - \varphi_d(\bar{u})}{\tau}.$$

We next give some pertinent properties to be used in interpreting these duality results.

Theorem 3.5. *Suppose that $\mathcal{P}_d$ is feasible and K_d is bounded. Then*

(a) $\operatorname{argmax} \mathcal{D}_d = -\partial\varphi_d(0)$,

(b) $d\varphi_d(0)(u) = \sup\{\langle v, u\rangle \mid v \in \partial\varphi_d(0)\}$.

Proof. The feasibility of $\mathcal{P}_d$ and boundedness of K imply that φ_d is a proper, convex, lsc function. (a) follows from [7, Theorem 16], employing the fact that $\min \mathcal{P}_d = \sup \mathcal{D}_d$ from Theorem 3.2. (b) follows from Theorem 8.30 in [13]. □

These two results indicate in particular that dual multipliers serve as a 'price system' that can be interpreted as measuring the cost of perturbations in the constraints in a given direction u, through 3.5 (b). We conclude this section by interpreting calmness from below in terms of subgradients and subderivatives.

Theorem 3.6. *If $\mathcal{P}_d$ is feasible and K_d is bounded, then the following are equivalent.*

(a) *φ_d is calm from below at 0,*

(b) *$\partial\varphi_d(0) \neq \emptyset$,*

(c) *$d\varphi_d(0)(u) > -\infty$ for any direction u.*

Proof. This follows from 3.1 above and Proposition 8.32 of [13]. The equivalence between (b) and (c) is also clear from Theorem 3.5 (b). □

From this theorem, the full meaning of calmness comes to light. A problem for which φ_d is calm from below at 0 is one in which a small shift in the constraints can produce a proportionally unbounded downward shift in the infimum, cf. [1]. We see that calmness from below at 0 is precisely the property that is necessary and sufficient to obtain the existence of subgradients to φ_d at 0, or equivalently the existence of optimal multipliers.

4. Direction-free feasibility

In this section, we consider finite-dimensional (deterministic) convex optimization problems of the form

$$\text{minimize } f_0(x) \text{ so that } x \in C, \;\; f_i(x) \leq 0, \; i = 1, \ldots, m \qquad (\mathcal{P}_d)$$

for a closed convex set $C \subset I\!R^n$, and finite convex functions $f_i : I\!R^n \to I\!R$, $i = 0, 1, \ldots, m$. Let's denote the feasible set of $\mathcal{P}_d$ by

$$K_d := \{x \in C \mid f_i(x) \leq 0, \; i = 1, \ldots, m\}.$$

The *recession cone* of a set $D \subset I\!R^n$ is the set of 'directions' contained in D, given by

$$D^\infty := \{v \in I\!R^n \mid x + \lambda v \in D \text{ for all } \lambda \geq 0, \; x \in D\}.$$

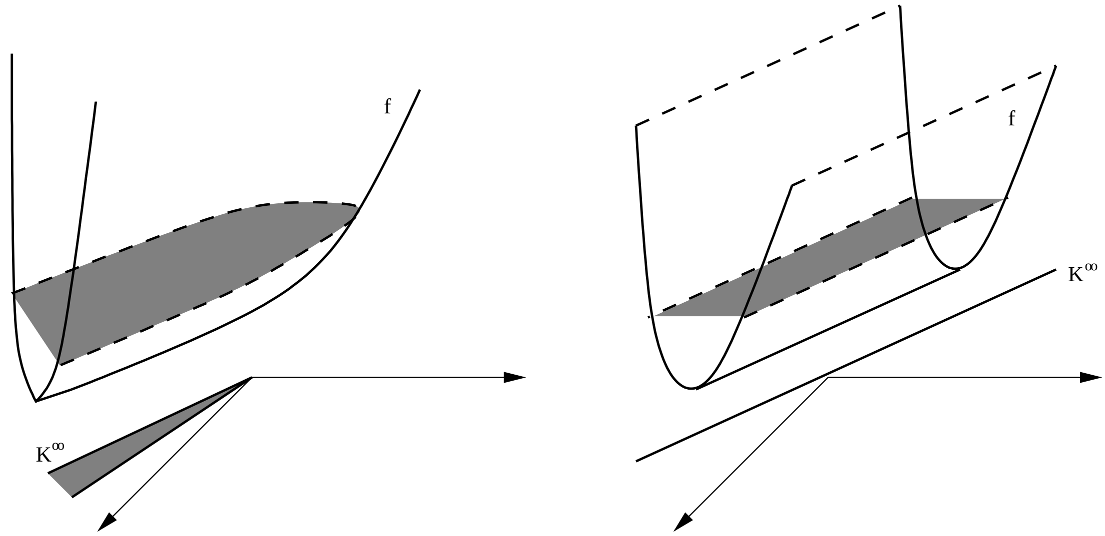

Figure 1: Recession cones of constraint sets. Right: direction-free feasibility.

In the proofs to follow we will use the easily verified fact that for a nonempty feasible set K_d, and any nonempty 'perturbed' set

$$K_d^{\alpha} := \{x \in C \mid f_i(x) \le \alpha_i,\ i = 1, \dots, m\},$$

$\alpha_i \in I\!R$, $i = 1, \dots, m$, it is always true that $K_d^{\infty} = K_d^{\alpha\infty}$, cf. [13, Chapter 3].

Definition 4.1. *A convex optimization problem $\mathcal{P}_d$ has direction-free feasibility if it is feasible, and K_d^{∞} is a subspace S of $I\!R^n$.*

Note that in particular, a problem $\mathcal{P}_d$ in which the feasible set K_d is nonempty and bounded (equivalently $K_d^{\infty} = \{0\}$) as in Theorem 3.2 satisfies direction-free feasibility. The property of direction-free feasibility is a generalization of feasibility with level-boundedness on C.

Theorem 4.2. *A convex optimization problem $\mathcal{P}_d$ has direction-free feasibility if and only if it is feasible and there exists a subspace S of $I\!R^n$ and a linear projection operator onto $S^{\perp}$, $\mathrm{prj}_{S^{\perp}} : I\!R^n \to I\!R^n$ such that $S \subset C^{\infty}$, $\max_i f_i|_{\mathrm{prj}_{S^{\perp}} C}$ is level-bounded on $\mathrm{prj}_{S^{\perp}} C$, and for $x \in C$, $f_i(x) = f_i(\mathrm{prj}_{S^{\perp}} x)$, $i = 1, \dots, m$. In that case, $K_d^{\infty} = S$.*

Proof. Let's first suppose $\mathcal{P}_d$ is feasible and that we have a subspace S of $I\!R^n$ with $S \subset C^{\infty}$, a linear projection operator $\mathrm{prj}_{S^{\perp}}$ onto $S^{\perp}$ with $\max_i f_i|_{\mathrm{prj}_{S^{\perp}} C}$ level-bounded on $\mathrm{prj}_{S^{\perp}} C$ and $f_i(x) = f_i(\mathrm{prj}_{S^{\perp}} x)$ on C. We show that for $K_d := \{x \in C \mid f_i(x) \le 0,\ i = 1, \dots, m\}$, $K_d^{\infty} = S$, which will imply that K_d^{∞} is a subspace. Let $v \in K_d^{\infty}$, which means that for any $x \in C$ satisfying $f_i(x) \le 0$ for $i = 1, \dots, m$, it follows that $x + \lambda v \in C$ and

$f_i(x+\lambda v) \leq 0$ for all $\lambda \geq 0$. Now, for fixed $\lambda > 0$, $f_i(x+\lambda v) = f_i(\operatorname{prj}_{S^\perp} x + \lambda \operatorname{proj}_{S^\perp} v)$. Since $\max_i f_i|_{\operatorname{prj}_{S^\perp} C}$ is level-bounded on $\operatorname{prj}_{S^\perp} C$, it follows that $\operatorname{prj}_{S^\perp} v = 0$, i.e., $v \in \ker(\operatorname{prj}_{S^\perp}) = S$. On the other hand for $v \in S$, $v \in C^\infty$ since $S \subset C^\infty$, so that $x + \lambda v \in C$. Also, $f_i(x+\lambda v) = f_i(\operatorname{prj}_{S^\perp}(x+\lambda v)) = f_i(\operatorname{prj}_{S^\perp} x) = f_i(x) \leq 0$.

For the opposite direction, suppose that $\mathcal{P}_d$ has direction-free feasibility. Direction-free feasibility implies that K_d^∞ is a subspace, let's call it S. Let $\operatorname{prj}_{S^\perp} : I\!R^n \to I\!R^n$ be the projection mapping onto $S^\perp$. Since S is the space of directions of unboundedness of K_d, it follows that $S^\perp$ has no directions of unboundedness of K_d, in other words $\max_i f_i|_{\operatorname{prj}_{S^\perp} C}$ is level-bounded. What remains is to show that for $i = 1, \ldots, m$, $f_i(x) = f_i(\operatorname{prj}_{S^\perp} x)$ on C. Fix an $x \in C$, and let $\operatorname{prj}_{S^\perp} x \in S^\perp$ and $x^0 \in S$ be the unique decomposition of x, i.e.

$$x = \operatorname{proj}_{S^\perp} x + x^0, \text{ where } x^0 = x - \operatorname{prj}_{S^\perp} x.$$

Then $f_i(x) = f_i(\operatorname{prj}_{S^\perp} x + x^0)$. It thus suffices to show that f_i is constant along directions in S. Fix $x \in C$ and let $x_0 \in S$. Recall that $S = K_d^\infty$, i.e. the directions of unboundedness in K_d. Suppose that $f_i(x) =: \alpha_i$, $i = 1, \ldots, m$. Then,

$$x \in \{x \in C \mid f_i(x) \leq \alpha_i,\ i = 1, \ldots, m\}.$$

On the other hand, $f_i(x+x_0) \leq \alpha_i$ since $x_0 \in S = K_d^\infty$, which implies $f_i(x+x_0) \leq f_i(x)$. A similar argument works for the reverse inequality: let $f_i(x+x_0) =: \beta_i$ and note that $-x_0 \in S = K_d^\infty$, so that $f_i(x) = f_i(x+x_0-x_0) \leq \beta_i$, whereby $f_i(x) \leq f_i(x+x_0)$. Thus we have established that $f_i(x) = f_i(x+x_0)$, $i = 1, \ldots, m$, i.e. f_i is constant along directions in S as claimed. Therefore $f_i(x) = f_i(\operatorname{prj}_{S^\perp} x + x^0) = f_i(\operatorname{prj}_{S^\perp} x)$. ◻

The method to be employed to derive duality results for problems with direction-free feasibility relies on passing to a 'projected' problem whose feasible set is bounded. Let's consider the modified problem given by

$$\text{minimize } F_0(x) \text{ so that } x \in \operatorname{prj}_{S^\perp} C,\ f_i(x) \leq 0,\ i = 1, \ldots, m \qquad (\mathcal{P}_d')$$

where $F_0 : I\!R^n \to \overline{I\!R}$ is defined by

$$F_0(x) := \inf_{x'} \{f_0(x') \mid \operatorname{prj}_{S^\perp} x' = x\}.$$

Note that $F_0(x) = \infty$ if $x \notin S^\perp$. The feasible set for $\mathcal{P}_d'$ is given by

$$K_d' := \{x \in \operatorname{prj}_{S^\perp} C \mid f_i(x) \leq 0,\ i = 1, \ldots, m\}.$$

Lemma 4.3. *F_0 is lsc, convex and satisfies $F_0|_{S^\perp} < \infty$.*

Proof. The convexity of F_0 follows from the joint convexity of the function $\psi : I\!R^m \times I\!R^m \to \overline{I\!R}$ given by

$$\psi(x, x') := \begin{cases} f_0(x') & \text{if } \operatorname{prj}_{S^\perp} x' = x, \\ +\infty & \text{otherwise,} \end{cases}$$

and the fact that $F_0(x) = \inf_{x'} \psi(x, x')$, cf. [7, Theorem 1]. $F_0|_{S^\perp} < \infty$ by virtue of the fact that $f_0 < \infty$. The lower semicontinuity of F_0 then follows by observing that F_0, because it is convex, must be either identically $-\infty$ on $S^\perp$ or finite and continuous on $S^\perp$. This combined with the fact that $F_0(x) = \infty$ for $x \notin S^\perp$ implies that F_0 is lsc. □

Lemma 4.4. *Suppose $\mathcal{P}_d$ has direction-free feasibility. Then K'_d is nonempty and bounded, and $\inf \mathcal{P}_d = \inf \mathcal{P}'_d$.*

First, observe that $\mathrm{prj}_{S^\perp} C = C \cap S^\perp$: Recall that under direction-free feasibility, $S \subset C^\infty$ (Theorem 4.2). For $x \in C \cap S^\perp$, it follows trivially that $x \in \mathrm{prj}_{S^\perp} C$. For $x \in \mathrm{prj}_{S^\perp} C$, we have $x \in S^\perp$ and there exists a $y \in C$, with $y = x + x^0$, for an $x^0 \in S \subset C^\infty$. Since S is a subspace, $-x^0 \in C^\infty$ also. Thus $x = y - x^0 \in C$. We have shown $x \in C \cap S^\perp$, hence $\mathrm{prj}_{S^\perp} C = C \cap S^\perp$. From this it follows that $K'_d = K_d \cap S^\perp$. This implies that K'_d is bounded, because $S^\perp$ contains no directions of unboundedness of K_d (equivalently, $K_d^\infty = S$).

To show that K'_d is nonempty, since K_d is nonempty by direction-free feasibility, it suffices to establish the equation, $K'_d = \mathrm{prj}_{S^\perp} K_d$. Let $x \in K'_d$. Then $x \in K_d \cap S^\perp$, so that trivially $x \in \mathrm{prj}_{S^\perp} K_d$. On the other hand, for $x \in \mathrm{prj}_{S^\perp} K_d$, we have that $x \in S^\perp$ and there exists $y \in K_d$ with $y = x + x^0$, $x^0 \in S = K_d^\infty$. Since S is a subspace, $-x^0 \in K_d^\infty$, whereby $x = y - x^0 \in K_d$. We have shown $x \in K_d \cap S^\perp = K'_d$, thus establishing $K'_d = \mathrm{prj}_{S^\perp} K_d$, and subsequently that K'_d is nonempty.

Based on the above relationships, it is easy to show that $\inf \mathcal{P}_d = \inf \mathcal{P}'_d$:

$$\begin{aligned} \inf \mathcal{P}_d &= \inf \{f_0(x) \mid x \in K_d\} \\ &= \inf \{f_0(x+y) \mid x \in K'_d,\ y \in S\} \\ &= \inf \{F_0(x) \mid x \in K'_d\} \\ &= \inf \mathcal{P}'_d, \end{aligned}$$

which completes the proof. □

We can now state the duality theorem relying on direction-free feasiblity. The dual problem for $\mathcal{P}_d$ is given by

$$\text{maximize } g(y) \text{ so that } y \geq 0, \qquad (\mathcal{D}_d)$$

where

$$g(y) := \inf_{x \in \boldsymbol{R}^n} L(x, y),$$

for $L : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ defined as in §3 with the usual inner product on $\mathbb{R}^m$:

$$L(x,y) := \begin{cases} f_0(x) + \sum_i f_i(x) y_i & \text{if } x \in C,\ y \geq 0 \\ +\infty & \text{if } x \notin C, \\ -\infty & \text{if } x \in C,\ y \ngeq 0. \end{cases}$$

Also recall the value function for $\mathcal{P}_d$, $\varphi : I\!R^m \to \overline{I\!R}$ given by

$$\varphi(u) := \inf \{f_0(x) \mid x \in C,\ f_i(x) \leq u_i,\ i = 1, \ldots, m\}.$$

Theorem 4.5. *Suppose $\mathcal{P}_d$ has direction-free feasibility. Then* $\inf \mathcal{P}_d = \sup \mathcal{D}_d$*. If the value function φ is calm from below at 0, then* $\inf \mathcal{P}_d = \max \mathcal{D}_d$.

Proof. This applies Theorem 3.2 to the problem $\mathcal{P}'_d$ utilizing Lemma 4.3 and the fact that $\inf \mathcal{P}_d = \inf \mathcal{P}'_d$ from Lemma 4.4. The dual problem for $\mathcal{P}'_d$ is precisely $\mathcal{D}_d$, as shown through writing down the Lagrangian for $\mathcal{P}'_d$,

$$L'(x,y) = \begin{cases} F_0(x) + \sum_i f_i(x) y_i & \text{if } x \in \operatorname{prj}_{S^\perp} C,\ y \geq 0, \\ +\infty & \text{if } x \notin \operatorname{prj}_{S^\perp} C, \\ -\infty & \text{if } x \in \operatorname{prj}_{S^\perp} C,\ y \ngeq 0, \end{cases}$$

and noting that

$$\begin{aligned} \inf\nolimits_{x \in \boldsymbol{R}^n} L'(x,y) &= \inf\nolimits_{x \in S^\perp} L'(x,y) \\ &= \inf\nolimits_{x \in \boldsymbol{R}^n} L(x,y). \end{aligned}$$

Thus,

$$\inf \mathcal{P}_d = \inf \mathcal{P}'_d = \sup \mathcal{D}'_d = \sup \mathcal{D}_d,$$

as claimed.

Now suppose that φ is calm from below at 0. The value function for $\mathcal{P}'_d$, $\varphi' : I\!R^m \to \overline{I\!R}$ is given by

$$\varphi'(u) := \{F_0(x) \mid x \in \operatorname{prj}_{S^\perp} C,\ f_i(x) \leq u_i,\ i = 1, \ldots, m\},$$

which is, by the definition of F_0, easily seen to be equal to φ. Thus φ' is calm from below at 0, and because there also exists a solution to the problem $\mathcal{P}'_d$, we may apply Theorem 3.4 to obtain the existence of a solution to $\mathcal{D}'_d$. Hence, a solution exists for $\mathcal{D}_d$, since $\mathcal{D}_d$ and $\mathcal{D}'_d$ are identical. □

5. Finite-dimensional duality theorems in stochastic programming

From §2, for an $\mathcal{S} \in \mathbf{G}$, $\mathcal{P}$ may be expressed equivalently as a deterministic convex optimization problem given by

$$\begin{array}{rll} \text{minimize} & f_{10}(x) + F_{20}(x) & \\ \text{so that} & f_{1i}(x) \leq 0, \quad i = 1, \ldots, m_1, \quad x \in C, & (\mathcal{P}_{\mathcal{S}}) \\ & F^A_{2i}(x) \leq 0, \quad i = 1, \ldots, m_2,\ A \in \mathcal{S}, & \end{array}$$

where

$$F_{20}(x) := E\{f_{20}(\boldsymbol{\xi}, x)\} \text{ and } F^A_{2i}(x) := E\{f^+_{2i}(\boldsymbol{\xi}, x) | A\}.$$

This puts us back in the finite-dimensional deterministic framework of §3.

In keeping with §1 and §3, for an $\mathcal{S} \in \mathbf{G}$, the *finite-dimensional optimal value function* $\varphi_{\mathcal{S}} : I\!R^{m_1} \times I\!R^{m_2|\mathcal{S}|} \to \overline{I\!R}$ for $\mathcal{P}$ is given by

$$\begin{aligned}\varphi_{\mathcal{S}}(u,v) \;:=\; \inf\Big\{ & f_{10}(x) + F_{20}(x) \mid x \in C,\ f_{1i}(x) \leq 0,\ i = 1,\ldots,m_1, \\ & F_{2i}^{A}(x) \leq 0,\ i = 1,\ldots,m_2,\ A \in \mathcal{S}\Big\}.\end{aligned}$$

Now let's consider the finite dimensional dual problem for $\mathcal{P}$ given by

$$\text{maximize } g(y,z) \text{ subject to } y \geq 0,\ z \geq 0, \qquad (\mathcal{D}_{\mathcal{S}})$$

where

$$g(y,z) := \inf_{x \in \boldsymbol{R}^n} L_{\mathcal{S}}(x,y,z),$$

and $L_{\mathcal{S}} : I\!R^n \times I\!R^{m_1} \times I\!R^{m_2|\mathcal{S}|} \to \overline{I\!R}$ is given by

$$L_{\mathcal{S}}(x,y,z) := \begin{cases} f_{10}(x) + F_{20}(x) + \sum_{i=1}^{m_1} y_i f_{1i}(x) & \\ + \sum_{A \in \mathcal{S}, i=1}^{m_2} z_i^A F_{2i}^A(x) & \text{if } x \in C,\ y \geq 0,\ z \geq 0, \\ +\infty & \text{if } x \notin C, \\ -\infty & \text{otherwise.} \end{cases}$$

We have the following duality theorem relating $\mathcal{P}$ and $\mathcal{D}_{\mathcal{S}}$.

Theorem 5.1. *Suppose $\mathcal{P}$ is feasible and K is bounded. Then* $\min \mathcal{P} = \sup \mathcal{D}_{\mathcal{S}}$.

Proof. The problems $\mathcal{P}$ and $\mathcal{P}_{\mathcal{S}}$ are equivalent from Theorem 2.1. Notice that $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}$ are primal and dual problems in the finite-dimensional setting of §3. Thus, the result follows from Theorem 3.2. □

We next concentrate on the existence of dual solutions to $\mathcal{D}_{\mathcal{S}}$, which relies on the property of *calmness* defined in §3. The existence of dual solutions in $\mathcal{D}_{\mathcal{S}}$ may be reduced to the question of calmness from below at $(0,0)$ of any of the finite-dimensional optimal value functions for $\mathcal{P}$.

Theorem 5.2. *For fixed $\mathcal{S} \in \mathbf{G}$, suppose $\mathcal{P}$ is feasible, K is bounded and the finite-dimensional optimal value function $\varphi_{\mathcal{S}}$ is calm from below at $(0,0)$. Then* $\min \mathcal{P} = \max \mathcal{D}_{\mathcal{S}}$.

Proof. This applies Theorem 3.4 to $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}$ to obtain the existence of a dual solution for $\mathcal{D}_{\mathcal{S}}$. Theorem 5.1 and the equivalence of $\mathcal{P}$ and $\mathcal{P}_{\mathcal{S}}$ in Theorem 2.1 then imply that $\min \mathcal{P} = \max \mathcal{D}_{\mathcal{S}}$. □

Thus we have managed to bypass the usual Slater conditions to obtain existence of dual multipliers for this class of stochastic programs. The next theorem relates calmness from below to subgradients and subderivatives as in 3.6.

Theorem 5.3. *If $\mathcal{P}$ is feasible and K is bounded, then for an $\mathcal{S} \in \mathbf{G}$, $\varphi_{\mathcal{S}}$ is calm from below at $(0,0)$ if and only if $\partial\varphi_{\mathcal{S}}(0,0) \neq \emptyset$. Additionally, this is equivalent to $d\varphi_{\mathcal{S}}(0,0)(u,v) > -\infty$ for all directions $(u,v) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2|\mathcal{S}|}$.*

Proof. This applies Theorem 3.6 under the equivalence of $\mathcal{P}$ and $\mathcal{P}_{\mathcal{S}}$ in 2.1. ◻

Remark. Because the dual multipliers may be interpreted economically as a price system for perturbations in the constraints (see Theorem 3.5 and the comments that follow it), the 'second-stage' dual multipliers $\bar{z}$ should be interpreted with respect to the specific constraints (choice of $\mathcal{S}$) considered. Thus, from the remark at the end of §2, we would have an infinite cost ($d\varphi_{\mathcal{S}}(0,0)(0,v) = +\infty$) if any component of v is negative, whereas for $v \geq 0$, $d\varphi_{\mathcal{S}}(0,0)(0,v)$ serves as a measure of the marginal cost that would be incurred for perturbations of f_{2i} in the average over sets $A \in \mathcal{S}$. Using this scheme, one could conceive of analyzing the sensitivity of the problem under various choices of finite partitions $\mathcal{S} \in \mathbf{G}$.

But let's go further now and derive similar results when the problem does not have bounded constraint sets, using the notion of direction-free feasibility developed in §4. Theorems 5.4, 5.5, and 5.6, parallel Theorems 5.1, 5.2, and 5.3 above. We will say that $\mathcal{P}$ has direction-free feasibility if $\mathcal{P}_{\mathcal{S}}$ does for some partition $\mathcal{S} \in \mathbf{G}$ (hence for all such partitions).

Theorem 5.4. *Suppose $\mathcal{P}$ has direction-free feasibility. Then $\inf \mathcal{P} = \sup \mathcal{D}_{\mathcal{S}}$.*

Proof. By forming the finite-dimensional problem, $\mathcal{P}_{\mathcal{S}}$, we put ourselves in the framework of §4. In fact, applying Theorem 4.5 to $\mathcal{P}_{\mathcal{S}}$ yields the desired result. ◻

Theorem 5.5. *Suppose $\mathcal{P}$ has direction-free feasibility. For fixed $\mathcal{S} \in \mathbf{G}$, suppose the finite-dimensional optimal value function $\varphi_{\mathcal{S}}$ is calm from below at $(0,0)$. Then $\inf \mathcal{P} = \max \mathcal{D}_{\mathcal{S}}$.*

Proof. The proof again relies on applying Theorem 4.5 to $\mathcal{P}_{\mathcal{S}}$, through the equivalence of $\mathcal{P}$ and $\mathcal{P}_{\mathcal{S}}$ in 2.1, and the fact that $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}$ are primal and dual problems as in §4. ◻

Theorem 5.6. *Suppose $\mathcal{P}$ has direction-free feasibility. Then, for an $\mathcal{S} \in \mathbf{G}$, $\varphi_{\mathcal{S}}$ is calm from below at $(0,0)$ if and only if $\partial\varphi_{\mathcal{S}}(0,0) \neq \emptyset$. Additionally, this is equivalent to $d\varphi_{\mathcal{S}}(0,0)(u,v) > -\infty$ for all directions $(u,v) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2|\mathcal{S}|}$.*

Proof. From the proof of Theorem 4.5, $\varphi'_{\mathcal{S}} = \varphi_{\mathcal{S}}$, where $\varphi'_{\mathcal{S}}$ is the optimal value function for the finite-dimensional optimization problem $\mathcal{P}'_{\mathcal{S}}$, which has a bounded constraint set. This puts us in the framework of Theorem 3.6, and the proof is immediate. ◻

6. Infinite-dimensional stochastic programming duality theorems

We are now prepared to extend these results to the setting of stochastic programming problems with infinite-dimensional dual problems.

The dual problem to $\mathcal{P}$ that we consider is derived from the Lagrangian $L : I\!R^n \times [I\!R^{m_1} \times \mathcal{L}^\infty(\Xi, \mathcal{F}, P; I\!R^{m_2})] \to \overline{I\!R}$,

$$L(x, y, z) := \begin{cases} f_{10}(x) + E\{f_{20}(\boldsymbol{\xi}, x)\} + \sum_{i=1}^{m_1} f_{1i}(x) y_i & \\ + \sum_{i=1}^{m_2} E\{f_{2i}(\boldsymbol{\xi}, x) z_i(\boldsymbol{\xi})\} & \text{if } x \in C,\ y \geq 0,\ z \geq 0\ P\text{-a.s.}, \\ +\infty & \text{if } x \notin C, \\ -\infty & \text{otherwise.} \end{cases} \tag{3}$$

The dual problem is given by

$$\text{maximize } g(y, z) \text{ so that } y \geq 0,\ z \geq 0\, P\text{-a.s.}, \tag{$\mathcal{D}$}$$

where

$$g(y, z) := \inf\{L(x, y, z) \mid x \in C\}.$$

We have the following duality theorem relating, for a given partition $\mathcal{S} \in \mathbf{G}$, the problems $\mathcal{P}$, $\mathcal{P}_\mathcal{S}$, $\mathcal{D}$, and $\mathcal{D}_\mathcal{S}$.

Theorem 6.1. *Suppose $\mathcal{P}$ has direction-free feasibility. Then*

$$\inf \mathcal{P} = \inf \mathcal{P}_\mathcal{S} = \sup \mathcal{D}_\mathcal{S} = \sup \mathcal{D}.$$

If $\mathcal{D}_\mathcal{S}$ has an optimal solution, then so does $\mathcal{D}$.

Proof. $\mathcal{P}$ and $\mathcal{P}_\mathcal{S}$ are just different ways of writing the same problem, so $\inf \mathcal{P} = \inf \mathcal{P}_\mathcal{S}$ is trivial. The equality, $\inf \mathcal{P}_\mathcal{S} = \sup \mathcal{D}_\mathcal{S}$ is a direct application of Theorem 4.5. To show that $\sup \mathcal{D}_\mathcal{S} = \sup \mathcal{D}$, first notice that for

$$\Lambda := \{\lambda \in \mathcal{L}^\infty(\Xi, \mathcal{F}, P; I\!R) \mid \lambda \in [0, 1]\ P\text{-a.s.}\},$$

which is weak* compact in $\mathcal{L}^\infty(\Xi, \mathcal{F}, P; I\!R)$ by the Banach-Alaoglu Theorem, we may rewrite F_{2i} as

$$F_{2i}(x) = \max_{\lambda \in \Lambda} \int_\Xi \lambda(\xi) f_{2i}(\xi, x) P(d\xi), \quad i = 1, \ldots, m_2.$$

This is a straightforward interchange between integration and maximization, as permitted when the maximization is taken over a *decomposable* space such as $\mathcal{L}^\infty(\Xi, \mathcal{F}, P; I\!R)$: for fixed x, i, let $h : \Xi \times I\!R \to \overline{I\!R}$ be defined by

$$h(\xi, \eta) := \begin{cases} \eta f_{2i}(\xi, x) & \text{if } \eta \in [0, 1], \\ -\infty & \text{otherwise.} \end{cases}$$

h is a *normal (usc) integrand* (upper semicontinuous in η for fixed ξ, and jointly measurable with respect to $\mathcal{F} \otimes \mathcal{B}$, where $\mathcal{B}$ denotes the Borel field on $I\!R$), and thus by [13, Theorem 14.60],

$$\begin{aligned}
\max_{\lambda\in\Lambda} \textstyle\int_\Xi \lambda(\xi) f_{2i}(\xi,x)P(d\xi) &= \max_{\lambda\in\mathcal{L}^\infty} \textstyle\int_\Xi h(\xi,\lambda(\xi))P(d\xi) \\
&= \textstyle\int_\Xi \max_{\eta\in\boldsymbol{R}} h(\xi,\eta)P(d\xi) \\
&= \textstyle\int_\Xi \max_{\eta\in[0,1]} \eta f_{2i}(\xi,x)P(d\xi) \\
&= \textstyle\int_\Xi f_{2i}^+(\xi,x)P(d\xi).
\end{aligned}$$

Through this expression for F_{2i}, $i = 1,\ldots,m_2$, we derive the expression for L':

$$L'(x,y,\zeta) = \max_{\lambda\in\Lambda^{m_2}} L(x,y,\zeta\circ\lambda),$$

where $\zeta\circ\lambda$ represents the componentwise vector of products $\zeta_i\lambda_i$. Recall

$$L(x,y,\zeta\circ\lambda) = \begin{cases} f_{10}(x) + E\{f_{20}(\boldsymbol{\xi},x)\} + \sum_{i=1}^{m_1} f_{1i}(x)y_i & \\ + \sum_{i=1}^{m_2} \zeta_i E\{f_{2i}(\boldsymbol{\xi},x)\lambda_i(\boldsymbol{\xi})\} & \text{if } x\in C,\ y\geq 0,\ \zeta\geq 0\ P\text{-a.s.}, \\ +\infty & \text{if } x\notin C, \\ -\infty & \text{otherwise.} \end{cases}$$

The weak* compactness of Λ along with the weak* upper semicontinuity and concavity of L in λ and the convexity of L in x allows the interchange

$$\inf_x \max_{\lambda\in\Lambda^{m_2}} L(x,y,\zeta\circ\lambda) = \max_{\lambda\in\Lambda^{m_2}} \inf_{x\in\boldsymbol{R}^n} L(x,y,\zeta\circ\lambda),$$

see for example the minimax theorems in [4, 15]. Thus through identification of $\zeta\circ\lambda$ with z, we obtain

$$\begin{aligned}
\sup\mathcal{D}_\mathcal{S} &= \sup_{y,\zeta} \inf_x L'(x,y,\zeta) \\
&= \sup_{y,\zeta} \inf_x \max_{\lambda\in\Lambda^{m_2}} L(x,y,\zeta\circ\lambda) \\
&= \sup_{y,\zeta} \max_{\lambda\in\Lambda^{m_2}} \inf_x L(x,y,\zeta\circ\lambda) \\
&= \sup_{y,z} \inf_x L(x,y,z) \\
&= \sup\mathcal{D}.
\end{aligned}$$

This argument also implies that attainment of an optimal solution in $\mathcal{D}_\mathcal{S}$ implies attainment in $\mathcal{D}$, since a solution $(\bar{y},\bar{\zeta})$ for $\mathcal{D}_\mathcal{S}$ and the weak* compactness of Λ would yield a solution $(\bar{y},\bar{\zeta}\circ\bar{\lambda})$ for $\mathcal{D}$. ◻

We next concentrate on the existence of dual solutions to $\mathcal{D}$. The next theorem relies on the property of *calmness* defined in §1. Here, the existence of dual solutions for $\mathcal{P}$ may be reduced to the question of calmness of the finite-dimensional optimal value function for $\mathcal{P}_\mathcal{S}$ for any $\mathcal{S}\in\mathbf{G}$.

Theorem 6.2. *Suppose $\mathcal{P}$ has direction-free feasibility and the value function for $\mathcal{P}_{\mathcal{S}}$,*

$$\varphi(u) := \inf_x \begin{cases} f_{10}(x) + F_{20}(x) & \text{if } x \in C,\ f_{1i}(x) \le u_{1i},\ i = 1, \ldots, m_1, \\ & F_{2i}(x) \le u_{2i},\ i = 1, \ldots, m_2, \\ +\infty & \text{otherwise,} \end{cases}$$

is calm from below at 0 for any $\mathcal{S} \in \mathbf{G}$. Then $\inf \mathcal{P} = \max \mathcal{D}$.

Proof. This applies 6.1 to obtain the string of equalities, $\inf \mathcal{P} = \inf \mathcal{P}_{\mathcal{S}} = \sup \mathcal{D}_{\mathcal{S}} = \sup \mathcal{D}$. Applying the latter part of 4.5 to $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{D}_{\mathcal{S}}$ yields the existence of a dual solution for $\mathcal{D}_{\mathcal{S}}$. Theorem 6.1 then implies that there exists a solution to $\mathcal{D}$ as well. □

7. The fundamental theorem of asset pricing

The following problem comes from the mathematical finance literature. It has been simplified here to a one-stage setting. Let $Z_0 \in I\!R^{J+1}_+$ and $Z_T \in \mathcal{L}^1_+(\Xi, \mathcal{F}, P; I\!R^{J+1})$ denote vectors of asset prices, where $(\Xi, \mathcal{F}, P)$ is an underlying probability space assumed to be P-complete. For simplicity it is assumed that the first asset Z^0_t is riskless, and equal to one (otherwise one could normalize the price vectors accordingly), for $t = 0, T$. A *contingent claim* is a contract to pay $F_T \in \mathcal{L}^1(\Xi, \mathcal{F}, P; I\!R)$ in the future (at time T), where it is assumed that $P(F_T > 0) > 0$, i.e. at least some events will yield a positive payout. (A futures contract is an example of a contingent claim which could have both positive and negative payouts.) The *writer* of the claim offers it at some price $F_0 \in I\!R_+$. The writer has two goals: one is to find a fair price F_0 for the contingent claim given that he can invest his receipts in the market, and the second is to optimize his portfolio of investments once this price is fixed. We will concentrate on the second of these problems, although they are intimately tied together.

The writer's portfolio optimization problem is to choose a portfolio of investments, $\theta \in I\!R^{J+1}$ to maximize his or her expected terminal wealth, given the ability to invest F_0, and the obligation to pay out F_T. One could also include a utility function in the formulation but that has been omitted here, for simplicity. We write the writer's problem (as a minimization problem in keeping with the preceding sections) as

$$\begin{array}{ll} \text{minimize} & -E\{Z_T \cdot \theta\} \\ \text{subject to} & Z_0 \cdot \theta \le F_0 \\ & Z_T \cdot \theta \ge F_T \quad P\text{-a.s.} \end{array} \qquad (\mathcal{P}_f)$$

and assume that the price F_0 and the payout F_T are such that $\mathcal{P}_f$ is feasible.

No arbitrage conditions on the market mean loosely that one cannot generate positive wealth from nothing. One form of this is to say that the market admits *no free lunches.*

A free lunch is a portfolio such that

$$\begin{aligned} Z_0 \cdot \theta &\leq 0 \\ Z_T \cdot \theta &\geq 0 \quad P\text{-a.s.} \\ P(Z_T \cdot \theta > 0) &> 0. \end{aligned}$$

Under the first two inequalities, this last condition may be equivalently stated as

$$E\{Z_T \cdot \theta\} > 0.$$

The *fundamental theorem of asset pricing* states that there are no free lunches if and only if there exists an equivalent martingale measure for the price process (such a measure is then used to price contingent claims; the 'fair price' is the expected value of F_T under the equivalent martingale measure). An *equivalent martingale measure* is a probability measure $Q \sim P$ (which means $P(E) = 0$ if and only if $Q(E) = 0$), such that the market price process is a martingale under Q:

$$E_Q\{Z_T\} := \int_\Xi Z_T(\xi) Q(d\xi) = Z_0.$$

In a duality framework, one can think of these measures Q as Radon-Nikodym derivatives of dual variables that lie, say, in $\mathcal{L}^1$. However, the usual duality theory for stochastic programs [9, 10] does not encompass this problem. Note that $\mathcal{P}_f$ does not satisfy relatively complete recourse since for $\theta = \begin{pmatrix} -\varepsilon \\ \vec{0} \end{pmatrix}$, $Z_T \cdot \theta = -\varepsilon \not\geq F_T$ P-a.s.. It also does not have a bounded feasibility set. Thus in the 'usual' stochastic programming setting, the best one can obtain is a dual problem whose variables lie in the dual of $\mathcal{L}^\infty$, cf. [11, 6]. In fact, this would yield finitely additive measures as dual multipliers, which does in itself have some interesting significance. But the goal here is to take the first steps to reproduce some of the mathematical finance results in a stochastic programming setting. The approach is to obtain the desired duality result through Theorem 6.1.

Lemma 7.1. *If the market admits no free lunches, then $\mathcal{P}_f$ has direction-free feasibility.*

Proof. Suppose that the market admits no free lunches. Then there is no θ such that

$$\begin{aligned} Z_0 \cdot \theta &\leq 0 \\ Z_T \cdot \theta &\geq 0 \quad P\text{-a.s.} \\ E\{Z_T \cdot \theta\} &> 0. \end{aligned}$$

Let

$$K_f := \{\theta \in I\!R^{J+1} \mid Z_0 \cdot \theta \leq F_0,\ Z_T \cdot \theta_0 \geq F_T\ P\text{-a.s.}\},$$

i.e. the feasible set for $\mathcal{P}_f$. If it can be shown that K_f^∞ is a subspace of $I\!R^{J+1}$, then $\mathcal{P}_f$ has direction-free feasibility. First observe that

$$K_f^\infty = \{\theta \in I\!R^{J+1} \mid Z_0 \cdot \theta \leq 0,\ Z_T \cdot \theta_0 \geq 0\, P\text{-a.s.}\}.$$

Let $\hat\theta \in K_f^\infty$. It suffices to show that $-\hat\theta \in K_f^\infty$. Since there are no free lunches, it must be true that $E\{Z_T \cdot \hat\theta\} = 0$, and hence that $Z_T \cdot \hat\theta = 0$ P-a.s.. It is also true that $Z_0 \cdot \hat\theta = 0$, because if $Z_0 \cdot \hat\theta = -\varepsilon < 0$, then $\bar\theta := \hat\theta + \begin{pmatrix}\varepsilon \\ \vec{0}\end{pmatrix}$ is a free lunch, a contradiction. Thus $Z_T \cdot (-\hat\theta) = 0$ and $Z_0 \cdot (-\hat\theta) = 0$, i.e. $-\hat\theta \in K_f^\infty$. This implies that K_f^∞ is a subspace of $I\!R^{J+1}$. Thus, if the market admits no free lunches, then $\mathcal{P}_f$ has direction-free feasibility. □

The theorem that follows is the simplified version of the fundamental theorem of asset pricing in a stochastic programming setting.

Theorem 7.2. *The following are equivalent:*
(a) *$\mathcal{P}_f$ is bounded*
(b) *$\mathcal{D}_f$ is feasible*
(c) *The market admits no free lunches*
(d) *There exists an equivalent martingale measure for the market price process.*

Proof. (a) $\iff$ (c) is readily apparent. (b) $\iff$ (d) comes from examing the dual problem:

$$\begin{array}{lrcl} \text{maximize} & F_0 y_0 - E\{F_T z_T\} & & \\ \text{subject to} & E\{Z_T(1+z_T)\} & = & Z_0 y_0 \\ & y_0 \geq 0,\ z_T & \geq & 0\, P\text{-a.s.} \end{array} \qquad (\mathcal{D}_f)$$

Here, feasibility of the dual is equivalent to the price process being a martingale under some measure Q equivalent to P ($Q \sim P$) given by

$$Q(E) = \int_E (1 + z_T(\xi))/y_0 P(d\xi)$$

for such a feasible point $(y_0, z_T) \in I\!R_+ \times \mathcal{L}_+^\infty(\Xi, \mathcal{F}, P; I\!R)$.

(b) $\Longrightarrow$ (a) since it is always true that $\inf \mathcal{P}_f \geq \sup \mathcal{D}_f$ and $\mathcal{D}_f$ feasible implies $\sup \mathcal{D}_f > -\infty$ which in turn implies $\inf \mathcal{P}_f > -\infty$, i.e. $\mathcal{P}_f$ is bounded. The main part of the proof is in showing that (a) $\Longrightarrow$ (b). This will certainly be true if we can show that $\inf \mathcal{P}_f = \sup \mathcal{D}_f$. If (a) holds, since (a) $\Longrightarrow$ (c), by Lemma 7.1 $\mathcal{P}_f$ has direction-free feasibility. Theorem 6.1 then implies $\inf \mathcal{P}_f = \sup \mathcal{D}_f$. $\mathcal{P}_f$ bounded means $\inf \mathcal{P}_f > -\infty$. Thus, $\sup \mathcal{D}_f > -\infty$, which means that $\mathcal{D}_f$ is feasible. Thus (a) $\Longrightarrow$ (b). □

This theorem is a very simplified version of the typical models appearing in the mathematical finance literature. Its significance lies in setting up a duality theory in a stochastic programming setting which may in principle be extended to cover multi-stage, and even possible continuous time models in a future development. It is noteworthy that one obtains the existence of an equivalent martingale measure whose Radon-Nikodym derivative is in $\mathcal{L}^\infty$ as opposed to $\mathcal{L}^1$. Moreover, the geometric consequence of the no free lunch condition (direction-free feasibility) provides a weaker condition than no free lunch, that if satisfied will yield (along with the boundedness of the problem) the existence of an equivalent martingale measure. This opens up the possibility to consider problems which may have additional types of variables and constraints, where no free lunch conditions may fail (or not even make sense), but direction-free feasibility still has meaning and could still be established.

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