Capital Growth with Security*

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Abstract

This paper discusses the allocation of capital over time with several risky assets. The capital growth log utility approach is used with conditions requiring that specific goals are achieved with high probability. The stochastic optimization model uses a disjunctive form for the probabilistic constraints, which identifies an outer problem of choosing an optimal set of scenarios, and an inner (conditional) problem of finding the optimal investment decisions for a given scenarios set. The multiperiod inner problem is composed of a sequence of conditional one period problems. The theory is illustrated for the dynamic allocation of wealth in stocks, bonds and cash equivalents.

1 Introduction

The problem of capital accumulation under uncertainty has occupied an important place in the theory of financial economics. Given a set of risky investment opportunities, a decision maker must choose how much of available capital to invest in each asset at each point in time. When the criterion for selecting an investment policy is maximizing the expected value of the logarithm of accumulated capital, the resulting policy, known as the Kelly or capital growth criterion, has many desirable properties. The maximum expected logarithm strategy asymptotically maximizes the log run expected growth rate of capital (Kelly 1956, Breiman 1961). Moreover, the optimal policy is myopic and period-by-period optimization can be used to compute the optimal decisions (Hakansson 1972). Breiman (1961) has shown that the expected time to reach asymptotically large wealth levels is minimized by this strategy. A theoretical exposition of the properties of the capital growth strategy in the intertemporally independent and weakly dependent cases appears in Algoet and Cover (1988). Rotando and Thorp (1992) apply the Kelly strategy to long-term investment in the U.S. stock market and demonstrate some of the benefits and liabilities of that strategy. MacLean, Ziemba and Blazenko (1992) discuss a theory of growth versus security using fractional Kelly strategies which are convex combinations of cash and the Kelly fraction and apply this to several speculative investment applications; see also MacLean and Ziemba (1991, 1999). MacLean, Ziemba and Li (2000) show that the fractional Kelly strategies lie on a growth-security efficient frontier if the assets are lognormally distributed. Without lognormality, the tradeoff of growth versus security is monotone but not necessarily efficient. Hakansson and Ziemba (1995) review the capital growth literature and various applications.

The optimality properties of the Kelly strategy are related to expected values, either of log wealth or first passage times. But the fraction of wealth invested may be unacceptably large because the Arrow-Pratt risk aversion index is the reciprocal of wealth and is essentially zero for reasonable wealth levels. Furthermore, if uncertainty in the return on investment is considerable then the probability of wealth becoming negligible at some point is high with the capital growth strategy.[1] This downside risk of investment strategies has led to a growing interest in risk control.

The traditional approach to risk control has been to include the variance of wealth in the decision problem, and solve for mean-variance efficient strategies (Markowitz 1959). More recently Value-at-Risk based risk management has emerged as the standard (Jorion 1997). The VaR is the floor below which wealth can fall in a specified time interval, with a prespecified small probability.

Investment models can be formulated in discrete or continuous time. In continuous time the dynamics of asset prices are usually defined by geometric Brownian motion, with the asset prices having log-normal distributions (Merton 1992). A continuous time model with VaR constraint is studied by Basak and Shapiro (1999), where the VaR risk managers' strategies are contrasted with portfolio insurance. The discrete time analogue to the Brownian motion model is a random walk. Computational approaches to obtaining optimal growth strategies in discrete time without assumptions on asset price distributions are presented by Cover (1991); see also Helmbold et al (1996).

In this paper the computation of strategies for investment in discrete time which achieve maximal capital growth subject to a VaR constraint is considered. The emphasis is on defining a multistage stochastic programming problem where the constraints are identified by a selection of scenarios sampled from the space of potential financial market outcomes. Using the VaR constraint, the outcomes are classified as critical and noncritical, with feasible investment strategies maintaining the critical outcomes at a small percentage. An algorithm for classifying scenarios is developed and illustrated with the optimal trade-off of cash, bonds and stocks over time.

2 Capital Accumulation Model

Consider an investor with initial wealth W_0 and the opportunity to invest in m risky securities. The following assumptions are made about capital markets: no transactions costs; no taxes; infinite divisibility of assets; assets have limited liability; borrowing and lending are allowed at the same rate; and short selling is permitted. These conditions will be referred to as the market assumptions.

The trading price of security i at time t is $P_i(t), i = 1, \cdots, m$. In discrete time the rate of return

[1] For example, Ziemba and Hausch (1986) ran a simulation where an investor has initial wealth \$1000 and makes 700 independent wagers, all of which have an expected value of \$1.14 per \$1 wagered and all of which have not small probabilities of winning. In 166 of 1000 replications the Kelly bettor had a final fortune of \$1,000,000 or more. However, the minimum final wealth was only \$18.

on a unit of capital invested in security i at time t is

$$\frac{P_i(t+1)}{P_i(t)} = R_i(t), \quad i = 1, \cdots, m. \tag{1}$$

It is assumed that the returns follow a log-normal distribution, so consider $Y_i(t) = \ln P_i(t), i = 1, \cdots, m$. Whereas $P_i(t)$ is a geometric process, $Y_i(t)$ is an arithmetic random walk with increments $Z_i(t) = \ln R_i(t), i = 1, \cdots, m$, with $Z_i(t)$ having a normal distribution. Therefore, the increments can be represented as a random effects linear model

$$Z_i(t) = \pi_i + \delta_i \epsilon_i(t), \quad i = 1, \cdots, m \tag{2}$$

where $\epsilon(t)^\top = (\epsilon_1(t), \cdots, \epsilon_m(t))$ is $N(0, I)$ and $\pi^\top = (\pi_1, \cdots, \pi_m)$ is $N(\mu, \Gamma)$, with $\mu^\top = (\mu_1, \cdots, \mu_m)$ and $\Gamma = (\gamma_{ij})$. So, γ_{ij} is the covariance between π_i and π_j, the random expected rates of return of securities i and j, respectively.

From the arithmetic random walk with normal increments the conditional distribution of log-prices at time t, given π and $\Delta = diag(\delta_1^2, \cdots, \delta_m^2)$ is $(Y(t)|\pi, \Delta) \sim N(\pi t, t\Delta)$. The marginal distribution of log-prices is $Y(t) \sim N(\mu t, \Sigma(t))$, where $\Sigma(t) = t\Delta + t^2\Gamma$.

Although this log-normal model for securities prices is specialized, the random rates of return provide the flexibility required to match the theoretical prices to observations. This discrete time model is the analogue to the geometric Brownian models in continuous time. The dynamics of price movements are clear from the distribution parameters. The underlying parameters (μ, Γ, Δ) generate the securities prices. If these parameters are known or estimated then the price distributions can be specified. Parameter estimation is discussed in Section 6.

From the initial values the forward price process evolves as a random walk with intertemporally independent increments defined by (2). Consider the rate of returns process $R_a(t)^\top = (R_1(t), \cdots, R_m(t)), t = 1, \cdots, T$, and denote the multivariate rate of returns distribution at time t by F_t. For the stochastic process $R_a(t)^\top = (R_1(t), \cdots, R_m(t))$, a trajectory or realization of the data process is associated with an outcome ω in the sample space Ω of all outcomes (trajectories). The distributions $F_1, \cdots, F_T$ generate a probability measure P on Ω and the associated probability space (Ω, B, P). The sample space can be represented as $\Omega = \Omega_1, \cdots, \Omega_T$, with $\omega_t \in \Omega_t$, the data for time t. The information available to the investor at time t is the data on the past, and is represented by the filtration $B_0 := \{\emptyset, \Omega\} \subset B_1 \subset \cdots \subset B_r := B$, where $B_t := \sigma(\omega^t)$ is the σ-field generated by the history ω^t of the data process ω to time t. So the stochastic process is adapted to $\{B_t; t = 1, \cdots, T\}$, the augmented filtration generated by ω.

In addition to the risky securities defined on $(\Omega, B, \{B_t\}, P)$ there is a riskless asset with rate of return $R_0(t) = 1 + r(t)$. Let $R(t)^\top = (R_0(t), R_a(t)^\top)$.

An investment decision at time t is the proportion of wealth to allocate to each asset, given by

$$X(t)^\top = (X_0(t), \cdots, X_m(t)). \tag{3}$$

It is assumed that $X(t)$ can depend on the data history ω^t but not on unknown future returns, so it is B_t predictable. The budget constraint at each time requires $\sum_{i=0}^m X(t) = 1$. The proportions of wealth invested in risky assets are unconstrained, since the proportion invested in the risk-free asset can always be chosen (with borrowing or lending) to satisfy the budget constraint.

An investment strategy is an $m+1$ vector process, $X = \{X(t), t = 1, \cdots, T\}$, where T is the planning horizon. With initial wealth W_0, rate of returns process R and investment process X, the capital accumulated to time t is

$$W(t) = W_0 \prod_{s=1}^{t} R(s)^\top X(s), \quad t = 1, \cdots, T. \tag{4}$$

The paths of the capital accumulation process are controlled by the investment strategy, and the investor selects a strategy based on anticipated performance as indicated by measures for growth and security (risk)

3 Downside Risk Control

For the geometric growth process of capital accumulation in (4) a natural performance measure is the geometric mean $E\left[W(T)^{\frac{1}{T}}\right] = E\left[\left(\prod_{t=1}^T R(t)^\top X(t)\right)^{\frac{1}{T}}\right]$. Since

$$W(T)^{\frac{1}{T}} = W_0 \left(\exp\left\{\frac{1}{T}\sum_{t=1}^{T} \ln(R(t)^\top X(t))\right\}\right),$$

$G(X) = \frac{1}{T}\sum_{t=1}^T \ln(R(t)^\top X(t))$ is the growth rate of capital. The geometric mean is maximized by the optimal expected growth rate strategy from

$$\max_X \quad E[G(X)] = \frac{1}{T}\sum_{t=1}^{T} E[\ln(R(t)^\top X(t))]. \tag{5}$$

The growth optimal strategy X^* from (5) is called the Kelly (1956) strategy. This strategy has been studied extensively. For advanced proofs of optimality properties with minimal assumptions see Algoet and Cover (1988).

The Kelly strategy is very risky. Although it provides optimum growth in the long run it is possible to experience negative growth in the medium term and in any period experience a substantial loss of capital (drawdown). Discussion of these properties appears in Table 1 of MacLean and Ziemba (1999).

Because of the volatility of financial markets it is prudent (and frequently it is a legal requirement) to include downside risk control in the decisions on investment strategy. To put risk measurement in context, consider the following definition, an adaptation of one provided in Breitmeyer et al (1999).

Definition 1. *(Downside Risk Measure.) Consider a wealth process* $\{W(t), t = 1, \cdots, T\}$ *with corresponding distributions* $\{H_t, t = 1, \cdots, T\}$*. Let* $q \in \Re$ *be an arbitrary number which partitions wealth trajectories into acceptable and unacceptable sets, denoted by* C *and* $\bar{C}$*, respectively. If* V *is the set of probability distributions for the wealth process, a downside risk measure* ϕ *is a function* $\phi : V \times R \to R$ *satisfying the axioms*

1. *(non-negativity):* $\phi(H, q) \geq 0$*,*
2. *(normalization): If* $H(\omega) = 0$ *for all* $\omega \in \bar{C}$*, then* $\phi(H, q) = 0$*,*
3. *(downside focus): If* H *and* $\bar{H}$ *are distributions over wealth trajectories such that* $H(\omega) = \tilde{H}(\omega)$*, for* $\omega \in \bar{C}$*, then* $\phi(H, q) = \phi(\bar{H}, q)$*.*

There are additional axioms which consider properties such as consistency, continuity and invariance (Breitmeyer et al. 1999, Artzner et al. 1999), but the properties in our definition are basic.

The standard measure for downside risk in the financial industry is *Value at Risk* (Jorion 1997). It is defined as *the loss*, which is exceeded with some given probability α, over a given time horizon. The intention is to control *VaR*, so a prespecified minimum value w^* is given and the value at risk with probability α must exceed w^*. Equivalently, the probability that wealth exceeds w^* at the VaR horizon is at least $1 - \alpha$.

Although VaR is widely used, it has some undesirable properties (see Artzner et al. 1999, and Basak and Shapiro 1999). Other proposed measures are the period-by-period drawdown (Grossman and Zhou 1993) and incomplete mean (Basak and Shapiro 1999). The drawdown used in this paper considers the potential fraction of wealth lost at each period. The incomplete mean is the partial expected value in the lower α percentile of the wealth distribution. Consider then the formal definition of the alternative risk measures.

Definition 2. *(Risk Measures.) Consider a wealth process* $\{W(t), t = 1, \cdots, T\}$ *with corresponding distributions* $\{H_t, t = 1, \cdots, T\}$*.*

1. *The value at risk measure for horizon* T *and wealth value* w *is*

$$\phi_1(H, w) = \Pr[W(T) \geq w] = 1 - H_T(w).$$

2. *The drawdown measure for decay fraction* $b \in [0, 1]$ *is*

$$\phi_2(H, b) = \Pr[W(t + 1) \geq bW(t), t = 1, \cdots, T].$$

3. *The incomplete mean measure for specified percentile* α *is*

$$\phi_3(H, \alpha) = \int_0^{w_\alpha} w dH_T(w)$$

where $\Pr[W(T) \leq w_\alpha] = \alpha$*.*

Each of these measures satisfies the basic axioms for downside risk. $\phi_1(H, w_\alpha)$ and $\phi_3(H, \alpha)$ have the same unacceptable sets, and ϕ_3 is a measure of the expected loss on the unacceptable set. With respect to ϕ_2 it is possible to link b to the value w in ϕ_1.

Let $b(w) = \left(\frac{w}{W(0)}\right)^{\frac{1}{T}}$, and see that the unacceptable set for ϕ_1 is contained in the unacceptable set for ϕ_2. Therefore $\phi_2(H, b(w)) \leq \phi_1(H, w)$. So ϕ_2 and ϕ_3 are more stringent risk measures than ϕ_1.

The purpose in defining a risk measure is to develop an investment strategy which achieves capital growth while controlling for risk. Subsequent discussion of risk will concentrate on the VaR measure ϕ_1. The approach followed is easily adapted to other risk measures.

Consider the general growth with security problem

$$\sup_X \{E[G(X)] | \phi_1(H(X), w) \geq 1 - \alpha\} \tag{6}$$

where $H(X) = (H_1(X), \cdots, H_T(X))$ is the distribution over wealth generated by investment strategy $X = \{X(t), t = 1, \cdots, T\}$, w is a prespecified wealth floor and $1 - \alpha$ is a confidence level. It is assumed that the measurability conditions previously discussed for the return process and the investment process are imposed.

Let $w^* = \frac{w}{W(0)}$. Problem (6) can be written more explicitly in terms of rate of returns as (CCP):

$$\sup_X \left\{ \sum_{t=1}^{T} E \ln(R(t)^\top X(t)) \middle| \Pr\left[\sum_{t=1}^{T} \ln(R(t)^\top X(t)) \geq \ln w^*\right] \geq 1 - \alpha \right\}. \tag{7}$$

The rate of return process $R = \{R(t), t = 1, \cdots, T\}$ and the investment process $X = \{X(t), t = 1, \cdots, T\}$ are defined on $(\Omega, B, \{B_t\}, P)$. For a given trajectory $\omega \in \Omega$ let the associated return path be $R(\omega) = \{R(\omega, t), t = 1, \cdots, T\}$ and the investment path be $X(\omega) = \{X(\omega, t), t = 1, \cdots, T\}$. The risk measure refers to acceptable and unacceptable sets of wealth trajectories. Consider sets of measure $1 - \alpha$ in the probability space, given by $B_\alpha = \{A | P(A) \geq 1 - \alpha\}$. There are associated sets of wealth trajectories for A and its complement $\bar{A}$. An equivalent formulation to (7) based on trajectories is (DP):

$$\sup_{A \in B_\alpha} \left[\sup_X \left\{ \sum_{t=1}^{T} E \ln(R(t)^\top X(t)) \middle| \sum_{t=1}^{T} \ln(R(\omega, t)^\top X(\omega, t)) \geq \ln w^*, \omega \in A \right\} \right]. \tag{8}$$

The disjunctive formulation in (8) defines a sequence of stochastic convex dynamic programming problems, referred to as *inner problems*, with a constraint for each $\omega \in A$.

Another reformulation of (7), found by introducing a weighing variable $\theta(\omega), \omega \in \Omega$, with $0 \leq \theta(\omega) \leq 1$, is (NCP):

$$\sup_{(x,\theta)} \left\{ \sum_{t=1}^{T} E \ln(R(t)^\top X(t)) \middle| \theta(\omega) \left[\sum_{t=1}^{T} \ln(R(t)^\top X(t)) \geq \ln w^*\right] \geq 0, E[\theta(\omega)] \geq 1 - \alpha \right\}. \tag{9}$$

At the optimal solution to (9) the weighing variable is an indicator function. That is, if (X^*, θ^*) is optimal then there is a set A such that $\theta^*(\omega) = 1$ for $\omega \in A$, $\theta(\omega) = 0$ for $\omega \in \bar{A}$. Hence, for the optimal investment strategy: (CCP)⇔(DP)⇔(NCP).

The growth with security problems (7-9) provide a framework for individual portfolio choice, where risk is controlled at a specific level. In multi-dimensional financial markets it is common to identify the underlying sources of systematic risk, and to define a small set of generating portfolios or mutual funds. These funds serve an intermediary role for fund managers to create products which satisfy investor preferences. It is important that the growth with security problem works within this intermediation theory.

To understand the generating portfolios (mutual funds) recall for the price process it is assumed there exist $p < m$ independent latent factors $U^\top = (U_1, \cdots, U_p), U_i \sim N(0, 1), i = 1, \cdots, p$, so that the log-prices are represented as

$$Y(t) = \mu t + t\Lambda U + \sqrt{t}\xi, \tag{10}$$

where $\mu^\top = (\mu_1, \cdots, \mu_m), \lambda = (\lambda_{ij})$ for loadings $\lambda_{ij}, i = 1, \cdots, m$ and $j = 1, \cdots, p$, and the covariance $Cov(\xi) = diag(\delta_1^2, \cdots, \delta_p^2)$. Assume that $E[\xi] = 0$, and U and ξ are independent. As before, $Y(t)$ has the distribution $N(\mu t, \Sigma(t))$, where $\Sigma(t) = t\Delta + t^2\Gamma$ and $\Gamma = \Lambda\Lambda^\top$. The factor model (10) and the associated conditions are referred to as the structural model assumptions.

In this structure imposed on the price process, the factors $U^\top = (U_1, \cdots, U_p)$ are the standardized log-prices for the mutual funds.

Theorem 1. *(Financial Intermediation) Suppose there exist m risky securities and a risk-free asset satisfying the market and structural model assumptions. Then there exists a risk-free and $p < m$ risky mutual funds, so that growth with security investors as defined by (7) are indifferent between choosing portfolios from among the original securities and choosing portfolios from the mutual funds.*

Proof. Consider the return in period t, $R(t)^\top X(t) = \sum_{i=0}^m R_i(t)X_i(t)$. From the model for securities prices $R_i(t) = \exp(\mu_i + \Lambda_i^\top U + \delta_i\epsilon_i)$, where Λ_i is the i^{th} row of the $m \times p$ loading matrix Λ in (10). Let $A_0 = (a_{ij})$, where $a_{ij} = \frac{\lambda_{ij}^2}{(\sigma_i^2 - \delta_i^2)}$, and $A = \begin{pmatrix} 1 & 0 \\ 0 & A_0^\top \end{pmatrix}$. Define $q(t) = A \cdot X(t)$. From (10), $\sum_{j=1}^p \lambda_{ij}^2 = \sigma_i^2 - \delta_i^2$ and

$$\begin{aligned} \sum_{j=0}^p q_j(t) &= X_0(t) + \sum_{j=1}^p \sum_{i=1}^m \left(\frac{\lambda_{ij}^2}{(\sigma_i^2 - \delta_i^2)} X_i(t) \right) \\ &= X_0(t) + \sum_{i=1}^m \left(\sum_{j=1}^p \frac{\lambda_{ij}^2}{(\sigma_i^2 - \delta_i^2)} X_i(t) \right) \\ &= \sum_{i=0}^m X_i(t) = 1. \end{aligned}$$

With A^-, the generalized inverse of A, let $X(t) = A^- q(t)$. Hence, the return in period t is $R(t)^\top X(t) = (R(t)^\top A^-)q(t) = M(t)^\top q(t) = \sum_{j=0}^{p} M_j(t) q_j(t)$, with $M_j(t)$ the return on risky mutual fund j, $j = 1, \cdots, p$ and $M_0(t) = 1 + r(t)$. Since the returns in each period are the same for the mutual funds and the original securities, the statement in the theorem holds.

The investment process $X = \{X(t), t = 1, \cdots, T\}$ is defined on the space of trajectories $(\Omega, B, \{B_t\}, P)$, where $X(t)$ is B_t predictable. An important property of the investment process is path independence (Cox and Leland, 1982), so that it depends only on the reinvested return on the securities, not on the whole price history. This property is satisfied by the growth with security investor.

Theorem 2. *(Path Independence) The optimal growth with security investment strategy X^* is path independent, i.e. $X(t)^*$ depends on the level of wealth at time t, w_t, but not on the particular path to achieve that wealth.*

Proof. Consider the growth with security problem in the disjunctive form

$$\text{DP:} \quad \sup_{A \in B_\alpha} \left[\sup_X \left\{ \sum_{t=1}^{T} E \ln(R(t)^\top X(t)) \middle| \sum_{t=1}^{T} \ln(R(\omega,t)^\top X(\omega,t)) \geq \ln w^*, \omega \in A \right\} \right].$$

If I_A is the indicator function for the set A and $\lambda(\omega) \geq 0$ is a multiplier so that $\lambda \in L(\Omega, B, P)$, the space of Lebesgue integrable functions on Ω, then a Lagrangian for (DP) is

$$\mathcal{L}(X, \lambda, A) = E\left[\sum_{t=1}^{T} \ln(R(\omega,t)^\top X(\omega,t)) + I_A \lambda(\omega) \left(\sum_{t=1}^{T} \ln(R(\omega,t)^\top X(\omega,t)) - \ln w^* \right) \right]. \tag{11}$$

If a solution exists for (DP) then there exist elements (A^*, λ^*) so that the solution to (DP) is given by $\sup_X \mathcal{L}(X, \lambda^*, A^*)$. The Lagrangian can be written as

$$\mathcal{L}(X, \lambda^*, A^*) = \sum_{t=1}^{T} E[(1 + I_{A^*}\lambda^*) \ln(R(t)^\top X(t))].$$

Consider $\eta_0^* = E[1 + I_{A^*}\lambda^*]$ and $\eta_t^* = E_{T-t}(1 + I_{A^*}\lambda^*)$, where E_{T-t} is the expectation over the data process from time $t+1$ to time T. Then, $\eta_t^* = \eta_{t-1}^* \cdot \beta_t^*$, where β_t^* depends only on the additional data in period t. To see this note that $\ln \eta(\omega^t) = \sum_{s=1}^{t} m_s(\omega_s)$, where $m_s(\omega_s) = E_{t-s} m(\omega^t) - E_{t-s+1} m(\omega^t)$. Then

$$\begin{aligned}
\sup_X \sum_{t=1}^{T} E[(1 + I_{A^*}) \ln(R(t)^\top X(t))] &= \sup_X \sum_{t=1}^{T} E[\eta_{t-1}^* \cdot \beta_t^* \ln(R(t)^\top X(t))] \\
&= \sup_X \sum_{t=1}^{T} E\left[\beta_t^* \cdot E_{t-1}[\eta_{t-1}^* \ln(R(t)^\top X(t))]\right]
\end{aligned}$$

where E_{t-1} is the expectation with respect to data for periods 1 to $t-1$. With $\bar{\eta}^*_{t-1} = E_{t-1}\eta^*_{t-1}$, and

$$\begin{aligned} E_{t-1}[\eta^*_{t-1}\ln(R(t)^\top X(t))] &= \bar{\eta}^*_{t-1} E_{t-1}\left[\frac{\eta^*_{t-1}}{\bar{\eta}^*_{t-1}}\ln(R(t)^\top X(t))\right] \\ &\leq \bar{\eta}^*_{t-1}\ln\left(E_{t-1}\left[R(t)^\top \frac{\eta^*_{t-1}}{\bar{\eta}^*_{t-1}} X(t)\right]\right) \end{aligned}$$

from Jensen's inequality and B_{t-1} measurability of η^*_{t-1} and $X(t)$. But $R_i(t) = \exp(Z_i(t))$, where the increments $Z_i(t)$ are intertemporally independent. With $\tilde{X}(t) = E_{t-1}\left[\frac{\eta^*_{t-1}}{\bar{\eta}^*_{t-1}} X(t)\right]$, the problem becomes

$$\sup_X \sum_{t=1}^T \bar{\eta}^*_{t-1} E[\beta^*_t \ln(R(t)^\top \tilde{X}(t))],$$

where $\tilde{X}(t)$ does not depend on the path $\omega^{t-1}, t = 1, \cdots, T$.

The path independence property is within the context of planning for T periods with a given distribution over returns. As will be described in Section VI, at the start of the next T period plan the history of prices is used to update the distribution over returns. So decisions depend on trajectories through the estimation (filtration) process, which is separated from the optimization.

4 Scenario Selection

The growth with security problem in its disjunctive form (8) or equivalent Lagrangian form (11) involves complicated multivariate integration. Some form of discrete approximation to distributions for securities prices is required to proceed with computation.

Consider the distributions F_t on the returns $R_a(t), t = 1, \cdots, T$. Assume that in $\Re^m$ a grid is constructured with rectangles so that the probability mass from F_t in each rectangle is equal. Sample a point from each rectangle, generating the empirical distribution F^n_t. Let the space of trajectories corresponding to the empirical distribution F^n_t, $t = 1, \cdots, T$, be $\Omega^{(n)}$, and the corresponding probability space be $(\Omega^{(n)}, B^{(n)}, P^{(n)})$. So $\Omega^{(n)} = \{\omega_1, \cdots, \omega_I\}$ and $P(\omega_i) = 1/I$.

With this lattice structure, a discrete approximation to the growth with security problem is (DP_n):

$$\sup_{A^{(n)}\in B^{(n)}}\left\{\sup_{X_n}\left\{E\sum_{t=1}^T \ln(R^n(t)^\top X^n(t)) \,\middle|\, \sum_{t=1}^T \ln(R^n(\omega,t)^\top X^n(\omega,t)) \geq \ln w^*, \omega \in A^{(n)}\right\}\right\}. \tag{12}$$

The inner problem and the set selection problem are now discrete. For a given A and the corresponding $A^{(n)}$ the convergence of the solution of the discrete inner problem to the solution of the continuous inner problem has been established and bounds on the error for given n developed (Pflug 1999). The identification of the optimal set $A^{(n)}$, where $P^{(n)}(A^{(n)}) \geq 1-\alpha$, is required.

Assume that the specified target w^* at the horizon T is such that problem (12) has a solution for $A^{(n)} = \Omega^{(n)}$. The process for identifying the optimal set of scenarios is *backward elimination* (Culioli 1996). If k is the largest integer such that $k/I \leq \alpha$, then up to k constraints corresponding to scenarios in $\Omega^{(n)}$ get eliminated and the remaining constraints yield the optimal $A^{(n)}$. The elimination is one constraint at a time starting from $\Omega^{(n)}$ and working backward. In describing the elimination procedure the notation $A^{(n)}(i_j|i_{j-1},\cdots,i_1)$ refers to a set of scenarios in the family of sets $A_j^{(n)}(i_{j-1},\cdots,i_1)$ defined by the $j-1$ elimination steps, where scenarios $\omega_{i_1},\cdots,\omega_{i_{j-1}}$ have been eliminated sequentially. The basis for the algorithm is the inner problem

$$P(A^{(n)}) : \sup_{X^n} \left\{ \sum_{t=1}^{T} E\ln(R^n(t)^\top X^n(t)) \middle| \sum_{t=1}^{T} \ln(R^n(\omega,t)^\top X^n(\omega,t)) \geq \ln w^*, \omega \in A^{(n)} \right\}.$$

The algorithm proceeds as follows

[1] Set $j=0$ and $A_o^{(n)} = \{\Omega^{(n)}\}$.

[2] (Solution step) For each $A^{(n)}(i_j|\cdot) \in A_j^{(n)}(\cdot)$, solve the problem $P(A^{(n)}(i_j|\cdot))$, denoting the optimal (*strategy*, *value*) by $(X(A^{(n)}(i_j|\cdot)),\ G^*(A^{(n)}(i_j|\cdot)))$, and the set of scenarios corresponding to the active constraints by $C_{j+1}(A^{(n)}(i_j|\cdot))$. If $j=k$ or $C_{j+1}(A^{(n)}(i_j|\cdot)) = \emptyset$, then designate the problem $P(A^{(n)}(i_j|\cdot))$ as k-reduced and go to step [4].

[3] (Reduction step) For each $A^{(n)}(i_j|\cdot) \in A_j^{(n)}(\cdot)$, generate a family of reduced sets, indexed by i_{j+1},

$$A_{j+1}^{(n)}(i_j,\cdots,i_1) = \left\{ A^{(n)}(i_{j+1}|\cdot) \middle| A^{(n)}(i_{j+1}|\cdot) = A^{(n)}(i_j|\cdot)/\{\omega\}, \omega \in C_{j+1}(A^{(n)}(i_j|\cdot)) \right\}.$$

Return to step [2] with j updated to $j+1$.

[4] From the set of all k-reduced problems select the set $A^{(n)}$ with the maximum optimal value $G^*(A^{(n)})$. The optimal strategy for the growth with security problem (DP_n) is $X^*(A^{(n)})$.

The reduction step in the elimination algorithm is founded on the following simple result which establishes the existence of candidate scenarios for elimination.

Theorem 3. *(Scenario Elimination) Assume there exists an "optimal" set $A^{(n)}$ for the problem (DP_n) and consider the j step reduced problem $P(A^{(n)}(i_j|\cdot))$ where j active constraints have been sequentially eliminated. That is, the corresponding scenarios are in $\Omega^{(n)}/A^{(n)}$. If $C_{j+1}^{(n)}(A^{(n)}(i_j|\cdot)) \neq \emptyset$ are the scenarios corresponding to active constraints in the solution of the reduced problem, then*

$$C_{j+1}(A^{(n)}(i_j|\cdot)) \bigcap \left[\Omega^{(n)}/A^{(n)}\right] \neq \emptyset.$$

Proof. It suffices to consider the first elimination. In the solution to the problem with all constraints, $P(\Omega^{(n)})$, there are sets of scenarios $C_1(\Omega^{(n)})$ and $\Omega^{(n)}/C_1(\Omega^{(n)})$ corresponding to active and inactive constraints, respectively, where $C_1(\Omega^{(n)}) \neq \emptyset$. Since the inactive constraints exceeds the goal $\ln w^*$

$$C_1(\Omega^{(n)}) \nsubseteq A^{(n)} \quad \text{and} \quad C_1(\Omega^{(n)}) \bigcap \left[\Omega^{(n)}/A^{(n)}\right] \neq \emptyset.$$

.

The procedure for identifying $A^{(n)}$ has a tree structure for reduced problems. If, for example, the number of active constraints in each reduced problem is $r \geq 2$, then the k eliminations require solving $\sum_{j=1}^{k} r^j$ problems. This could be unmanageable. A simple heuristic is to solve one reduced problem at each stage corresponding to eliminating the active constraint in the previous stage with the best result (highest expected log wealth). With this heuristic method at most rk problems are solved. The heuristic is equivalent to the exact method if there is at most one active constraint at each stage.

5 Application to the Fundamental Problem of Asset Allocation over Time

The computation of growth with security strategies is now illustrated with the determination of optimal fractions over time in cash, bonds and stocks. Consider the yearly asset returns corresponding to the S&P500, the Salomon Brothers Bond index and U.S. T-bills for 1980-1990 with data from Data Resources, Inc. Without loss of generality cash returns are set to one in each period and the mean returns for other assets are adjusted for this shift. The standard deviation for cash is small and is set to 0 for convenience.

Table 1: Yearly Rate of Return on Assets Relative to Cash (%)

Parameter	Stocks	Bonds	Cash
Mean: μ	108.75	103.75	100
Standard deviation: σ	12.36	5.97	0
Correlation: ρ	0.32		

A simple grid was constructed from the assumed lognormal distribution for stocks and bonds by partitioning $\Re^2$ at the centroid along the principal axes. A sample point was selected from each quadrant with the goal of approximating the parameter values. The sample points are shown in Table 2.

Table 2: Rates of Return Scenarios

Scenarios	Stocks	Bonds	Cash	Probability
1	95.00	101.50	100	0.25
2	106.50	110.00	100	0.25
3	108.50	96.50	100	0.25
4	125.00	107.00	100	0.25

The planning horizon is $T = 3$, so that there are 64 scenarios each with probability 1/64. Problems are solved with the VaR constraint and then for comparison, with the stronger drawdown constraint.

(i) VaR Control with $w^* = a$

Consider the problem

$$\max_X \left\{ E \sum_{t=1}^{3} \ln(R(t)^\top X(t)) \middle| \Pr\left[\sum_{t=1}^{3} \ln(R(t)^\top X(t)) \geq 3 \ln a\right] \geq 1 - \alpha \right\}.$$

With initial wealth $W(0) = 1$, the value at risk is a^3.

Table 3 presents the optimal investment decisions and optimal growth rate for several values of a, the secured average annual growth rate and $1 - \alpha$, the security level.

The heuristic was used to determine A, the set of scenarios for the security constraint. Since only a single constraint was active at each stage the solution is optimal. The mean return structure for stocks is quite favorable in this example, as is typical over long horizons (see Keim and Ziemba, 2000), and the Kelly strategy, not surprisingly, is to invest all capital in stock most of the time. It is only when security requirements are high that some capital is in bonds. As the requirements increase the fraction invested in the more secure bonds increases. The three-period investment decisions become more conservative as the horizon approaches. Although this example is simplified the patterns observed illustrate the effect of security constraints on decisions and growth.

(ii) Secured Annual Drawdown: b

The VaR condition only controls loss at the horizon. At intermediate times the investor could experience substantial loss, and in practice be unable to continue. The more stringent risk control constraint, referred to as drawdown , considers the loss in each period. Consider, then, the problem

$$\max_X \left\{ E \sum_{t=1}^{3} \ln(R(t)^\top X(t)) \middle| \Pr\left[\ln(R(t)^\top X(t)) \geq \ln b, t = 1, 2, 3\right] \geq 1 - \alpha \right\}.$$

Table 3: Growth with Secured Rate

Secured Growth Rate a	Secured Level $1-\alpha$	Period									Optimal Growth Rate (%)
		1			2			3			
		Stocks	Bonds	Cash	Stocks	Bonds	Cash	Stocks	Bonds	Cash	
0.95	0	1	0	0	1	0	0	1	0	0	23.7
	0.85	1	0	0	1	0	0	1	0	0	23.7
	0.9	1	0	0	1	0	0	1	0	0	23.7
	0.95	1	0	0	1	0	0	1	0	0	23.7
	0.99	1	0	0	0.492	0.508	0	0.492	0.508	0	19.6
0.97	0	1	0	0	1	0	0	1	0	0	23.7
	0.85	1	0	0	1	0	0	1	0	0	23.7
	0.9	1	0	0	1	0	0	1	0	0	23.7
	0.95	1	0	0	1	0	0	1	0	0	23.7
	0.99	1	0	0	0.333	0.667	0	0.333	0.667	0	18.2
0.99	0	1	0	0	1	0	0	1	0	0	23.7
	0.85	1	0	0	1	0	0	1	0	0	23.7
	0.9	1	0	0	1	0	0	1	0	0	23.7
	0.95	1	0	0	0.867	0.133	0	0.867	0.133	0	19.4
	0.99	0.456	0.544	0	0.27	0.73	0	0.27	0.73	0	12.7
0.995	0	1	0	0	1	0	0	1	0	0	23.7
	0.85	1	0	0	0.996	0.004	0	0.996	0.004	0	23.7
	0.9	1	0	0	0.996	0.004	0	0.996	0.004	0	23.7
	0.95	1	0	0	0.511	0.489	0	0.442	0.558	0	19.4
	0.99	0.27	0.73	0	0.219	0.59	0.191	0.218	0.59	0.192	12.7
0.999	0	1	0	0	1	0	0	1	0	0	23.7
	0.85	1	0	0	0.956	0.044	0	0.956	0.044	0	23.4
	0.9	1	0	0	0.956	0.044	0	0.956	0.044	0	23.4
	0.95	1	0	0	0.381	0.619	0	0.51	0.49	0	19.1
	0.99	0.27	0.73	0	0.008	0.02	0.972	0.008	0.02	0.972	5.27

Table 4: Growth with Secured Maximum Drawdown

Draw-down b	Secured Level $1-\alpha$	Period 1			Period 2			Period 3			Optimal Growth Rate (%)
		Stocks	Bonds	Cash	Stocks	Bonds	Cash	Stocks	Bonds	Cash	
0.96	0	1	0	0	1	0	0	1	0	0	23.7
	50	1	0	0	1	0	0	0.846	0.154	0	23.1
	75	1	0	0	0.846	0.154	0	0.846	0.154	0	22.5
	100	0.846	0.154	0	0.846	0.154	0	0.846	0.154	0	21.9
0.97	0	1	0	0	1	0	0	1	0	0	23.7
	50	1	0	0	1	0	0	0.692	0.308	0	22.5
	75	1	0	0	0.692	0.308	0	0.692	0.308	0	21.3
	100	0.692	0.308	0	0.692	0.308	0	0.692	0.308	0	20.1
0.98	0	1	0	0	1	0	0	1	0	0	23.7
	50	1	0	0	1	0	0	0.538	0.462	0	21.2
	75	1	0	0	0.538	0.462	0	0.538	0.462	0	18.6
	100	0.538	0.462	0	0.538	0.462	0	0.538	0.462	0	16.1
0.99	0	1	0	0	1	0	0	1	0	0	23.7
	50	1	0	0	1	0	0	0.385	0.615	0	21.2
	75	1	0	0	0.385	0.615	0	0.385	0.615	0	18.6
	100	0.385	0.615	0	0.385	0.615	0	0.385	0.615	0	16.1
0.999	0	1	0	0	1	0	0	1	0	0	23.7
	50	1	0	0	1	0	0	0.105	0.284	0.611	17.7
	75	1	0	0	0.105	0.284	0.611	0.105	0.284	0.611	11.8
	100	0.105	0.284	0.611	0.105	0.284	0.611	0.105	0.284	0.611	5.84

The simple form of the constraint follows from the arithmetic random walk $\ln W(t)$, where

$$\begin{aligned}\Pr[W(t+1) \geq bW(t), t = 0,1,2] &= \Pr[\ln W(t+1) - \ln W(t) \geq \ln b, t = 0,1,2] \\ &= \Pr[\ln R(t)^\top X(t) \geq \ln b, t = 1,2,3].\end{aligned}$$

In Table 4 the optimal investment decisions and growth rate for several values of b, the drawdown and $1-\alpha$, the security level are presented. The heuristic is used in determining scenarios in the solution. The security levels are different in the table since constraints are active at different probability levels in this discretized problem.

As with the VaR constraint, investment in the more secure bonds and cash increases as the drawdown rate and/or the security level increases. Also the strategy is more conservative as the horizon approaches. For similar requirements (compare $a = 0.97, 1-\alpha = 0.85$ and $b = 0.97, 1-\alpha = 0.75$), the drawdown condition is more stringent, with the Kelly strategy (all stock) optimal for

VaR constraint, but the drawdown constraint requires substantial investment in bonds in the second and third periods. In general, consideration of drawdown requires a heavier investment in secure assets and at an earlier time point. It is not a feature of this aggregate example, but both the VaR and drawdown constraints are insensitive to large losses, which occur with small probability. Control of that effect would require the lower partial mean violations condition or a model with a convex risk measure that penalizes more and more as larger constraint violations occur, see e.g. Cariño and Ziemba (1998). These results can be compared with those of Grauer and Hakansson (1998) who do calculations with the standard capital growth-Kelly model and Brennan and Schwartz (1998) who use a Merton, continuous time model with the instantaneous mean returns dependent upon fundamental factors. Each of these three models tend to lead to hair trigger type behavior, very sensitive to small changes in mean values (see Chopra and Ziemba 1993).

6 Portfolio Rebalancing

The approach to investment planning with downside risk control is to develop an optimal strategy over a T period planning horizon using projections of the multivariate returns distributions on securities. Although securities prices are dynamic, the changes are generated from a pricing model with seed parameters (μ, Γ, Δ).

It is anticipated that the planning horizon is short, and the values of the seed parameters will be reconsidered at the end of the horizon. An important feature of the proposed pricing model is the ability to revise the seed parameters using data collected during the planning period.

Consider the observations $\{Y(1), \cdots, Y(T)\}$. Since $(Y(T)|\pi) \sim N(\pi T, T\Delta)$ and $\pi \sim N(\mu, \Gamma)$, from Bayes Theorem

$$(\pi(T)|Y(T)) \sim N(\pi_T, \Gamma_T), \tag{13}$$

where

$$\begin{aligned}
\pi_T &= \mu + (I - \Delta_T \Sigma_T^{-1})(\bar{Y}_T - \mu), \\
\Gamma_T &= \frac{1}{T^2}(I - \Delta_T \Sigma_T^{-1})\Delta_T, \\
\bar{Y}_T &= \frac{1}{T} Y(T), \\
\Delta_T &= T\Delta, \quad \text{and} \\
\Sigma_T &= T\Delta + T^2\Gamma.
\end{aligned} \tag{14}$$

Furthermore, from the increments $Z_i(t) = Y_i(t+1) - Y_i(t), t = 1, \cdots, T-1$, the covariance matrix S_T can be computed. With the number of factors (mutual funds) set at $p < m$, perform a factor analysis on S_T obtaining a loading matrix L_T and the specific variance matrix $D_T = diag(D_1^2, \cdots, D_m^2)$. Let $S_T^* = L_T L_T^\top + D_T$. Then S_T^* is an estimate of Σ_T, and D_T is an estimate of

Δ_T. If there is a common mean, $\mu^\top = (\mu, \cdots, \mu)$ and $\bar{\bar{Y}}_T = \frac{1}{m}\sum_{i=1}^{m} \bar{Y}_i(T)$, then $\bar{\bar{Y}}_T$ is an estimate of μ. All parameters in (13) and (14) are now estimated and the rate of return distribution for the next T period planning cycle can be specified.

The revision of the rate of return distributions produces a rebalanced portfolio in the next planning cycle in light of new information on securities prices. The formula for π_T in (14) displays reversion to the grand mean. Considering the impact of errors in estimating mean values (Chopra and Ziemba 1993), the improved estimates lead to more reliable investment decisions.

An alternative to blending the mean estimate with a grand mean would be to blend the mean with the prior Bayes estimate. That is, for successive planning periods $\{1, \cdots, T_1, T_1 + 1, \cdots, T_1 + T_2\}$ the revised estimate for the mean rate of return is $\pi_{T_2} = \pi_{T_1} + (I - \Delta_{T_2}\Sigma_{T_2}^{-1})(\bar{Y}_{T_2} - \pi_{T_1})$. This approach provides smoothed estimates where the full history of prices is considered with the past being weighted in the manner of exponential smoothing.

7 Conclusion

This paper considers the problem of investment in risky securities with the objective of achieving maximal capital growth while controlling for downside risk. Working in discrete time, a geometric random walk model for asset prices was developed. The model has two important feature. The increments in the random walk have a Bayes framework, so that the asset prices dependent on hyper parameters. In addition the correlation in the asset price distributions was related to a structure al model and therefore the hyper parameters are identified from data.

A variety of risk measures were defined and corresponding capital growth with security problems were presented. The emphasis was on the Value at Risk, but control of period-by-period drawdown was also considered in the application.

An algorithm for the computation of growth with security strategies was presented for the problems using discrete approximations to the distributions on asset returns. The computational procedure is general and applies to any price distributions, although it was presented in the context of the geometric random walk and log normal asset prices.

The methods were applied to an example where investment capital is allocated to stocks , bonds and cash over time. At low levels of risk control the capital growth or Kelly strategy is optimal. As the risk control requirements are tightened the strategy becomes more conservative, particularly close to the planning horizon. The solved problems are discrete in time and state and in contrast to the continuous time lognormal model (MacLean, Ziemba, and Li 2000) a fractional Kelly strategy is not optimal.

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