A stochastic programming model for asset liability management of a Finnish pension company

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Abstract

This paper describes a stochastic programming model that was developed for asset liability management of a Finnish pension company. In many respects the model resembles those presented in the literature, but it has some unique features stemming from the statutory restrictions for Finnish pension companies. Particular attention is paid to modeling the stochastic factors, implementation and to numerical testing. Out-of-sample tests clearly favor the strategies suggested by our model over fixed-mix strategies.

1 Introduction

Stochastic programming has proven to be an efficient tool in designing good strategies in wealth- and asset liability management in practice. This is due to its ability to cope with the dynamics and complex constraint structures usually inherent in such problems. Stochastic programming is not tied to any particular form of objective function or model of stochastic factors. Successful applications of stochastic programming to asset liability management have been reported in Nielsen and Zenios [16], Cariño and Ziemba [3], Cariño, Myers and Ziemba [2], Høyland [11], Consigli and Dempster [5], Kouwenberg [14], and Geyer, Herold, Kontriner and Ziemba [10]. For a general introduction to stochastic programming we refer the reader to the official (COSP) stochastic programming site: www.stoprog.org.

This paper describes a stochastic programming model and its computer implementation for asset liability management of a Finnish pension company. Finnish pension companies are responsible for huge investment funds and, like most pension companies in Europe, they are facing a large number of retiring employees at around 2010–2020. Our model describes a long term dynamic investment problem where the aim is to cover the uncertain future liabilities with dynamic investment strategies in an uncertain environment. The assets are considered at the level of the larger investment classes of cash, bonds, stocks, property and loans to policyholders. In addition to investment decisions, our model looks for optimal bonus payments and it takes explicitly into account various portfolio and transaction restrictions as well as some legal restrictions coming from the complex pension system in Finland which is based on the so called defined benefit rule. The legal restrictions form a unique part of the model not present in earlier applications of stochastic programming.

We pay particular attention to describing the uncertain factors in the model which include investment returns, cash-flows, and the so called technical reserves used in the definition of the statutory restrictions. This is important since the output of a stochastic programming model depends usually heavily on the underlying model for the stochastic factors. Our approach consists of first building a time series model, which is then discretized into scenario trees appropriate for numerical solution of the optimization model. This is convenient for the user who only needs to come up with an appropriate econometric description of the stochastic factors. The discretization is fully automated and hidden from the user.

The model was implemented and tested against fixed-mix strategies that are simple (usually suboptimal) decision rules. We used the so called out-of-sample testing procedure recommended e.g. by Dardis and Mueller [6] of Tillinghast-Towers Perrin. In the tests, the strategies based on our stochastic programming model clearly outperform the fixed-mix ones. Similar results have been obtained by Fleten, Høyland and Wallace [7] in the case of a Norwegian mutual life insurance company.

The rest of the paper is organized as follows. A mathematical model of the ALM problem is presented in Section 2. A model for the underlying stochastic factors and its discretizations (scenario trees) are described in Section 3. Section 4 outlines a computer implementation of our model and reports the results of some numerical tests including an extensive out-of-sample simulation.

2 The optimization model

Our model is a multistage stochastic program where a sequence of decisions (asset allocations etc.) is interlaced with a sequence of observations of random variables (asset returns etc.). At each stage, decisions are made based on the information revealed so far, so the decision variables at a stage are functions of the random variables observed up to that stage. This is why the decision variables in a stochastic program are sometimes called decision rules or recourse functions. This kind of interdependent dynamics of information and decisions is essential in sequential decision making under uncertainty, which is what ALM and many other wealth management problems are; see for example Ziemba and Mulvey [19] or Föllmer and Schied [8].

The decision stages will be indexed by $t = 0, 1, \ldots, T$, where $t = 0$ denotes the present time, and the set of assets is indexed by $j \in J$, with

$$J = \{\text{cash, bonds, stocks, property, loans to policyholders}\}.$$

The decision variables describe the asset management strategy as well as the company's solvency situation and the bonus strategy. Uncertainties result from random future investment returns as well as from random cash flows and technical reserves described below. There are several constraints coming from the regulations of the Finnish pension system. The objective is to optimize the development of the company's solvency situation as described by the Ministry of Social Affairs and Health as well as the amount of bonuses paid to policyholders.

We will first describe the asset management model, followed by the model of statutory restrictions and finally the objective. Decision variables are random variables for all t except for $t = 0$. For parameters, randomness will be indicated explicitly.

2.1 Asset management

Asset management constitutes a central part of the model. The following formulation is more or less standard in asset management applications of stochastic programming.

Inventory constraints describe the dynamics of holdings in each asset class:

$$\begin{aligned} h_{0,j} &= h_j^0 + p_{0,j} - s_{0,j} \\ h_{t,j} &= R_{t,j} h_{t-1,j} + p_{t,j} - s_{t,j} \quad t = 1, \ldots, T-1, \quad j \in J, \end{aligned}$$

where

$$\begin{aligned} h_j^0 &= \text{initial holdings in asset } j, \\ R_{t,j} &= \text{return on asset } j \text{ over period } [t-1, t] \ (\textit{random}) \end{aligned}$$

are parameters, and

$$\begin{aligned} p_{t,j} &= \text{(nonnegative) purchases in asset } j \text{ at time } t, \\ s_{t,j} &= \text{(nonnegative) sales in asset } j \text{ at time } t, \\ h_{t,j} &= \text{holdings in asset } j \text{ in period } [t, t+1] \end{aligned}$$

are decision variables. As usual, we do not allow portfolio rebalancing at the horizon, which is why the index t goes only up to $T - 1$ in the inventory constraints. Also, the company does not have control over the loans since the amount invested in them is determined by the policyholders. Holdings in loans is stochastic and we will assume them to be proportional to the technical reserves; see Section 2.2.1 below.

Budget constraints guarantee that the total expenses do not exceed revenues:

$$\sum_{j\in J}(1+c_j^p)p_{0,j} + H_{-1} \leq \sum_{j\in J}(1-c_j^s)s_{0,j} + F_0,$$
$$\sum_{j\in J}(1+c_j^p)p_{t,j} + \tau_t H_{t-1} \leq \sum_{j\in J}(1-c_j^s)s_{t,j} + \sum_{j\in J} D_{t,j}h_{t-1,j} + F_t \quad t = 1,\ldots,T-1,$$

where

$$\begin{aligned} c_j^p &= \text{transaction costs for buying asset } j,\\ c_j^s &= \text{transaction costs for selling asset } j,\\ \tau_t &= \text{length of period } [t-1,t] \text{ in years},\\ D_{t,j} &= \text{dividend paid on asset } j \text{ over period } [t-1,t]\ (\textit{random}),\\ F_t &= \text{cash flows in period } [t-1,t]\ (\textit{random}) \end{aligned}$$

are parameters and

$$H_t = \text{transfers to the bonus reserve per year during period } [t, t+1]$$

are decision variables. The net cash flow F_t is the difference between pension contributions and expenditure during period $[t-1,t]$. The company can pay a proportion of its accumulated wealth as bonuses to its policyholders. These bonuses are paid as reductions of the pension contributions. The amount of the total bonuses is determined at the end of each year, and the sum is transferred to the so called bonus reserve. The whole bonus reserve is then paid out during the following year. For periods longer than one year, we assume that H_t is kept constant throughout the period, hence $\tau_t H_{t-1}$ gives the value of bonuses paid to policyholders during period $[t-1,t]$.

Portfolio constraints give bounds for the allowed range of portfolio weights:

$$l_j w_t \leq h_{t,j} \leq u_j w_t \quad t = 0,\ldots,T-1, \quad j \in J,$$

where

$$w_t = \sum_{j\in J} h_{t,j} = \text{total wealth at time } t = 0,\ldots,T-1,$$

and

$$\begin{aligned} l_j &= \text{lower bound for the proportion of } w_t \text{ in asset } j,\\ u_j &= \text{upper bound for the proportion of } w_t \text{ in asset } j \end{aligned}$$

are parameters whose values are given in Table 1.

Table 1: Lower and upper bounds for investment proportions

j	l_j	u_j
Cash	0.01	1

Bonds	0	1
Stocks	0	0.5
Property	0	0.4

The upper bounds for stocks and property are statutory restrictions. The lower bound for cash investments is set to guarantee sufficient liquidity.

Note that the total wealth w_t at stage $t = 0, \ldots, T-1$ is computed after portfolio rebalancing. At the horizon, there is no rebalancing so we define it as

$$w_T = \sum_{j \in J} (R_{T,j} + D_{T,j}) h_{T-1,j} + F_T - \tau_T H_{T-1}.$$

Transaction constraints bound the sales and purchases to a given fraction of w_t:

$$\begin{aligned} p_{t,j} &\leq \tau_t b_j^p w_t \quad t = 0, \ldots, T-1, \quad j \in J, \\ s_{t,j} &\leq \tau_t b_j^s w_t \quad t = 0, \ldots, T-1, \quad j \in J, \end{aligned}$$

where

b_j^p = upper bound for purchases of asset j per year as a fraction of total wealth,

b_j^s = upper bound for sales of asset j per year as a fraction of total wealth

are parameters. The values of b_j^p and b_j^s are displayed in Table 2. The tight re balancing restrictions for property are set because of illiquidity of the Finnish property markets. For other asset classes the yearly rebalancing is restricted to be at most 20% of the total wealth. These restrictions model the policies of the company as well as the requirement that the size of transactions should be kept at levels that do not affect market prices.

Table 2: Upper bounds for transactions

j	b_j^p	b_j^s
Cash	0.2	0.2
Bonds	0.2	0.2
Stocks	0.2	0.2
Property	0.01	0.01

2.2 Statutory restrictions

The statutory restrictions for Finnish pension companies are quite strict, and they form a unique part of our stochastic programming model. Besides imposing constraints on the decision variables, these rules form the basis for defining the objective function in our model.

2.2.1 Solvency capital

The Finnish pension companies are obliged to comply with several restrictions described in the legislation, government decrees or regulations given by the Ministry of Social Affairs and Health. A fundamental restriction is that, the assets of a company must always cover its *technical reserves* L_t, which gives the present value of future pension expenditure discounted with the so called "technical interest rate". The assets include, besides the total amount of investments w_t, the transitory item of the net amount of other debts and credits in the balance sheet. This relatively small amount is calculated approximately as a fixed proportion c^G of the technical reserves. The difference

$$C_t = w_t + c^G L_t - L_t = w_t - (1 - c^G) L_t$$

of assets and the technical reserves is called the *solvency capital*. If at any time, C_t becomes negative, the company is declared bankrupt.

2.2.2 Solvency limits

Besides bankruptcy, $C_t = 0$, there are several target levels that have been set to characterize pension companies solvency situation. These levels form an early warning system, so that the company and the supervising authorities can take action before a bankruptcy actually happens. A fundamental concept in the system is the *solvency border* $\tilde{B}_t$, defined in (1) below. If the solvency capital C_t falls below this limit, the financial position is considered to be at risk, and the company is required to present to the authorities a plan for recovering a safe position. In addition, the company is not allowed to give any bonuses to its policyholders.

The target zone for the ratio $C_t/\tilde{B}_t$ is $[2, 4]$. In this zone, the financial position of a company is considered to be quite good. There is still discussion about how strictly the upper limit should be observed (in practice, no company has yet exceeded the upper limit). Therefore, we will ignore the upper limit in the model.

The concept of the solvency border corresponds to the solvency requirements in the European Union (EU) insurance directives. There is, however, an essential difference in the calculation method. The Finnish solvency border is based on the investment portfolio of a company. The fluctuation of the solvency capital is mainly caused by the investment market, and therefore the risk of going bankrupt is strongly dependent of the company's investment risk. The starting point of the Finnish system is that the probability of ruin in one year at the solvency border should be approximately 2.5%, and therefore the value of the border is required to be dependent on the investment portfolio. In contrast, the EU directives take no account of the company's investments. It is widely regarded that the EU regulations are insufficient, and in fact a project is now established to renew the EU solvency requirements. The solvency border $\tilde{B}_t$ is given by

$$\tilde{B}_t = \frac{0.9}{100} \left(-1.08 \sum_{j \in J} m_j h_{t,j} + 1.98 \sqrt{\sum_{j,k \in J} \sigma_{j,k} h_{t,j} h_{t,k}} \right) \frac{(L_t + H_t)}{w_t}, \quad (1)$$

where the parameters

$$m = \begin{bmatrix} 0.18 \\ 0.66 \\ 6.20 \\ 3.70 \\ 0.72 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0.93 & 0.01 & 3.08 & 1.05 & -0.02 \\ 0.01 & 11.47 & 12.80 & -3.62 & 11.19 \\ 3.08 & 12.80 & 460.51 & 91.50 & 9.67 \\ 1.05 & -3.62 & 91.50 & 176.55 & -1.31 \\ -0.02 & 11.19 & 9.67 & -1.31 & 11.18 \end{bmatrix}$$

give the mean one-year return over the technical interest rate on asset j according to the government decree and $\sigma_{j,k}$ is the covariance between one-year excess returns of assets j and k according to the government decree. For asset classes like stocks, the parameter $\sigma_{j,j}$ is substantially larger that for safer classes like bonds. Note that $\tilde{B}_t$ is a nonconvex function of the variables in the model.

2.2.3 Upper bound for bonuses

Finnish pension companies compete with each other by paying out bonuses to their policyholders. Companies would like to maximize the amount of bonuses to attract new customers, but because the pension system is statutory, the government has aimed to restrict the amount of bonuses so that a sufficient proportion of the assets is preserved in the system to guarantee future pensions. Therefore, the Ministry of Social Affairs and Health has confirmed a formula for the maximum amount of each year's bonus transfers. The maximum depends on the solvency capital C_t and the solvency border $\tilde{B}_t$ of the company according to the formula

$$\tilde{H}_t^{\max} = \beta(C_t/\tilde{B}_t)\,(C_t - \tilde{B}_t)$$

where $\beta(z)$ is a piecewise linear function which has the minimum value of 0 when $z \leq 1$ and the maximum value of 0.04 when $z \geq 4$. It follows that $H_t^{\max}$ is also a nonconvex function of the variables in the model.

2.2.4 Convex approximations

In the optimization model, the nonconvex solvency border is replaced by

$$B_t = \frac{0.9}{100}\left(-1.08\sum_{j\in J} m_j h_{t,j} + 1.98\sqrt{\sum_{j,k\in J}\sigma_{j,k}h_{t,j}h_{t,k}}\right),$$

which is convex in the variables. We have $B_t \geq \tilde{B}_t$ since $(L_t + H_t)/w_t \leq 1$ unless the company is bankrupt. Replacing $\tilde{B}_t$ by B_t in the model, makes the constraints in the model more restrictive, so we will always stay on the safe side.

We will also replace the nonconvex function $\tilde{H}_t^{\max}$ by a convex approximation, namely,

$$H_t^{\max} = 0.03\max\{C_t - B_t, 0\}.$$

This is based on the fact that the historical average of $\beta(z)$ has been close to 0.03.

2.3 Objective function

There are many possibilities for measuring the performance of a company by an objective function. Natural candidates would be expected utility from wealth or solvency capital under various utility functions. Here, we will describe a utility function that takes explicitly into account the unique features of the Finnish pension system.

As described in Section 2.2.2, the Ministry of Social Affairs and Health measures pension companies' solvency situation by the ratio $C_t/\tilde{B}_t$ of the solvency capital and the solvency border. The Ministry defines four zones according to which companies' solvency situation is classified:

$$\begin{aligned}
C_t/\tilde{B}_t &\in [2,\infty) : \text{target} \\
C_t/\tilde{B}_t &\in [1,2) : \text{below target} \\
C_t/\tilde{B}_t &\in [0,1) : \text{crisis} \\
C_t/\tilde{B}_t &\in (-\infty,0) : \text{bankrupt.}
\end{aligned}$$

We replace $\tilde{B}_t$ throughout by its convex approximation B_t given above, and we define three shortfall variables:

$$\begin{aligned}
SF_{t,1} &\geq 2B_t - C_t & t &= 1,\dots,T-1, \\
SF_{t,2} &\geq B_t - C_t + H_t/0.03 & t &= 1,\dots,T-1, \\
SF_{t,3} &\geq -C_t & t &= 1,\dots,T,
\end{aligned}$$

each of which gives the amount by which a zone is missed. These will be penalized in the objective function. The defining inequality for $SF_{t,2}$ incorporates the constraint

$$H_t \leq H_t^{\max}$$

for bonus transfers. The penalty for $SF_{t,2}$ will be chosen large enough to guarantee that, at the optimum, the upper bound is satisfied.

For $t = 0,\dots,T-1$, the state of the company will be evaluated by the following utility function

$$u(C_t, B_t, H_t, L_t) = C_t/L_t - \sum_{z=1}^{3} \beta_z SF_{t,z}/L_t + u^b(H_t/L_t),$$

where β_z are positive parameters and u^b is a nondecreasing concave function that will be specified according to the preferences of the company. At stage T, the utility is measured by

$$u_T(C_T, L_T) = C_T/L_T - \beta_3 SF_{T,3}/L_T.$$

The overall objective function in our model is the discounted expected utility

$$E^P\left\{\sum_{t=1}^{T-1} d_t u(C_t, B_t, H_t, L_t) + d_T u_T(C_T, L_T)\right\},$$

where d_t is the discount factor for stage t. The problem is to maximize this expression over all the decision variables and subject to all the constraints described above.

2.4 Problem summary

Deterministic parameters:

$$
\begin{aligned}
h_j^0 &= \text{initial holdings in asset } j, \\
c_j^p &= \text{transaction costs for buying asset } j, \\
c_j^s &= \text{transaction costs for selling asset } j \\
l_j &= \text{lower bound for wealth in asset } j \text{ as a fraction of total wealth}, \\
u_j &= \text{upper bound for wealth in asset } j \text{ as a fraction of total wealth}, \\
b_j^p &= \text{upper bound for purchases of asset } j \text{ per year as a fraction of total wealth}, \\
b_j^s &= \text{upper bound for sales of asset } j \text{ per year as a fraction of total wealth}, \\
c^G &= \text{the amount of the transitory item as a fraction of the technical reserves}, \\
m_j &= \text{mean yearly return on asset } j \text{ according to the government decree}, \\
\sigma_{j,k} &= \text{ covariance of one-year returns according to the government decree}, \\
\tau_t &= \text{length of period } [t-1,t] \text{ in years}, \\
\beta_z &= \text{parameters in the objective function},
\end{aligned}
$$

Stochastic parameters:

$$
\begin{aligned}
R_{t,j} &= \text{return on asset } j \text{ over period } [t-1,t], \\
D_{t,j} &= \text{dividend paid on asset } j \text{ over period } [t-1,t], \\
F_t &= \text{cash flows from period } [t-1,t], \\
L_t &= \text{technical reserves at time } t,
\end{aligned}
$$

Decision variables:

$$
\begin{aligned}
h_{t,j} &= \text{holdings in asset } j \text{ from period } t \text{ to } t+1, \\
p_{t,j} &= \text{purchases in asset } j \text{ at time } t, \\
s_{t,j} &= \text{sales in asset } j \text{ at time } t, \\
w_t &= \text{total wealth at time } t, \\
H_t &= \text{transfers to bonus reserve at time } t, \\
C_t &= \text{solvency capital at time } t, \\
B_t &= \text{solvency border at time } t, \\
SF_{t,z} &= \text{shortfall from zone } z \text{ at time } t.
\end{aligned}
$$

Our stochastic programming model is

$$
\begin{aligned}
\text{maximize} \quad & E^P\left\{\sum_{t=1}^{T-1} d_t u(C_t, B_t, H_t, L_t) + d_T u_T(C_T, L_T)\right\} \\
h_{0,j} &= h_j^0 + p_{0,j} - s_{0,j}, \\
h_{t,j} &= R_{t,j} h_{t-1,j} + p_{t,j} - s_{t,j}, \\
p_{t,j}, s_{t,j} &\geq 0, \\
\sum_{j \in J}(1 + c_j^p) p_{0,j} + H_{-1} &\leq \sum_{j \in J}(1 - c_j^s) s_{0,j} + F_0, \\
\sum_{j \in J}(1 + c_j^p) p_{t,j} + \tau_t H_{t-1} &\leq \sum_{j \in J}(1 - c_j^s) s_{t,j} + \sum_{j \in J} D_{t,j} h_{t-1,j} + F_t, \\
w_t &= \sum_{j \in J} h_{t,j}, \\
l_j w_t &\leq h_{t,j} \leq u_j w_t, \\
p_{t,j} &\leq \tau_t b_j^p w_t, \\
s_{t,j} &\leq \tau_t b_j^s w_t, \\
C_t &= w_t - (1 - c^G) L_t, \\
B_t &\geq \frac{0.9}{100}\left(-1.08 \sum_{j \in J} m_j h_{t,j} + 1.98 \sqrt{\sum_{j,k \in J} \sigma_{j,k} h_{t,j} h_{t,k}}\right), \\
SF_{t,1} &\geq 2B_t - C_t, \\
SF_{t,2} &\geq B_t - C_t + 100/3 H_t, \\
SF_{t,3} &\geq -C_t, \\
\text{for all} \quad & t = 1, \ldots, T-1, \quad j \in J, \\
w_T &= \sum_{j \in J}(R_{T,j} + D_{T,j}) h_{T-1,j} + F_T - \tau_T H_{T-1}. \\
C_T &= w_T - (1 - c^G) L_T, \\
SF_{T,3} &\geq -C_T, \\
(h, b, s, w, H, C, B, SF) &\in \mathcal{N}
\end{aligned}
$$

where P is the probability distribution of the random parameters, E^P denotes the expectation operator, and the constraints are required to hold almost surely with respect to P. The symbol $\mathcal{N}$ stands for the subspace on nonanticipative decision rules, so the decision variables (recourse functions) of a given stage are not allowed to depend on random variables whose values are observed only in later stages. Note that our model is a convex program that is nonlinear both in the objective and the constraints. There are 19 decision variables in each stage $t = 0, \ldots, T-1$ (recall that for loans to policyholders, $h_{t,j}$, $p_{t,j}$ and $s_{t,j}$ are determined by L_t) and 3 in the last stage.

3 Scenario tree generation

The probability distribution P of the random parameters is an important input to the model, and the solution will depend on it in an essential way. We assume that the random parameters follow the stochastic model developed in Koivu, Pennanen and Ranne [13]. Numerical solution of the optimization problem is then done through discretization of the continuous distribution as in Pennanen and Koivu [17]. This results in a description of the stochastic elements in the form of a scenario tree. The stochastic model for assets and liabilities is briefly described in Section 3.1 and Section 3.2 outlines the discretization methods that produce the scenario trees.

3.1 Modeling the stochastic factors

The formulas for calculating R_t and D_t for each asset class are displayed in Table 3, where sr, br, S, Div, P and $Rent$ denote the short term interest rate, long term bond yield, stock price index, dividend index, property price index and rental index, respectively, and τ_t denotes the length of the time period in years. The parameter D_M denotes the average duration of the company's bond portfolio.

Table 3: Return and dividend formulas

Asset class	R_t	D_t
Cash	$((1+sr_t)(1+sr_{t-1}))^{\frac{\tau_t}{2}}$	1
Bonds	$\left(\frac{1+br_{t-1}}{1+br_t}\right)^{D_M}$	$\frac{1}{2}(br_{t-1}+br_t)\tau_t$
Stocks	$\frac{S_t}{S_{t-1}}$	$\frac{1}{2}(\frac{Div_{t-1}}{S_{t-1}}+\frac{Div_t}{S_t})\tau_t$
Property	$\frac{P_t}{P_{t-1}}$	$\left(\frac{1}{2}(\frac{Rent_{t-1}}{P_{t-1}}+\frac{Rent_t}{P_t})-0.03\right)\tau_t$
Loans	1	$\frac{1}{2}(br_{t-1}+br_t)\tau_t$

The return for cash investments is approximated by the geometric average of the short term interest rate during the holding period. The formula for bond returns is based on a duration approximation as in [13]; see also Campbell, Lo and MacKinley [1, Chapter 10]. The parameter D_M is set equal to five years. The dividends for stock and property investments present the average dividend and rental yield, respectively, during the holding period. For property investments the maintenance costs, which are assumed to be a constant 3% of the property value, are deducted from the rental yield. Similarly to bonds, the cash income for loans is approximated by an average of bond yield. This is based on the fact that the interest on newly given loans is usually set equal to the current bond yield. The return for loans is equal to one because these instruments are not traded in the market.

The Finnish earnings-related pension scheme follows the defined benefit principle, where the pension company guarantees the pension payments which are tied to the development of the policyholder's salaries. It follows that, the technical reserves L and cash flows F depend on policyholder's wages and population dynamics. These are assumed independent, so that their development can be modeled separately. The values of L and F depend also on the technical interest rate, which determines the total growth rate for the reserves. In the model, the technical interest rate is calculated based on recent asset returns and it is an important part of the model because, to a great extent, it determines the correlations between the investment variables and the reserves. The development of wages is described by the general Finnish wage index W. For a more detailed description of the development of L and F, see [13].

The stochastic variables in the model can thus be approximately expressed in terms of the seven economic factors, sr, br, S, Div, P, $Rent$ and W. The quarterly development of

$$x_t = \begin{bmatrix} \ln sr_t \\ \ln br_t \\ \ln S_t \\ \ln Div_t \\ \ln P_t \\ \ln Rent_t \\ \ln W_t \end{bmatrix}$$

will be described with a Vector Equilibrium Correction (VEqC) model of the form

$$\Delta_\delta x_t = \sum_{i=1}^{k} A_i \Delta_\delta x_{t-i} + \alpha(\beta' x_{t-1} - \mu) + \epsilon_t, \tag{3}$$

where $A_i \in \mathbb{R}^{7\times 7}$, $\beta \in \mathbb{R}^{7\times l}$, $\mu \in \mathbb{R}^l$, $\alpha \in \mathbb{R}^{7\times l}$, Δ_δ denotes the shifted difference operator

$$\Delta_\delta x_t := \Delta x_t - \delta$$

with $\delta \in \mathbb{R}^7$, and ϵ_t are independent normally distributed random variables with zero mean and variance matrix $\Sigma \in \mathbb{R}^{7\times 7}$. When the model is stationary the parameter vector δ determines the average drift for the time series. The term $\alpha(\beta' x_{t-1} - \mu)$ takes into account the long-term behavior of x_t around statistical equilibria described by the linear equations $\beta' x = \mu$. It is assumed that, in the long run,

$$E[\beta' x_t] = \mu,$$

and that if x_t deviates from the equilibria it will tend to move back to them. The matrix α determines the speed of adjustment toward the equilibria. In a sense, VEqC-models incorporate long-run equilibrium relationships (often derived from economic theory) with short-run dynamic characteristics deduced from historical data.

We take δ and μ as user specified parameters. This enables the incorporation of expert information in specifying the expected growth rates for x_t as well as long term equilibrium values for such quantities as mean reversion levels, interest rate spread and dividend yield. In particular, this gives control over mean returns which have been shown to have a big impact on the optimal portfolio choice, see Chopra and Ziemba [4]. The appropriate lag-length k is determined and the remaining parameters are estimated from quarterly data from Finland and EU-area. For a more detailed description of the model; see [13].

3.2 Discretization

In our optimization model, we are interested in the conditional distributions of x_{t+h}, given x_t, typically for $h \geq 4$. This can be calculated conveniently as follows. After specifying the model (3), we write it as a Vector Auto-Regressive (VAR) model in levels

$$x_t = (I + A_1 + \Gamma)x_{t-1} + \sum_{i=2}^{k}(A_i - A_{i-1})x_{t-i} - A_k x_{t-k-1} + c + \epsilon_t,$$

where $\Gamma = \alpha\beta'$ and $c = -\alpha\mu + (I - \sum_{i=1}^{k} A_i)\delta$. This, in turn, can be written in the companion form

$$\bar{x}_t = \bar{A}\bar{x}_{t-1} + \bar{c} + \bar{\epsilon}_t,$$

where

$$\bar{x}_t = \begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-k} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} I + A_1 + \Gamma & A_2 - A_1 & \cdots & A_k - A_{k-1} & -A_k \\ I & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

$$\bar{c} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \bar{\epsilon}_t = \begin{bmatrix} \epsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It follows that

$$\bar{x}_{t+h} = \bar{A}^h \bar{x}_t + \sum_{i=1}^{h} \bar{A}^{h-i}\bar{c} + e_h, \tag{4}$$

where $e_h = \sum_{i=1}^{h} \bar{A}^{h-i}\bar{\epsilon}_i$. The random term e_h is normally distributed with zero mean, and from the independence of $\bar{\epsilon}_i$ it follows that e_h has the variance matrix

$$\bar{\Sigma}_h = \sum_{i=1}^{h} \bar{A}^{h-i} \begin{bmatrix} \Sigma & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} (\bar{A}^T)^{h-i}.$$

A convenient feature of (4) is that the dimension of the random term never exceed $7(k+1)$ even if h is increased. In the model of [13], $k = 1$, so the dimension will be at most 14.

We discretize the model (4) using integration quadratures as described in [17]. This results in scenario trees that converge weakly to the original process as the number of branches is increased. This technique is just as easy to implement as the better known method of conditional sampling. Indeed, a scenario tree with a given period structure $(\tau_1, \dots, \tau_T)$ and branching structure $(\nu_1, \dots, \nu_T)$ can be generated as follows. For each $t = 0, \dots, T$, denote by $\mathcal{N}_t$ the set of nodes in the scenario tree at stage t. The set $\mathcal{N}_0$ consists only of the root node which is labeled by 0. The rest of the nodes will be labeled by positive integers in the order they are generated. The number $h_t = 4\tau_t$ gives the length of period $[t-1, t]$ in quarters.

Set $m := 0$, $\bar{x}_m :=$ the current state of the world, and $\mathcal{N}_0 := \{m\}$.
for $t := 1$ **to** T
  $\mathcal{N}_t := \emptyset$
  for $n \in \mathcal{N}_{t-1}$
    Draw a random sample of ν_t points $\{e^i_{h_t}\}_{i=1}^{\nu_t}$ from $N(0, \bar{\Sigma}_{h_t})$
    for $i := 1$ **to** ν_t
      $m := m + 1$
      $\bar{x}_m = \sum_{i=1}^h \bar{A}^{h-i}\bar{c} + \bar{A}^h \bar{x}_n + e^i_{h_t}$
      $\mathcal{N}_t := \mathcal{N}_t \cup \{m\}$
    end
  end
end

The random samples required above are easily generated by computing the spectral decomposition

$$\bar{\Sigma}_h = \sum_{i=1}^{7(k+1)} \lambda^i_h u^i_h (u^i_h)^T,$$

where λ^i_h are the eigenvalues of $\bar{\Sigma}_{h_t}$ in decreasing order and u^i_h are the corresponding eigenvectors. If $\bar{\Sigma}_{h_t}$ has rank d_t, we have

$$\bar{\Sigma}_h = C_h C_h^T,$$

where $C_h = [\sqrt{\lambda^1_h} u^1_h, \ldots, \sqrt{\lambda^{d_t}_h} u^{d_t}_h]$, and then the desired sample is obtained as

$$e^i_{h_t} := C_h F_{d_t}^{-1}(u^i_{h_t}),$$

where $\{u^i_{h_t}\}_{i=1}^{\nu_t}$ is a random sample from U_{d_t}, the d_t-dimensional uniform distribution on $[0,1]^{d_t}$ and F_{d_t} is the distribution function of the d_t-dimensional standard normal distribution. An advantage of computing the spectral decomposition (instead of the Cholesky decomposition as e.g. in Høyland, Kaut and Wallace [12]) is that when $\bar{\Sigma}_{h_t}$ is singular, d_t gives the true dimension of the random term. For example, when $h = 1$, $d_t = 7$.

The random samples $\{u^i_{h_t}\}_{i=1}^{\nu_t}$ above can be viewed as discrete approximations of U_{d_t}. As in [17], we will replace these random samples by point-sets given by modern integration quadratures that have been designed to give good approximations of U_{d_t}. In the numerical tests in the next section we will use point-sets from the Sobol sequence; see for example Press, Teukolsky, Vetterling and Flannery [18]. This produces a scenario tree with the same branching structure as the above conditional sampling procedure but potentially better approximation of the original stochastic process. See Pennanen and Koivu [17] for a numerical study of such scenario trees.

4 Numerical results

4.1 Implementation

Figure 1 sketches the structure of the overall optimization system. The scenario generator (written in C programming language) takes as inputs the period and branching structures of the scenario tree and the time series model for the stochastic factors and generates the scenario tree for the assets and liabilities. The tree can be visually and otherwise inspected e.g. in spreadsheet programs until the outcomes are satisfactory. The scenario tree is then written into a text file in AMPL format described in Fourer, Gay and Kernighan [9]. The optimization model written in AMPL modeling language and the data from the scenario generator are processed in AMPL and fed to MOSEK [15] which is an interior-point algorithm for convex (nonlinear) programming. The solution details and statistics produced by AMPL/MOSEK can again be visualized e.g. in spreadsheet programs. The system can be used under most Unix and Windows platforms.

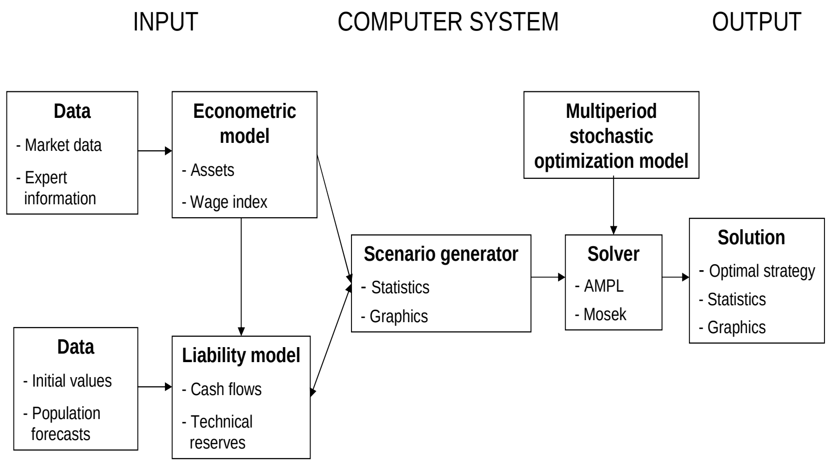


Figure 1: Stochastic optimization system

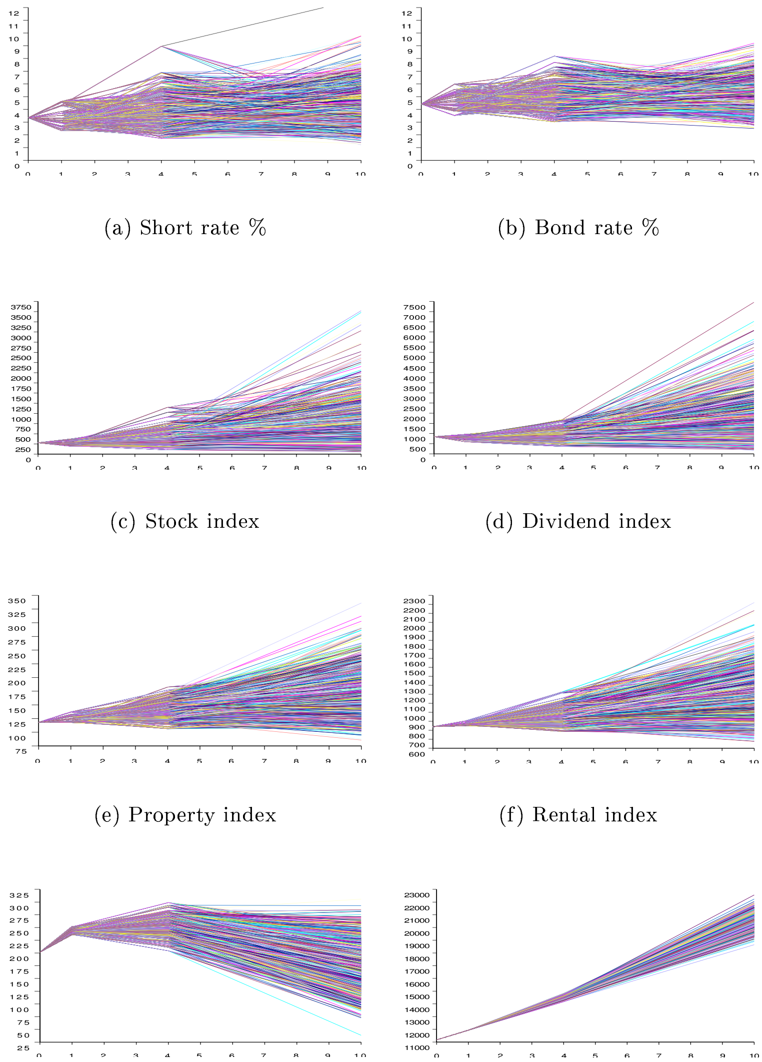
As an example, we generate a scenario tree with period structure $(1, 3, 6)$ years and branching structure $(25, 10, 10)$ (2500 scenarios). This takes less than a second on Intel Pentium 4, 2.33GHz, with 1Gb of SDRam. Figure 2 plots the values of some important parameters on the scenario tree. We solved the corresponding stochastic programming model for five sets of shortfall penalty coefficients presented in Table 4.

Table 4: Shortfall penalty coefficients in the example

	β_1	β_2	β_3
SP 1	1	10	10
SP 2	0.5	10	10
SP 3	1	1	1
SP 4	0.1	10	10
SP 5	0	0	0

In all cases we used the piecewise linear utility function

$$u^b(\cdot) = 1.5\beta_2 \min\{\cdot, 0.01\}$$



(a) Short rate % (b) Bond rate %

(c) Stock index (d) Dividend index

(e) Property index (f) Rental index

(g) Yearly cash flows (Meuro) (h) Technical reserves (Meuro)

Figure 2: Scenario tree of the example.

for bonuses. The solution of the corresponding optimization models takes less than 10 seconds each. Figure 3 displays the optimal portfolio weights in stage $t = 0$. One can also examine the development of the optimized decision variables along the scenario tree. Figures 4(a) and 4(b) plot the optimized C_t/L_t and H_t/L_t ratios, respectively, for SP1 of Table 4. The solvency capital C_t is always nonnegative (no bankruptcy) in every scenario while the bonustransfer/liability ratio H_t/L_t is equal to 0.01 in almost every scenario.

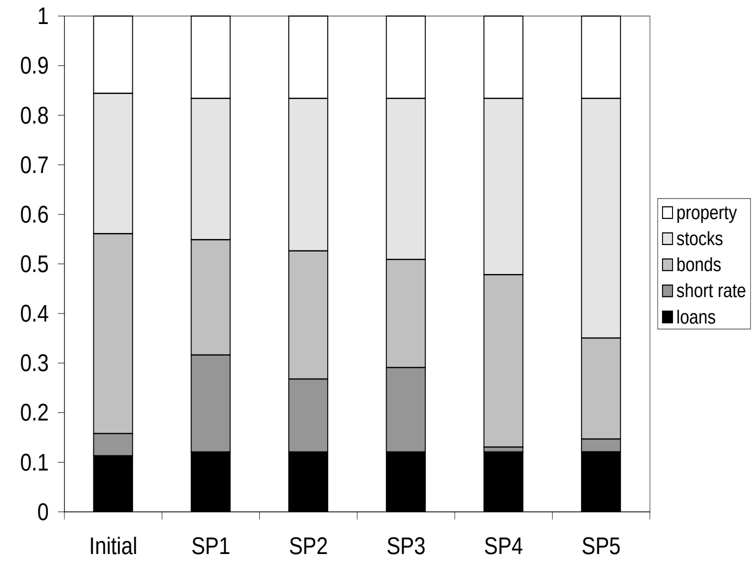
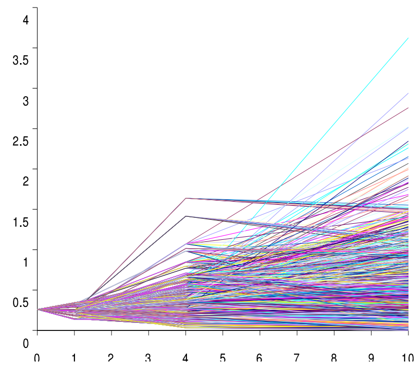
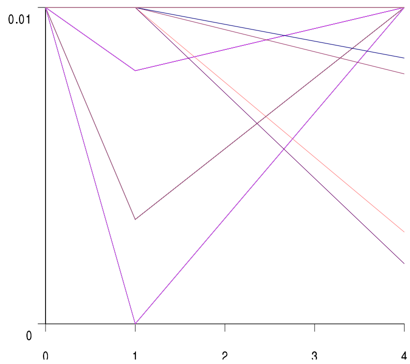


Figure 3: Initial portfolio h^0 and the optimal portfolios corresponding to the parameter values in Table 4.



(a) C_t/L_t.



(b) H_t/L_t.

Figure 4: Optimized solvency capital and bonus ratios along the scenario tree for SP1.

4.2 Convergence of discretizations

Being forced to approximate the continuous distribution of the uncertain parameters by finite distributions, it is natural to ask how the corresponding optimization problems depend on the number of scenarios. A simple test is to study the behavior of the optimal values as the number of scenarios is increased. We will do the test for SP1 of Table 4 using the Sobol sequence as described in Section 3.2. For simplicity, we only considered fully symmetric scenario trees where each node has an equal number of branches, i.e. branching structure is (k, k, k) for $k = 1, 2, 3 \ldots$. The solid line in Figure 5 plots the objective value as a function of the size of the scenario tree. For low values of k, the optimal value

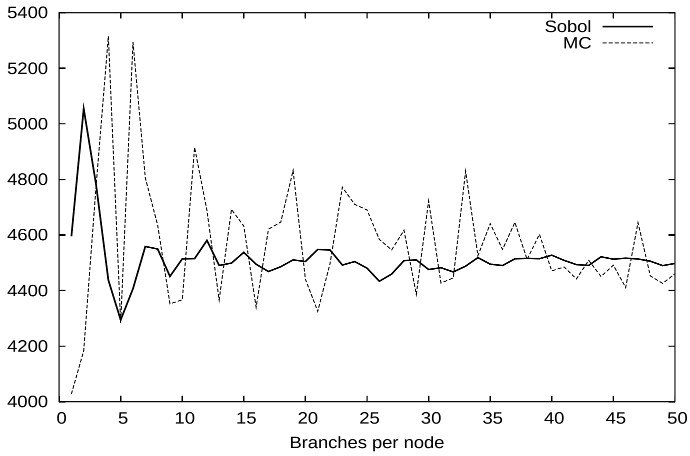


Figure 5: Convergence of the optimal value

goes through large variations, but as k is increased the optimal value seems to stabilize. In fact, it stabilizes close to 4504 which is what we obtained with the branching structure $(25, 10, 10)$ in the above example.

For comparison, we did the same test using Monte Carlo sampling in generating the scenario trees. This resulted in the dotted line in Figure 5. The optimal values obtained with Monte Carlo seem to converge too but not nearly as fast as the optimal values obtained with the Sobol sequence.

4.3 Out-of-sample test

We implemented an out-of-sample testing procedure to evaluate the performance of our stochastic programming model. Optimized strategies corresponding to the five sets of shortfall penalty coefficients in Table 4 were compared to 1328 fixed-mix strategies meeting the statutory restrictions of Table 1. To simplify the comparison of different strategies, bonus transfers were set to zero in each model. In addition, transaction costs were ignored in the fixed-mix models to simplify computations. The scenario trees used in optimization had the same structure as in the example of Section 4.1, that is, period structure $(1, 3, 6)$ years and branching structure $(25, 10, 10)$.

In the test, we evaluated the performance of each strategy over 325 randomly simulated scenarios of the stochastic parameters over 20 years. Portfolio rebalancing was made every year, i.e. fixed-mix portfolios are rebalanced to fixed proportions and stochastic programming problems were solved with a new scenario tree based on the current values of the stochastic parameters along the simulated scenario. The following describes the testing procedure. As outlined in Section 3, the stochastic factors in each year can be expressed in terms of a 14-dimensional vector. Below, $\bar{x}_{s,y}$ denotes the value of this vector in year y along a randomly generated scenario s.

for $s := 1$ **to** 325
    Set $\bar{x}_{s,0} = \bar{x}_0$ (the current state of the world).
    for $y := 0$ **to** 19
        Generate a scenario tree rooted at $\bar{x}_{s,y}$.
        Solve the corresponding optimization problems and rebalance all the portfolios.
        Randomly sample $\bar{x}_{s,y+1}$ from the time series model and calculate the resulting portfolios and cash-flows at time $y + 1$.
    end
end

Figure 6 plots the performance of all the 1328 fixed-mix strategies and the 5 stochastic programming strategies with respect to the average solvency capital at the end of the simulation period versus the bankruptcy probability during the period. Considering the main risk of the company, bankruptcy, and average

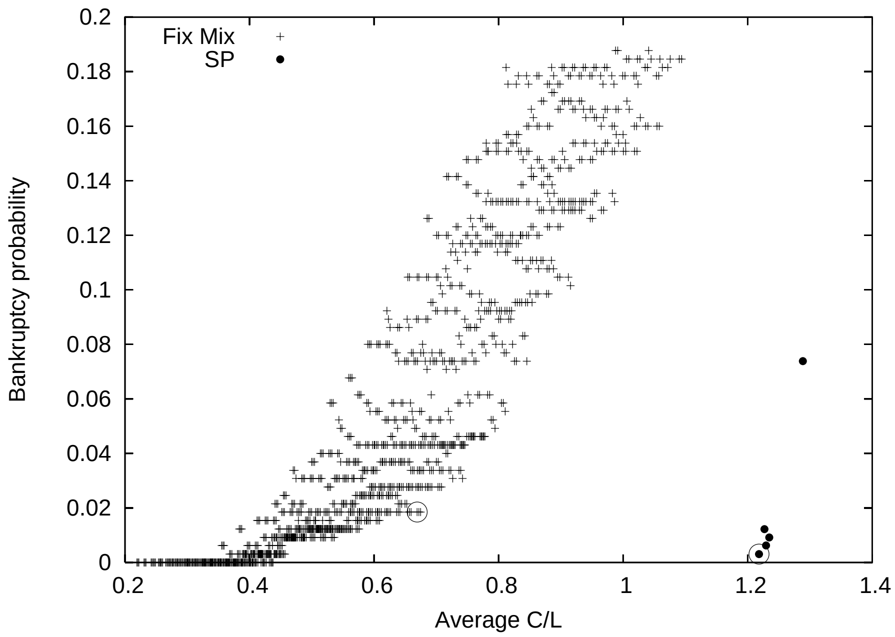
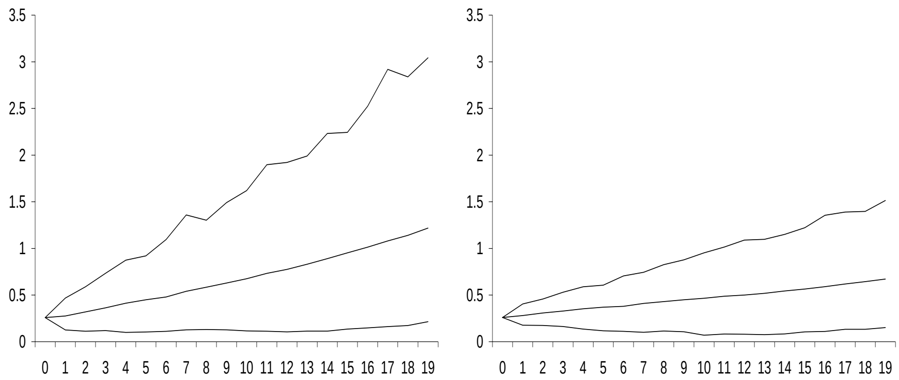


Figure 6: C_T/L_T vs. bankruptcy probability.

solvency capital, the stochastic programming strategies clearly dominate the fixed-mix strategies, even though the probability of bankruptcy was not explicitly minimized. The riskiest stochastic programming strategy, SP5 of Table 4, went bankrupt in 25 simulations out of the 325 and the safest, SP1, in only one. We will compare SP1 more closely with the fixed-mix strategy circled in Figure 6. The development of the solvency capital-reserves ratio for both strategies is described in Figure 7. The three lines represent the development of the sample average and the 95% confidence interval computed from the 325 scenarios. A higher mean and upwards skewed distribution indicates that the stochastic programming model can hedge against risks without losing profitability.



(a) Stochastic programming.

(b) Fixed-mix.

Figure 7: C_t/L_t averages ja 95% confidence intervals.

Figure 8 shows the distribution of the solvency capital-solvency border ratio C_t/B_t at the beginning of the second year. Due to the aim for high investment returns, the stochastic programming strategy avoids unnecessarily high levels of C_t/B_t, and consequently, it hits the lower border of the target zones frequently.

Figure 9 displays the development of the distribution of C_t/B_t in the 325 scenarios over the four zones defined in Subsection 2.3. The stochastic programming strategy achieves target levels better in the long run by choosing a slightly riskier portfolios at the beginning.

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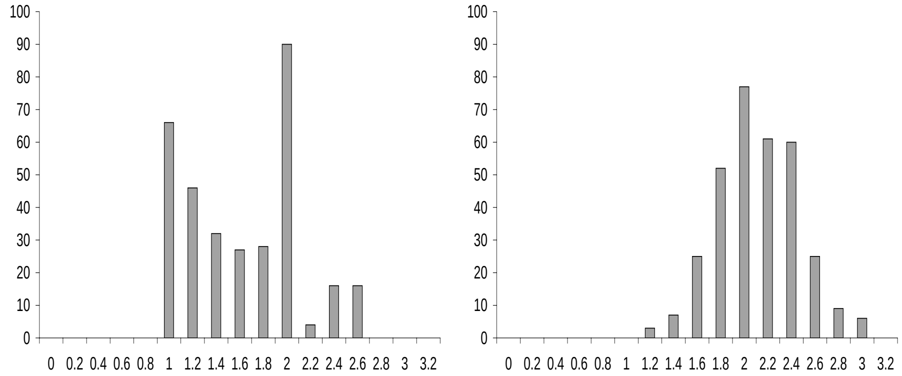
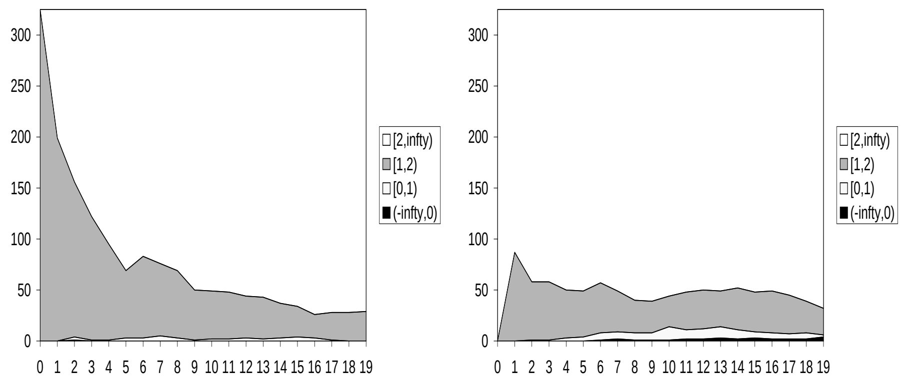


(a) Stochastic programming (b) Fixed-mix

Figure 8: Distribution of C_2/D_2 at the beginning of the second period



(a) Stochastic programming. (b) Fixed-mix

Figure 9: Development of the distribution of C_t/D_t over the different zones

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