An ALM Model for Pension Funds using Integrated Chance Constraints

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Abstract

We discuss integrated chance constraints in their role of short-term risk constraints in a strategic ALM model for Dutch pension funds. The problem is set up as a multi-stage recourse model, with special attention for modeling the guidelines proposed by the regulating authority for Dutch pension funds. The paper concludes with a numerical illustration of the importance of such short-term risk constraints.

1 Introduction

The strategic Asset Liability Management (ALM) problem for pension funds is a *dynamic* decision problem under *uncertainty*. Management of assets involves decisions on the strategic portfolio mix, and the liabilities – consisting of (future) pension payments – depend on indexation policies. Because of the long time horizon over which the liabilities range, typically in the order of 30 years, the problem is inherently dynamic. Moreover, uncertainty plays a major role because e.g. asset investments yield unknown pay-offs; by way of the valuation of future liabilities at market value, this is also a source of uncertainty on the liability side.

The goal of the ALM process is to enable payment of current and future pensions. This should be done at minimal funding costs, consisting of contributions by active participants of the fund and the sponsor (e.g., the company backing the fund), and subject to laws and the rules specified by the regulation authority for pension funds. In addition, the outcome has to confirm to the long-term policy rules of the pension fund.

Thus, in general terms the ALM problem is to select *decisions* on allocation of the assets, the contributions, indexation of future payments (relative to e.g. wage inflation), etcetera, which are *optimal* in some sense, subject to a number of constraints and taking care of uncertainty in an explicit way. All these aspects are taken care of in a so-called *multistage recourse model*, which is a model for decision making under uncertainty belonging to the field of stochastic programming. Indeed, multistage recourse models have been applied successfully to a wide range of financial and other problems, see e.g. [13] and [10].

As detailed below, multistage recourse models comprise additional decisions which allow to react conditionally on new information, becoming available as the future unfolds. The corresponding additional or recourse variables come at certain unit costs, so that the risk associated with a current decision is modeled by assigning additional costs due to uncertainty. Alternatively, one may simply disallow decisions which are too risky (in some well-defined sense). We will use the latter approach to explicitly model *short-term risk* within a multistage recourse setting.

For the pension fund problem that we study in this paper, long-term solvency goals go together with short-term constraints on the *funding ratio*, defined as the ratio of assets over (discounted) future liabilities. Due to the uncertainty involved, such a constraint has to be stated in probabilistic terms. For example, it could be formulated as: The probability that the funding ratio one year from now falls below 105% should be at most 5%. In stochastic programming terminology, such restrictions on the feasible decisions are called probabilistic or *chance constraints*. They are closely related to the well-known *Value-at-Risk* concept used in financial applications.

The inclusion of chance constraints in multistage recourse models for pension fund ALM problems was pioneered by Dert [1]. In this paper we focus on an alternative formulation of short-term risk constraints in ALM models, known as *integrated chance constraints* (ICC). As detailed below, our motivation to advocate the use of ICCs comes both from modeling as well as computational considerations.

The remainder of this paper is organized as follows. First we outline the environment in which the problem is set. In Section 2 we then formalize an ALM decision problem for pension funds, arriving at a multistage recourse model. In Section 2.2 we motivate and describe in some detail the role and implementation of integrated chance constraints in this model. In Section 3 we present numerical results for a small example problem. Finally, in Section 4 we outline a possible approach to modeling mid-term risk in such a multistage recourse model.

1.1 Dutch pension funds and regulation

Before entering upon these detailed issues, we first present some general background information on Dutch pension funds, since this is the setting of our ALM application. In The Netherlands, as in several other countries, old-age pensions consist of a state allowance complemented by payments out of pension savings. These savings are accumulated during each worker's active career by paying contributions (a fraction of the wages) to a pension fund, both by the employee and the employer. A pension fund may be related to or owned by a single company, a branch of industry, or a specific group of professionals.

Currently, there are about 830 pension funds in The Netherlands. Their total asset value is of the same order of magnitude as the Dutch GDP, see Table 1.1. While these data

Total assets/GDP (in %)	
France	6.87
Germany	14.51
Italy	2.14
Netherlands	113.02
Spain	3.97
UK	79.06
US	104.95

Table 1.1: Total asset value of pension funds as a percentage of GDP (in 1997, source: OECD).

suggest that Dutch pension savings are at a relatively high level, the more recent absolute

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numbers of Table 1.2 indicate that there is ample reason for concern: following a period of very rapid growth, total asset value has been declining since 2000. On the other hand, liabilities are forecasted to grow steadily for several more years, mainly due to demographic developments. Indeed, although no recent official statistics are available, current estimates

Year	Total asset value
1997	333
1998	387
1999	451
2000	464
2001	456

Table 1.2: Total asset value ($\times 10^9$ €) of Dutch pension funds (source: DNB).

indicate that the funding ratio of 25% of the Dutch pension funds is too low, and that the accumulated funding shortage of this group of funds amounts to 23 billion Euros.

Thus, the situation has changed dramatically over the last few years. The Dutch regulating authority for pension funds (DNB, see `http://www.dnb.nl`) has reacted by adapting the rules by which pension funds have to operate. The technical details have been in progress for quite some time. In our model, we propose implementations of the following three conceptual criteria:

(i) *Short term:* With high reliability, the funding ratio in the next year should be at least at some level specified by the regulating authority.

(ii) *Mid term:* Seen over a number of years, the funding ratio may fall short occasionally, but if this happens too often or if the shortage is too large (as defined by the regulating authority), some remedial action is required.

(iii) *Long term:* The solvency of the pension fund should be sufficiently high, from a going-concern and/or liquidation perspective.

Next we will see that all three criteria are covered in the multistage recourse ALM model that is outlined below. In particular, we will focus on the representation of the short-term criterion by means of integrated chance constraints.

2 Multistage recourse ALM for pension funds: a reference model

As explained in the introduction, our strategic ALM problem is a dynamic decision problem under uncertainty. We are asked to come up with decisions such as the contribution rate and the portfolio mix, and possibly a remedial action in case the funding ratio is insufficient. These decisions need to be taken right now, in the face of uncertainty about investment yields and other problem parameters.

We consider a large Dutch company which has its own pension fund. It will be assumed that the pension plan is benefit defined, and that the company acts as *sponsor* of the fund: if the funding ratio is too low, the company may be obliged to provide (or guarantee) additional money. Thus, the pension fund has three sources of funding its liabilities: revenues from its asset portfolio, regular contributions made by the company and the participants of the fund, and remedial contributions made by the company. The pension fund has to decide periodically how to distribute the investments over different asset classes and what the contribution rate should be in order to meet all its obligations.

Within a pension fund, there are typically different groups of participants, who all have their own interests. For example, active participants (workers) prefer to pay a low contribution rate, whereas passive participants (retirees) are concerned about the level of current payments (indexation). The company (also called the sponsor) prefers low contribution rates, and wishes to avoid remedial contributions. The ALM model should adequately reflect the - often conflicting - objectives of all interest groups. This is achieved by means of constraints and/or penalty terms in the objective function.

In addition, as already mentioned in the Introduction, there are a number of regulatory rules which further restrict the decisions of the pension fund. The long-term solvency requirements are reflected by the multi-stage character of the model. How to model the short-term criterion is discussed in detail in Section 2.2, and an approach to modeling the mid-term criterion (not considered in the numerical example of Section 3) is outlined in Section 4.

In the following we will look at a multi-stage recourse model representing this problem. We will focus on the asset side of the problem, so the indexation policies are left out of consideration.

2.1 Basic model

We split the planning horizon in T subperiods of one year each, and denote the resulting time stages by an index t. Time $t = 0$ is the current time and $t = T$ is the length of the horizon. By year t ($t = 1, \ldots, T$), we mean the span of time $[t - 1, t)$.

The uncertainty is modelled through a large but finite number S of scenarios. Each scenario represents a possible realization of all uncertain parameters in the model. Let ω_t represent the vector of random parameters whose values are revealed in year t. Then the set of all scenarios is the set of all realizations $(\omega_1^s, \ldots, \omega_T^s)$, $s \in \mathcal{S} := \{1, \ldots, S\}$, of $(\omega_1, \ldots, \omega_T)$. Each scenario s has a probability p^s, where $p^s > 0$ and $\sum_{s=1}^{S} p^s = 1$. Since in a dynamic model information on the actual values of the uncertain parameters is revealed in stages, a suitable representation of the set of scenarios is given by a scenario tree. An example of a scenario tree is given in Figure 1. Each path from $t = 0$ to $t = T$ represents one scenario.
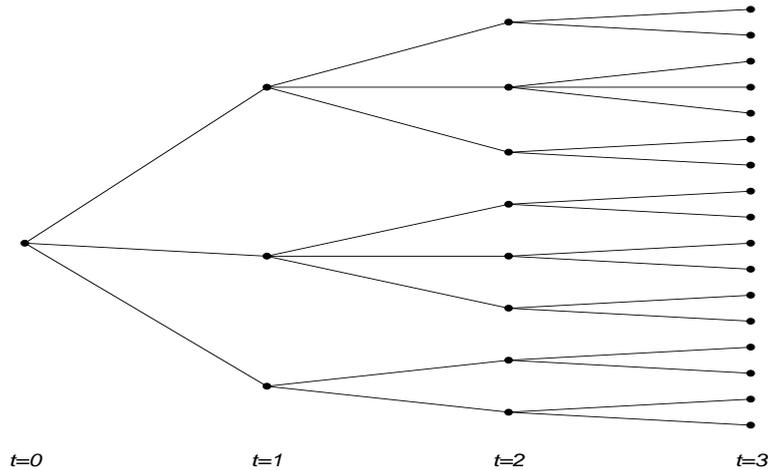

Figure 1: Example of a scenario tree, with $T = 3$ and $S = 17$.

At every node (t, s) of the tree, decisions are to be made which are optimal given the history up to then and under uncertainty about the remaining future. In the corresponding

scenario representation of the problem, explicit constraints are added to enforce nonanticipativity of the decisions.

2.2 Integrated chance constraints

The short-term criterion proposed in Section 1.1 reads that next years funding ratio should be on or above a given level α, say $\alpha = 105\%$, with *high reliability*. The formulation of this criterion clearly indicates its stochastic nature, which should be reflected in the way it is modeled. Starting from traditional chance constraints, we will arrive at our implementation of an adapted short-term criterion by means of integrated chance constraints.

2.2.1 Conceptual motivation

In our ALM model, a direct translation of the short-term criterion would be the *chance constraint*

$$\Pr\{F^*_{t+1} \geq \alpha | (t, s)\} \geq \gamma_t,$$

where F^*_{t+1} is the funding ratio just before a possible remedial contribution by the sponsor, γ_t is the required reliability at time t (e.g. $\gamma_t = 0.95$), and the notation indicates that probability is measured conditional on (t, s) being the current node. Using that the funding ratio is defined as the ratio of the assets A^*_{t+1} over liabilities L_{t+1}, an equivalent formulation is

$$\Pr\{A^*_{t+1} - \alpha L_{t+1} \geq 0 | (t, s)\} \geq \gamma_t. \tag{1}$$

Note that the risk measure underlying such a chance constraint is *qualitative*, in the sense that it measures the probability of a shortfall of the funding ratio, but the magnitude of the shortage is not taken into account. In other applications this may be justified or even preferable, but in an ALM model the size of the funding shortage is obviously relevant.

Of course, this criticism actually applies to the underlying short-term criterion suggested the DNB, which should take the magnitude of funding shortages into account. Thus, we propose to replace the current criterion by a *quantitative* one-year risk measure. In our ALM model described below, this role is played by *integrated chance constraints*. To introduce this concept, we return to the chance constraint, and will see that it comes up in a natural way when we look at computational issues.

2.2.2 Computational motivation and definition

In the chance constraint (1), both A^*_{t+1} and L_{t+1} are random quantities because they depend on underlying random parameters such as asset yields. Moreover, they depend linearly on the current decisions x^s_t which involve asset allocation and contributions. To stress these relations and in order to simplify the notation, for the time being we will use the generic representation of an (individual) chance constraint

$$\Pr\{Bx - d \geq 0\} \geq \gamma,$$

where x is an n-vector of decision variables and the n-vector B and the scalar d are both random parameters. As in our ALM model, we assume that (B, d) follow a finite discrete distribution with realizations (B^s, d^s) and corresponding probabilities p^s, $s \in \mathcal{S} := \{1, \ldots, S\}$.

All other constraints in our ALM model are linear, and also the chance constraint can be represented by linear constraints, as follows.

$$B^s x + \delta^s M \geq d^s, \quad s \in \mathcal{S}$$

$$\sum_{s \in \mathcal{S}} p^s \delta^s \leq 1 - \gamma$$

$$\delta^s \in \{0, 1\}, \quad s \in \mathcal{S},$$

where M is a sufficiently large number. Note however, that this formulation necessarily uses *binary variables* δ^s, $s \in \mathcal{S}$, to indicate realizations (B^s, d^s) which are unfavorable for x, e.g., which would result in underfunding in the ALM model. The probability weighted average of these binary variables then equals the risk of underfunding associated with the decision x, which should be at most $1 - \gamma$. Because of these binary variables, the requested inclusion of a chance constraint at every node (t, s) (except at the end nodes) of our multistage recourse model would have severe consequences for the computational tractability of the model.

For problems involving binary (or general integer) decision variables, a natural approach is to relax the integrality restrictions and solve the resulting relaxation. In our case, such a relaxation transforms the mixed-integer linear representation of the chance constraint into a system of linear constraints in continuous variables, which is equivalent to

$$
\begin{aligned}
& B^s x + y^s \geq d^s, \quad s \in \mathcal{S} \\
& \sum_{s \in \mathcal{S}} p^s y^s \leq \beta \\
& y^s \geq 0, \quad s \in \mathcal{S},
\end{aligned}
\tag{2}
$$

where the parameter β is non-negative. By the first set of inequalities, each of the nonnegative variables y^s is not less than the shortfall $d^s - B^s x$ (if any). The next inequality therefore puts an upper bound β on the *expected shortfall*. That is, the system (2) is equivalent to

$$
\sum_{s \in \mathcal{S}} p^s \left(B^s x - d^s\right)^- \leq \beta,
$$

where $(a)^- := \max\{-a, 0\}$ is the negative part of $a \in \mathbb{R}$, or

$$
\mathbb{E}\left[(Bx - d)^-\right] \leq \beta
\tag{3}
$$

with $\mathbb{E}$ denoting expectation with respect to the distribution of (B, d). Such constraints, bounding an expected shortfall, were named *integrated chance constraints* by Klein Haneveld [6], since it can be shown that

$$
\mathbb{E}\left[(Bx - d)^-\right] = \int_{-\infty}^0 \Pr\{Bx - d < u\}du.
\tag{4}
$$

The integrand in (4) is the complement of the probability $\Pr\{Bx - d \geq u\}$; it appears in an equivalent risk version

$$
\Pr\{Bx - d < u\} \leq 1 - \gamma(u)
$$

of the underlying chance constraint, where the parameter $1 - \gamma(u)$ denotes the maximum acceptable risk of not meeting the target level u.

Loosely speaking, the identity (4) shows that an integrated chance constraint corresponds to some aggregation of the infinitely many chance constraints which – in theory – could be defined for all possible target levels $u \leq 0$. By indicating, through the corresponding risk parameters $1 - \gamma(u)$, that larger shortages are even less acceptable than smaller ones, this indeed results in a quantitative risk measure defined in terms of traditional, qualitative chance constraints.

In [8] it is shown that integrated chance constraints are closely related to constraints on the *conditional surplus-at-risk* (CSaR), which is a variant of conditional value-at-risk, see e.g. [11]. Essentially, the difference is that in an ICC constraint the shortage is measured with respect to some a priori chosen threshold parameter, whereas in a CSaR constraint the threshold is equal to the surplus-at-risk, which is itself an outcome of the optimization process.

2.2.3 Implementation as short-term risk constraints

We conclude that integrated chance constraints provide a suitable way to model short-term risk in an ALM model, both from conceptual as well as computational point of view. Returning to the specific notation of our ALM model, we thus include an integrated chance constraint

$$\mathbb{E}\left[\left(A_{t+1}^* - \alpha L_{t+1}\right)^- \mid (t,s)\right] \le \beta_t \tag{5}$$

in every subproblem (t,s), $t < T$, of our multistage recourse model. They reflect our alternative short-term criterion, stating that *next years funding ratio should be such that the expected funding shortfall is at most β_t*, given that the current state is (t,s).

The parameters β_t, $t = 0, \ldots, T-1$, giving the maximal acceptable expected funding shortage, of course need to be specified numerically. It should be noted that in general it is harder to come up with these values than it would be for the reliability parameters α_t, as required for traditional chance constraints. The latter parameters are scale free, and correspond to a risk notion which is more familiar to e.g. managers of pension funds.

With S_t^s denoting the number of possible realizations from (t,s) in year $t+1$, it follows from (2) that inclusion of the integrated chance constraint (5) in subproblem (t,s) comes at the price of S_t^s additional continuous variables and $S_t^s + 1$ additional linear constraints. In many applications, e.g. in the numerical example below, S_t^s is in the order of 5 to 10, so that this extension of the model provides no computational hardship.

In case the number of realizations is substantially larger, say 1000 or more, the linear programming (LP) representation (2) becomes inefficient. In [8] it is shown that the induced feasible set $C(\beta)$, corresponding to an integrated chance constraint (3) with an underlying finite discrete distribution on S points, is given by

$$C(\beta) = \bigcap_{K \subset \mathcal{S}} \left\{ x \in \mathbb{R}^n : \sum_{k \in K} p^k \left(d^k - B^k x\right) \le \beta \right\}. \tag{6}$$

Since there are $2^S - 1$ non-empty subsets of $\mathcal{S} = \{1, \ldots, S\}$, it follows that $C(\beta)$ is a polyhedral set defined by as many linear constraints. For any non-trivial number of realizations S, it is obviously not sensible (if at all possible) to explicitly include all of them in the model. However, the representation (6) underlies a very efficient cutting plane algorithm for solving LP problems with (variants of) integrated chance constraints. For a small example problem with 1000 realizations, an optimal solution is found after generating only 9 out of the approximately 10^{300} constraints defining the set $C(\beta)$. Further numerical evidence shows that the cutting plane algorithm is much faster than the straightforward LP approach on larger problem instances.

3 Numerical illustration

In this section we present numerical results for a small instance of a multistage ALM model. The main purpose is to illustrate the effect of including short-term risk constraints, modeled as integrated chance constraints. In this example we include such a constraint only at $t = 0$,

$$\mathbb{E}\left[\left(A_1^* - \alpha L_1\right)^-\right] \le \beta,$$

thus requiring current decisions which are not too risky, measured by the expected shortfall below the target funding level at $t = 1$.

3.1 Model description

We define the following indices, variables, random parameters and deterministic parameters. (To simplify the notation, we omit the scenario index s.)

Indices
t time index, $t = 0, 1, \ldots, T$
i index of asset classes, $i = 1, \ldots, N$

Variables
Z_t remedial contribution by the sponsor at time t
X_{it} value of investments in asset class i, at the beginning of year t
c_{t+1} contribution rate for year $t + 1$
A_t total asset value at time t
A_t^* total asset value at time t *just before* a possible remedial contribution Z_t
Δ_{it}^+ value of assets in class i bought at time t
Δ_{it}^- value of assets in class i sold at time t
U_T underfunding at time horizon with respect to the original funding ratio A_0/L_0

Random parameters
r_{it} random return on asset class i in year t
W_t random total wages of active participants in year t
P_t random total benefit payments in year t
L_t random value of liabilities after year t

Deterministic parameters
k_i proportional transaction cost for asset class i
w_i^l lower bound on the value of asset class i as a fraction of the total asset portfolio
w_i^u upper bound on the value of asset class i as a fraction of the total asset portfolio
c^l lower bound on the contribution rate
c^u upper bound on the contribution rate
α lower bound on the funding ratio
γ_t discount factor for a cash flow in year t
λ_Z penalty parameter for a remedial contribution
λ_U penalty parameter for underfunding at time horizon

The decisions at time $t \in \mathcal{T}_0 := \{0, 1, \ldots, T - 1\}$ are denoted by the vector

$$x_t = (Z_t, X_{1t}, \ldots, X_{Nt}, c_{t+1}).$$

At the time horizon $t = T$, only the decision Z_T occurs. The variables A_t, $\Delta_{1t}^+, \ldots, \Delta_{Nt}^+$, $\Delta_{1t}^-, \ldots, \Delta_{Nt}^-$, and U_T are state variables. They are determined by the parameters and the decision variables.

The state variable A_t^*, denoting the value of the assets *just before* a possible remedial payment Z_t, is used to model the short-term risk criterion, and to determine the size of the remedial contribution (if any): it must hold that $Z_t \geq (A_t^* - \alpha L_t)^-$.

The randomness of the asset returns and the total wages of active participants is obvious. The randomness of the liabilities results from their valuation at market value. Demographic factors cause the randomness in the benefit payments. For $t \in \mathcal{T}_1 := \{1, 2, \ldots, T\}$, we define the random vector ω_t with realizations

$$\omega_t^s = (r_{1t}^s, \ldots, r_{Nt}^s, W_t^s, P_t^s, L_t^s), \quad s \in \mathcal{S}.$$

The constraints in the model can be divided into two types of constraints: actuarial principals, and policies. In each scenario s and at every time t (that is, in each node (t, s)), the constraints must be satisfied. As before, we omit the index s in our presentation of the constraints below.

Let us first formulate the constraints based on actuarial principles:

$$A_t = \sum_{i=1}^{N} (1 + r_{it}) X_{it} + c_t W_t - P_t + Z_t, \quad t \in \mathcal{T}_1$$

The total value of the assets at time t. Note that $A_t = A_t^* + Z_t$.

$$X_{i,t+1} = (1 + r_{it}) X_{it} - \Delta_{it}^- + \Delta_{it}^+ - k_i(\Delta_{it}^- + \Delta_{it}^+), i = 1, \ldots, N, \quad t \in \mathcal{T}_0$$

The value of the investments in asset class i, at the beginning of the year $t + 1$.

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$$\sum_{i=1}^{N}(X_{i,t+1} + k_i(\Delta_{it}^- + \Delta_{it}^+)) = A_t, \quad t \in \mathcal{T}_0$$

All assets should be allocated.

The policies of the pension fund are reflected by the following constraints:

$$w_i^l \sum_{j=1}^{N} X_{jt} \leq X_{it} \leq w_i^u \sum_{j=1}^{N} X_{jt}, \quad t \in \mathcal{T}_1$$

Lower and upper bounds on the value of asset class i as a fraction of the total asset portfolio.

$$c^l \leq c_t \leq c^u, \quad t \in \mathcal{T}_1$$

Lower and upper bounds on the contribution rate.

$$A_t \geq \alpha L_t, \quad t \in \mathcal{T}_1$$

Lower bound on the funding ratio $F_t := A_t/L_t$. The level α is prescribed by the regulator.

$$A_T + U_T \geq F_0 L_T$$

It is desired that the funding ratio at the time horizon is not below the funding ratio at $t = 0$.

Finally, the integrated chance constraint $\mathbb{E}\left[\left(A_1^* - \alpha L_1\right)^-\right] \leq \beta$, reflecting the short-term risk criterion, is implemented as a system of linear constraints analogous to (2).

The pension fund aims to minimize the total expected cost of funding, i.e., the contribution rates for the active participants and the company, and the remedial contributions. Moreover, penalty costs are assigned to the undesirable events: remedial contributions, and a low funding ratio at the planning horizon. All these components together constitute the objective function:

$$\mathbb{E}\left[\sum_{t=0}^{T} \gamma_t(c_t W_t + \lambda_Z Z_t) + \gamma_T \lambda_U U_T\right].$$

3.2 Data

In this example model we consider four asset classes ($N = 4$), whose deterministic properties are described in Table 3.1. All amounts are in millions of euros.

asset class	i	w_i^l	w_i^u	k_i	initial investments
stocks	1	0.30	0.60	0.00425	7656
bonds	2	0.30	0.60	0.00150	4048
real estate	3	0.10	0.25	0.00425	3080
cash	4	0	0.20	0.00050	1056

Table 3.1: Data on the asset classes.

The values of the other deterministic parameters and parameters observed at $t = 0$ shown in Table 3.2.

The model has 4 stages, allowing for decisions at $t = 0$ (now) up to the time horizon $T = 3$. For each year $[t - 1, t)$, $t = 1, 2, 3$, the asset returns, wages, benefit payments, and liabilities are modeled by random parameters. For future reference, some statistics on the distribution of the asset returns for $t = 1$ are presented in Table 3.3.

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$$
\begin{array}{lll}
c^l = 0 & \gamma_0 = 1 & W_0 = 10520 \\
c^u = 0.21 & \gamma_1 = 0.9693 & P_0 = 1115 \\
\alpha = 1.05 & \gamma_2 = 0.9308 & L_0 = 15200 \\
\lambda_U = 1.20 & \gamma_3 = 0.8873 & c_0 = 0.17 \\
\lambda_Z = 1.25 & &
\end{array}
$$

Table 3.2: Values of some deterministic parameters, and observed random parameters and contribution rate at $t = 0$

parameter	mean	standard deviation	correlations			
			r_{11}	r_{12}	r_{13}	r_{14}
r_{11}	0.068	0.159	1.00	0.50	-0.11	0.13
r_{12}	0.058	0.060	0.50	1.00	0.17	-0.22
r_{13}	0.065	0.112	-0.11	0.17	1.00	-0.31
r_{14}	0.032	0.017	0.13	-0.22	-0.31	1.00

Table 3.3: Summary statistics for the random returns in year 1.

For further details on the distribution and the construction of the scenario tree for this example, we refer to the forthcoming PhD thesis of S.J. Drijver.

In each state of the world before the horizon, uncertainty about the future is represented by 10 different realizations of the random parameters in each of the remaining years. Thus, there are $S = 10^T = 1000$ scenarios in total. In each node, the conditional probabilities are 0.1, so that each scenario s has probability $p^s = 0.001$.

Using the data of the Tables 3.1, we compute that the current asset value A_0 equals 16513.4 million euro. Thus, the initial funding ratio $F_0 = A_0/L_0$ equals 1.09, which is above the threshold value $\alpha = 1.05$.

3.3 Results

All numerical results were obtained using the model management system SLP-IOR (see e.g. [4, 5]). Because of the small size of the recourse model, the equivalent LP model could be solved by any of the LP codes available in SLP-IOR in negligible time.

Initially, we solved the ALM model without an integrated chance constraint at $t = 0$. It appears that for the optimal solution, the expected shortfall of the assets with respect to α times the liabilities at $t = 1$ equals 647 million euro (3.9% of A_0). Subsequently, we solved the model including the integrated chance constraint, restricting the expected shortfall at $t = 1$ by several values of the right-hand side parameter β. The resulting optimal values and first-stage solutions are presented in Table 3.4.

Of course, the optimal value increases with decreasing values of β. However, as illustrated in Figure 3.1, the increase is relatively small (less than 2%) for values of β as small as 300 (1.8% of A_0). The influence on the composition of the total costs is much stronger: over this range, direct costs increase sharply (72%), whereas expected future costs fall (26%).

Next we consider the effect of the integrated chance constraint on the first-stage decisions, i.e., on the contribution rate, the asset mix, and the remedial contribution at $t = 0$. For β decreasing to 450, we see that the request for higher short-term reliability is satisfied by increasing the contribution rate. But then, as β keeps decreasing, also the asset mix changes gradually: less is invested in relatively risky stocks and real estate, and more in bonds and eventually also cash. For extremely low values of β, this pattern is less clear due to binding lower and upper bounds on the relative amounts of stocks and cash, respectively.

| β	opt.val	direct	future	X_{10}	X_{20}	X_{30}	X_{40}	c_1	Z_0
∞	4686	1318	3368	7427	4951	4126	0	0.128	0
600	4687	1418	3268	7427	4951	4126	0	0.138	0
550	4688	1540	3148	7427	4951	4126	0	0.150	0
500	4690	1661	3029	7427	4951	4126	0	0.161	0
450	4694	1782	2911	7427	4951	4126	0	0.173	0
400	4703	1945	2757	7158	5220	4126	0	0.199	0
350	4720	2107	2612	6887	5489	4125	0	0.205	0
300	4767	2268	2498	6949	6465	3183	0	0.210	92
250	4829	2469	2359	6660	7016	3080	0	0.210	252
200	4896	2674	2222	6078	7447	3080	310	0.210	416
150	4998	2937	2060	5477	6713	3080	1855	0.210	626
100	5108	3171	1936	5974	5661	2335	3342	0.210	814
50	5233	3472	1761	5264	6561	2213	3509	0.210	1055
0	5529	4147	1381	5428	5967	3080	3619	0.210	1595

Table 3.4: Results for different values of β.
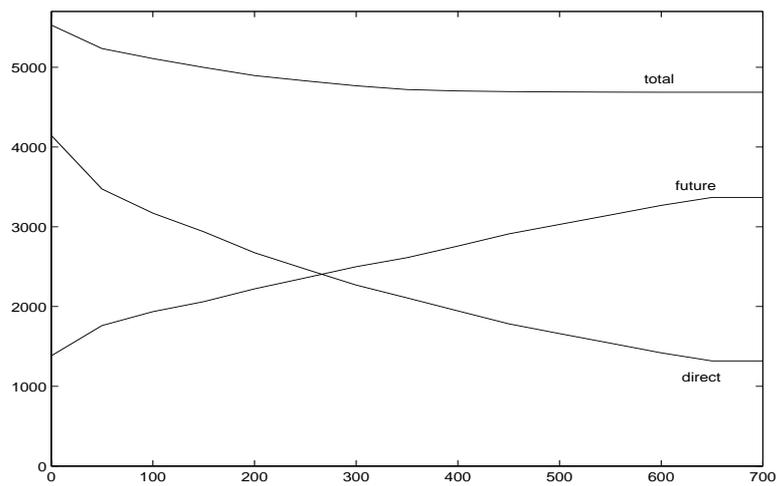
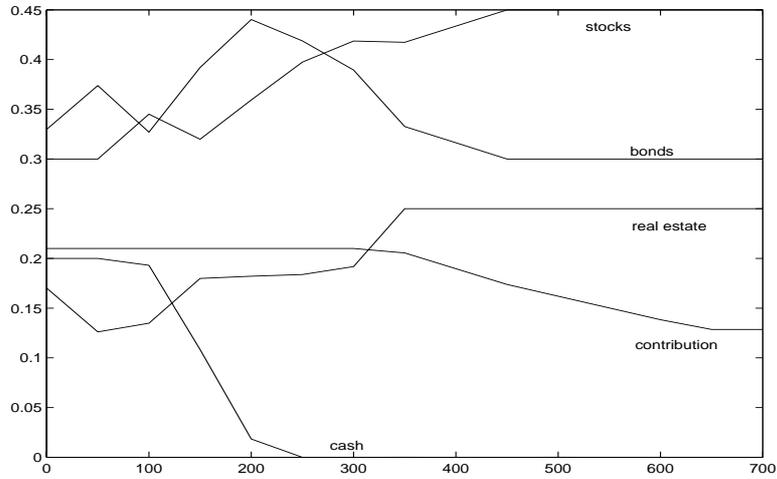Figure 3.1: Optimal value (direct and expected future costs) as function of β.

Figure 3.2: Asset mix and contribution rate as function of β.

For β lower than 350, remedial contributions Z_0 are part of the optimal solution at $t = 0$. On first sight, they appear to be larger than necessary. For example, to accommodate a decrease of β from 300 to 250, one could reason that an increase of Z_0 with 50 should be sufficient. However, due to the penalty parameter $\lambda_Z = 1.25$, this would increase direct costs by 62.5. In the presented optimal solution, in which Z_0 is increased by 160 at additional direct costs of 200, this is (more than) compensated by the resulting decrease of the expected future costs, since the additional assets allow for lower funding costs in later stages. Obviously, other choices for the parameter λ_Z will result in a different balance between direct and expected future costs.
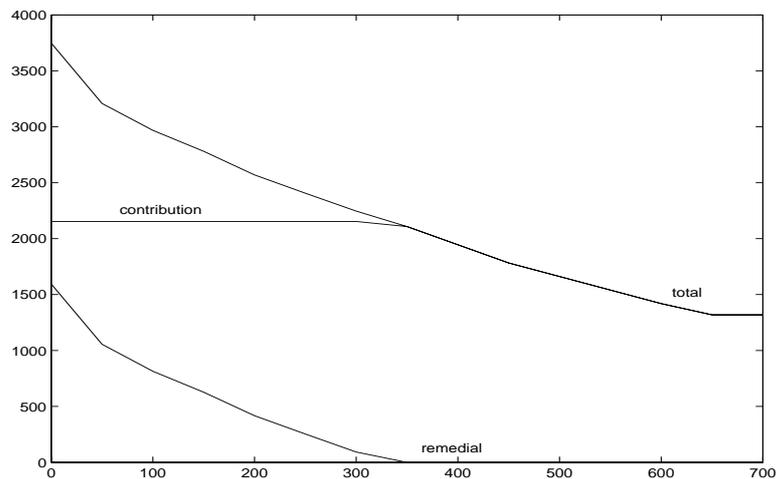

Figure 3.3: Funding costs at $t = 0$ as function of β.

We conclude that, at least in this example problem, the integrated chance constraint on the expected funding shortage appears to be a suitable tool for modeling short-term risk. In particular, for appropriately chosen values of the parameter β, it may lead to a solution which has lower short-term risk in exchange for marginally higher expected total costs. It

should be stressed, however, that selecting a good value for β *a priori* can be difficult in practice. Given the low computational costs of including integrated chance constraints, it may be feasible to find a reasonable parameter setting by numerical experiments.

Finally, let us briefly consider the reliability of the optimal solutions in terms of the probability $\Pr\{A_1^* - \alpha L_1 \geq 0\}$, i.e., the reliability concept underlying traditional chance constraints. The solution of the model without integrated chance constraint turns out to have reliability 0.5, whereas e.g. the solution of the ICC model with $\beta = 450$ is reliable at the 0.7 level, and $\beta = 200$ yields reliability 0.8. In this example, it appears that the reliability of the solutions increases (step-wise) with decreasing β. In general, however, this need not be the case. For example, a solution which yields small shortages for every realization is feasible with respect to the integrated chance constraint for some small β, whereas it has reliability 0.

In chance-constrained models, the reliability is usually required to be at least 0.9, which is obtained in this model by setting $\beta = 72$. For the current model, higher aspiration levels actually imply 100% reliability (corresponding to the extreme setting $\beta = 0$), due to the fact that each realization at $t = 1$ has probability 0.1. Indeed, this is a general modeling weakness of chance constraints based on a small number of possible realizations, as is often the case for single-period risk constraints in multi-stage recourse models of realistic size.

4 Modeling mid-term risk in the ALM model

In this final section, we briefly describe how the mid-term risk criterion introduced in Section 1.1 can be modeled. This approach is implemented in the ALM model described in the forthcoming PhD thesis of Drijver. As we will see, the approach involves binary decision variables after all.

Foremost, the introduction of binary variables in the model is justified by the fact that they are necessary to include several realistic features in our multistage recourse ALM model. In addition, we have a long-standing interest in (computational) properties of recourse models with (mixed-)integer variables, see e.g. [9, 7, 12]. In fact, our interest in this ALM application was raised initially by the observation that integer variables appear naturally in these models.

4.1 Modeling remedial action in case of underfunding

According to our interpretation of the mid-term criterion, the funding ratio may occasionally drop below the required level α, but this should not happen too often nor should the shortage be too large. Given the long-term character of the liabilities, this appears to be very reasonable: there is no compelling reason to react immediately on a possibly temporary drop in e.g. stock prices. Indeed, such an immediate reaction could very well turn out to be harmful in the longer run. On the other hand, if the funding ratio is too far below the aspired level α, then immediate action might be necessary to prevent a further decline.

In the current version of the ALM model, 'not too often' is understood as 'not in two consecutive years'. However, with only minor modifications, the approach outlined below also applies to the more general interpretation 'at most in n out of m consecutive years'. To simplify the exposition, we do not account here for immediate action in case of a very low funding ratio. It is not difficult to see how this refinement can be modeled.

Essentially, the approach is based on keeping track of years in which underfunding occurs. This calls for the use of binary *indicator variables* in the model. In turn, these indicator variables are used to determine when a remedial action is necessary in order to restore the funding ratio to the required level α. In the model, remedial contributions by the sponsor of the pension fund are used to this end.

With A_t^{*s} and L_t^s denoting assets and liabilities in node (t, s) as before, the constraint

$$M\delta_t^s \geq \alpha L_t^s - A_t^{*s},$$

where M is again a sufficiently large number, forces the indicator variable δ_t^s to take on the value 1 in case of a funding shortage. If this happens in two consecutive years in the same scenario s, then the constraint

$$\pi_t^s \geq \delta_{t-1}^s + \delta_t^s - 1$$

forces the indicator variable π_t^s to become 1, triggering a remedial contribution Z_t^s, equal to the observed current shortage $\alpha L_t^s - A_t^{*s}$, which is modeled by the constraint

$$Z_t^s \geq \alpha L_t^s - A_t^{*s} - M(1 - \pi_t^s).$$

In addition, the model contains constraints which ensure that a remedial contribution is only allowed in case of underfunding. Because both funding shortages and remedial contributions are undesirable events, corresponding fixed penalty costs are included in the objective function. On top of that, proportional costs for remedial contributions are assigned.

On the opposite side, if the pension fund has a structural funding surplus, a restitution to the sponsor may be required. This is modeled analogously to the shortage case above. (Unfortunately, this issue is currently not as relevant as a few years ago, when a first version of the model was developed.) Further realistic features of the ALM model include a detailed modeling of indexation of future pension payments (again using indicator variables), and the use of *soft constraints*, for example to model a preference for gradual changes in the contribution rate from one year to the next. For a detailed exposition of all features included in this model we refer to [3, 2] and the forthcoming PhD thesis of Drijver.

5   Summary and concluding remarks

We motivated and described the role played by integrated chance constraints in an ALM model for Dutch pension funds. To set the stage, we outlined the practical setting as well as our modeling approach for this dynamic decision problem under uncertainty.

Integrated chance constraints are appropriate for modeling single-period risk constraints, in particular if a quantitative risk measure is preferable, as is the case here. Moreover, they are computationally attractive in the given multistage recourse setting, since they can be formulated in terms of a limited number of linear constraints without the need to introduce additional binary decision variables.

These claims are supported by the numerical results on a small example problem. No computational results on (semi-)realistic data are available at this time. However, such data have been made available to us by a major Dutch pension fund. Initial outcomes of an ALM model including integrated chance constraints and implementing the mid-term risk criterion as described in the previous section, obtained with a special purpose heuristic, will be reported in the forthcoming PhD thesis of Drijver, and in other publications.

We expect that such multistage recourse models, including implementations of various risk criteria, will prove to be a useful tool in strategic ALM studies for pension funds. Even though we believe that this approach allows to model important aspects in a realistic way, the final judgement on such models will have to come from the analysis of numerical results for (semi-)realistic problems.

References

[1] C.L. Dert. *Asset Liability Management for Pension Funds, A Multistage Chance Constrained Programming Approach*. PhD thesis, Erasmus University, Rotterdam, The Netherlands, 1995.

[2] S.J. Drijver, W.K. Klein Haneveld, and M.H. van der Vlerk. ALM model for pension funds: numerical results for a prototype model. Research Report 02A44, SOM, University of Groningen, `http://som.rug.nl`, 2002.

[3] S.J. Drijver, W.K. Klein Haneveld, and M.H. van der Vlerk. Asset liability management modeling using multistage mixed-integer stochastic programming. In B. Scherer, editor, *Asset and Liability Management Tools: A Handbook for Best Practice*, chapter 16, pages 309–324. Risk Books, London, 2003.

[4] P. Kall and J. Mayer. SLP-IOR: An interactive model management system for stochastic linear programs. *Math. Program.*, 75(2):221–240, 1996.

[5] P. Kall and J. Mayer. Modeling support for multistage recourse problems. In *Dynamic stochastic optimization (Laxenburg, 2002)*, volume 532 of *Lecture Notes in Econom. and Math. Systems*, pages 21–41. Springer, Berlin, 2004.

[6] W.K. Klein Haneveld. *Duality in stochastic linear and dynamic programming*, volume 274 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1986.

[7] W.K. Klein Haneveld and M.H. van der Vlerk. Stochastic integer programming: General models and algorithms. *Ann. Oper. Res.*, 85:39–57, 1999.

[8] W.K. Klein Haneveld and M.H. van der Vlerk. Integrated chance constraints: reduced forms and an algorithm. Research Report 02A33, SOM, University of Groningen, `http://som.rug.nl`, 2002. To appear in *Computational Management Science*.

[9] F.V. Louveaux and M.H. van der Vlerk. Stochastic programming with simple integer recourse. *Math. Program.*, 61:301–325, 1993.

[10] A. Ruszczynski and A. Shapiro, editors. *Stochastic Programming*, volume 10 of *Handbooks in Operations Research and Management Science*. North-Holland, 2003.

[11] S.P. Uryasev, editor. *Probabilistic constrained optimization*. Kluwer Academic Publishers, Dordrecht, 2000. Methodology and applications.

[12] M.H. van der Vlerk. Convex approximations for complete integer recourse models. *Math. Program.*, 99(2):297–310, 2004.

[13] W.T. Ziemba and J.M. Mulvey, editors. *World Wide Asset and Liability Management*. Cambridge Univ. Press, 1998.