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A Branch-and-Cut Algorithm for the Stochastic Uncapacitated Lot-Sizing Problem*

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Abstract. This paper addresses a multi-stage stochastic integer programming formulation of the uncapacitated lot-sizing problem under uncertainty. We show that the classical (ℓ, S) inequalities for the deterministic lot-sizing polytope are also valid for the stochastic lot-sizing polytope. We then extend the (ℓ, S) inequalities to a general class of valid inequalities, called the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities, and we establish necessary and sufficient conditions which guarantee that the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities are facet-defining. A separation heuristic for $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities is developed and incorporated into a branch and cut algorithm. A computational study verifies the usefulness of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities as cuts.

Key words. Stochastic Lot-Sizing – Multi-stage Stochastic Integer Programming – Polyhedral Study – Branch and Cut

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1. Introduction

The deterministic uncapacitated lot-sizing problem is to determine a minimum cost production and inventory holding schedule for a product so as to satisfy its demand over a finite discrete-time planning horizon. A standard mixed-integer programming formulation for the single item, uncapacitated, lot-sizing problem is (cf. [17]):

$$
\begin{aligned}
\text{(LS)} : \min \ & \sum_{i=1}^{T} (\alpha_i x_i + \beta_i y_i + h_i s_i) \\
\text{s.t.} \ & s_{i-1} + x_i = d_i + s_i \qquad i = 1, \ldots, T, \\
& x_i \le M_i y_i \qquad i = 1, \ldots, T, \\
& x_i, s_i \ge 0, \ y_i \in \{0, 1\} \qquad i = 1, \ldots, T, \\
& s_0 = 0,
\end{aligned}
$$

where x_i represents the production in period i, s_i represents the inventory at the end of period i, and y_i indicates if there is a production set-up in period i. Problem parameters α_i, β_i, h_i, and d_i represent the production cost, set-up cost, holding cost, and the demand in period i, respectively. Since there is no

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restriction on the production level, the parameter M_i is a sufficiently large upper bound on x_i. In the absence of backlogging, this bound can be set as $M_i = \sum_{j=i}^{T} d_j$. We denote the set of feasible solutions of (LS) as X_{LS}.

Although (LS) is solvable in strongly polynomial time using specialized dynamic programming algorithms (cf. [1,10,22,23]), such algorithms are not applicable when (LS) is embedded, as it frequently is, in various multi-period production planning problems. This has motivated the polyhedral study of X_{LS} in order to improve integer programming approaches for such production planning problems. Barany, Van Roy and Wolsey [6,7] proved that a complete polyhedral description of the convex hull of X_{LS} is given by some of the original inequalities together with the (ℓ, S) inequalities

$$\sum_{i\in S} x_i + \sum_{i\in \overline{S}} d_{i\ell} y_i \geq d_{1\ell},$$

where $\ell \in \{1, 2, \ldots, T\}$, $S \subseteq \{1, 2, \ldots, \ell\}$, $\overline{S} = \{1, 2, \ldots, \ell\} \setminus S$, and $d_{ij} = \sum_{k=i}^{j} d_k$. The authors reported good computational results for multiple item capacitated lot-sizing problems using the (ℓ, S) inequalities within a branch-and-cut scheme. Following Barany et al.'s work, polyhedral structures of many variants of (LS) have been investigated. These include variants of (LS) involving sales and safety stocks [14], start-up costs [21], piecewise linear and concave production costs [2], and constant [13,19], as well as dynamic [5,16,18] production capacities, only to name a few.

The lot-sizing model (LS) assumes that the cost and demand parameters are known with certainty for all periods of the planning horizon. However, in many applications, these parameters are uncertain, and, at best, only some distributional information may be available. In this case, (LS) can be extended to explicitly address uncertainty by adopting a stochastic programming [20] approach. Haugen, Løkketangen and Woodruff [12] proposed a heuristic strategy for such stochastic lot-sizing problems. Ahmed, King and Parija [3] proposed an extended reformulation of the uncapacitated stochastic lot-sizing problem whose LP relaxation is significantly tighter than the standard formulation. They also point out that the Wagner-Whitin optimality conditions for deterministic uncapacitated lot-sizing problems, i.e., no production is undertaken if inventory is available, do not hold in the stochastic case. The stochastic lot-sizing problem has also been considered as subproblems embedded in some classes of stochastic capacity expansion problems [4], stochastic batch-sizing problems [15], and stochastic production planning problems [8].

In this paper, we study the polyhedral structure of the uncapacitated stochastic lot-sizing problem. We show that the (ℓ, S) inequalities are also valid for the stochastic lot-sizing polytope. We generalize the (ℓ, S) inequalities to a new class of valid inequalities for the stochastic lot-sizing polytope. We provide necessary and sufficient conditions that guarantee that the proposed inequalities are facet-defining, and develop separation algorithms. Our computational experiments demonstrate that the proposed inequalities are extremely useful within a branch-and-cut scheme for stochastic lot-sizing problems.

2. The Stochastic Lot-sizing Problem

A stochastic programming extension of the deterministic formulation (LS) is presented in [3]. This extension is described next. The problem parameters α_i, β_i, h_i, and d_i are assumed to evolve as discrete time stochastic processes with a finite probability space. This information structure can be interpreted as a scenario tree with T levels (or stages) where a node i in stage t of the tree gives the state of the system that can be distinguished by information available up to time stage t. Each node i of the scenario tree, except the root node (indexed as $i = 0$), has a unique parent $a(i)$, and each non-terminal node i is the root of a subtree $\mathcal{T}(i) = (\mathcal{V}(i), \mathcal{E}(i))$, which contains all descendants of node i. For notational brevity we use $\mathcal{T} = \mathcal{T}(0)$ and $\mathcal{V} = \mathcal{V}(0)$ for the whole tree. The set of leaf nodes of $\mathcal{T}$ is denoted by $\mathcal{L}$. The probability associated with the state represented by node i is p_i. The set of nodes on the path from the root node to node i is denoted by $\mathcal{P}(i)$. If $i \in \mathcal{L}$ then $\mathcal{P}(i)$ corresponds to a *scenario*, and represents a joint realization of the problems parameters over all periods $1, \ldots, T$. We define $\mathcal{P}(i, j) = \{k : k \in \mathcal{P}(j) \cap \mathcal{V}(i)\}$, thus $\mathcal{P}(i) = \mathcal{P}(0, i)$. Similarly, we let $d_{ij} = \sum_{k \in \mathcal{P}(i,j)} d_k$. We let $\mathcal{C}(i)$ denote the set of nodes those are immediate children of node i, i.e. $\mathcal{C}(i) = \{j : a(j) = i\}$; $t(i)$ denote the time stage or level of node i in the tree, i.e., $t(i) = |\mathcal{P}(i)|$; $\mathcal{L}(i)$ denote the leaf nodes of the subtree $\mathcal{T}(i)$.

Using this notation, a multi-stage stochastic integer programming formulation of the single-item, uncapacitated, stochastic lot-sizing formulation is:

$$\begin{aligned}
\text{(SLS1)} : \min \; & \sum_{i \in \mathcal{V}} p_i(\alpha_i x_i + \beta_i y_i + h_i s_i) \\
\text{s.t. } \; & s_{a(i)} + x_i = d_i + s_i \qquad i \in \mathcal{V}, \\
& x_i \le M_i y_i \qquad i \in \mathcal{V}, \\
& x_i, s_i \ge 0, \; y_i \in \{0,1\} \quad i \in \mathcal{V}, \\
& s_{a(0)} = 0,
\end{aligned}$$

where x_i represents the production in period $t(i)$ corresponding to the state defined by node i, similarly s_i represents the inventory at the end of period $t(i)$ and y_i is the indicator variable for a production set-up in period $t(i)$. An upper bound on x_i is given by

$$M_i = \max_{j \in \mathcal{L}(i)} d_{ij}.$$

Upon eliminating variables s_i from (SLS1), we obtain the reformulation:

$$\text{(SLS)} : \min \sum_{i \in \mathcal{V}} (\bar{\alpha}_i x_i + \bar{\beta}_i y_i) \tag{1}$$

$$\text{s.t. } \sum_{j \in \mathcal{P}(i)} x_j \ge d_{0i} \qquad i \in \mathcal{V}, \tag{2}$$

$$0 \le x_i \le M_i y_i \qquad i \in \mathcal{V}, \tag{3}$$

$$y_i \in \{0,1\} \qquad i \in \mathcal{V}, \tag{4}$$

where $\bar{\alpha}_i = p_i\alpha_i + \sum_{j\in\mathcal{V}(i)} p_j h_j$ and $\bar{\beta}_i = p_i\beta_i$. Throughout this paper, we use the formulation (SLS) for the stochastic lot-sizing problem. The set of feasible solutions to (SLS) defined by the constraints (2)-(4) is denoted by X_{SLS}.

3. Valid Inequalities for the Stochastic Lot-Sizing Problem

In this section, we provide valid inequalities for the stochastic lot-sizing problem. We first show that the well-known (ℓ, S) inequalities, for the deterministic lot-sizing problem, are valid for (SLS). These inequalities are based on a sequence of consecutive time periods that can be thought of as a path in the scenario tree $\mathcal{T}$. Next, we extend the (ℓ, S) inequalities to a general class, called the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities, which are derived from subtrees of $\mathcal{T}$.

3.1. The (ℓ, S) inequalities

Theorem 1. *Given $\ell \in \mathcal{V}$ and $S \subseteq \mathcal{P}(\ell)$, the (ℓ, S) inequality*

$$\sum_{i\in S} x_i + \sum_{i\in \overline{S}} d_{i\ell} y_i \geq d_{0\ell},$$

where $\overline{S} = \mathcal{P}(\ell) \setminus S$, is valid for X_{SLS}.

Proof. The proof is analogous to that of the deterministic case (cf. [6]). Given a point $(x, y) \in X_{\mathrm{SLS}}$, we consider two cases: (a) there exists $i \in \overline{S}$ such that $y_i = 1$, and (b) $y_i = 0$ for all $i \in \overline{S}$.

Case (a): Let $k = \operatorname{argmin}\{t(i) : \ i \in \overline{S},\ y_i = 1\}$. Then $y_i = 0$ and $x_i = 0$ for all $i \in \overline{S} \cap \mathcal{P}(a(k))$. Hence

$$\sum_{i\in S} x_i + \sum_{i\in \overline{S}} d_{i\ell} y_i \geq \sum_{i\in\mathcal{P}(a(k))} x_i + d_{k\ell} \geq d_{0a(k)} + d_{k\ell} = d_{0\ell}.$$

Case (b): If $y_i = 0$ for all $i \in \overline{S}$, then

$$\sum_{i\in S} x_i + \sum_{i\in \overline{S}} d_{i\ell} y_i = \sum_{i\in\mathcal{P}(\ell)} x_i \geq d_{0\ell}.$$

□

3.2. The $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities

In this section, we extend the (ℓ, S) inequalities to a general class called the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities.

Consider a subset $\mathcal{Q} \subset \mathcal{V} \setminus \{0\}$ satisfying the following properties:

(A1) If $i, j \in \mathcal{Q}$, then $d_{0i} \neq d_{0j}$.

(A2) If $i, j \in \mathcal{Q}$, then $i \notin \mathcal{P}(j)$ and $j \notin \mathcal{P}(i)$.

(A1) allows us to uniquely index the nodes in the set $\mathcal{Q}$ as $\{1, 2, ..., Q\}$ where $Q = |\mathcal{Q}|$, such that $d_{01} < d_{02} < \cdots < d_{0Q}$.

Define $\mathcal{T}_\mathcal{Q} = \{\mathcal{V}_\mathcal{Q}, \mathcal{E}_\mathcal{Q}\}$ to be the subtree of $\mathcal{T}$ whose leaf nodes are $\mathcal{Q}$, i.e, $\mathcal{V}_\mathcal{Q} = \cup_{i\in\mathcal{Q}}\mathcal{P}(i)$. Note that by (A2), *all* nodes in $\mathcal{Q}$ are leaf nodes of $\mathcal{T}_\mathcal{Q}$. Given $i \in \mathcal{V}_\mathcal{Q}$, we denote by $\mathcal{T}_\mathcal{Q}(i) = \{\mathcal{V}_\mathcal{Q}(i), \mathcal{E}_\mathcal{Q}(i)\}$ the subtree of $\mathcal{T}_\mathcal{Q}$ with i as the root node. Note that $\mathcal{V}_\mathcal{Q}(i) = \mathcal{V}(i) \cap \mathcal{V}_\mathcal{Q}$. We use $\mathcal{Q}(i) \subseteq \mathcal{Q}$ to denote the set of leaf nodes of the subtree $\mathcal{T}_\mathcal{Q}(i)$, i.e., $\mathcal{Q}(i) = \mathcal{V}_\mathcal{Q}(i) \cap \mathcal{Q}$. Property (A2) simply gives us a convenient way of defining the subtrees over which the $(\mathcal{Q}, S_\mathcal{Q})$ inequalities are defined. We will comment on (A1) and (A2) at the end of this section.

In addition to (A1) and (A2), we need the following property on the set $\mathcal{Q}$ for the validity of the $(\mathcal{Q}, S_\mathcal{Q})$ inequalities:

(A3) Given any node $k \in \mathcal{V}_\mathcal{Q}$, and nodes $i, j \in \mathcal{Q}$ such that $i < j$ and $i, j \in \mathcal{Q}(k)$, we have that $\{i, i+1, \ldots, j-1, j\} \subseteq \mathcal{Q}(k)$.

Given a subset $\mathcal{Q}$, define the following quantities for any node $i \in \mathcal{V}_\mathcal{Q}$:

$$\overline{D}_\mathcal{Q}(i) = \max\{d_{0j} : \ j \in \mathcal{Q}(i)\} \tag{5}$$

$$\widetilde{D}_\mathcal{Q}(i) = \begin{cases} 0, & \text{if } \{j : j \in \mathcal{Q} \setminus \mathcal{Q}(i) \text{ such that } d_{0j} \le \overline{D}_\mathcal{Q}(i)\} = \emptyset \\ \max\{d_{0j} : \ j \in \mathcal{Q} \setminus \mathcal{Q}(i) \text{ such that } d_{0j} \le \overline{D}_\mathcal{Q}(i)\}, & \text{otherwise} \end{cases} \tag{6}$$

$$M_\mathcal{Q}(i) = \max\{d_{ij} : j \in \mathcal{Q}(i)\} \tag{7}$$

$$\Delta_\mathcal{Q}(i) = \min\left\{\overline{D}_\mathcal{Q}(i) - \widetilde{D}_\mathcal{Q}(i), M_\mathcal{Q}(i)\right\}. \tag{8}$$

Given $k \in \mathcal{Q}$, let $\mathcal{Q}_k = \{1, 2, \ldots, k-1, k\}$ and $\mathcal{T}_{\mathcal{Q}_k} = \{\mathcal{V}_{\mathcal{Q}_k}, \mathcal{E}_{\mathcal{Q}_k}\}$ be the subtree of $\mathcal{T}$ with leaf nodes $\mathcal{Q}_k$. It is easily verified that, if $\mathcal{Q}$ satisfies (A1)-(A3) then every subset $\mathcal{Q}_k$ for $k = 1, \ldots, Q$ satisfies these properties as well.

Now, let $K \in \mathcal{Q}$, and suppose there exists a $j^* \in \mathcal{V}_{\mathcal{Q}_K}$ such that $j^* \in \mathcal{P}(K)$ and $\widetilde{D}_{\mathcal{Q}_K}(j^*) > 0$. Then there exists $r^* \in \mathcal{Q}$ such that $\widetilde{D}_{\mathcal{Q}_K}(j^*) = d_{0r^*}$. Clearly $1 \le r^* \le K$. Let $u^* = \operatorname{argmax}\{t(i) : i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \cap \mathcal{P}(K)\}$. From (A3), it follows that $u^* \in \mathcal{P}(r^*)$. If not (i.e., $u^* \notin \mathcal{P}(r^*)$), then there exists a $r' < r^*, r' \in \mathcal{Q}_{r^*}$ such that $u^* \in \mathcal{P}(r')$ since $u^* \in \mathcal{V}_{\mathcal{Q}_{r^*}}$. Thus, we have $r', K \in \mathcal{Q}(u^*)$. Then $r^* \in \mathcal{Q}(u^*)$ according to (A3) since $r' < r^* \le K$, which contradicts with $u^* \notin \mathcal{P}(r^*)$. Figure 1 illustrates the relative position of the nodes j^*, r^*, and u^*, and the set $\mathcal{V}_{\mathcal{Q}_{r^*}}$. In this figure $\mathcal{Q}_K = \{1, 2, 3, r^*, K-1, K\}$, $\mathcal{Q}_{r^*} = \{1, 2, 3, r^*\}$, $\mathcal{V}_{\mathcal{Q}_K}$ is the set of all nodes and $\mathcal{V}_{\mathcal{Q}_{r^*}}$ is the set of nodes within the dotted area as shown in the graph. For K, j^*, r^* and u^* defined as above, we need the following two lemmas.

Lemma 1. $\Delta_{\mathcal{Q}_K}(i) \ge \Delta_{\mathcal{Q}_{r^*}}(i)$ *for any* $i \in \mathcal{P}(u^*)$.

Proof. We have

$$\overline{D}_{\mathcal{Q}_K}(i) = d_{0K} \ge d_{0r^*} = \overline{D}_{\mathcal{Q}_{r^*}}(i) \quad \text{for any } i \in \mathcal{P}(u^*). \tag{9}$$

Furthermore, for any $i \in \mathcal{P}(u^*)$, we have $r^*, K \in \mathcal{V}_{\mathcal{Q}_K}(i)$. It then follows from (A3) that $\mathcal{Q}_K(i) = \mathcal{Q}_{r^*}(i) \cup \{r^*+1, \ldots, K\}$. Thus

$$\begin{aligned} \mathcal{Q}_K \setminus \mathcal{Q}_K(i) &= \{1, \ldots, K\} \setminus (\mathcal{Q}_{r^*}(i) \cup \{r^*+1, \ldots, K\}) \\ &= (\{1, \ldots, K\} \setminus \{r^*+1, \ldots, K\}) \setminus \mathcal{Q}_{r^*}(i) \\ &= \mathcal{Q}_{r^*} \setminus \mathcal{Q}_{r^*}(i). \end{aligned} \tag{10}$$

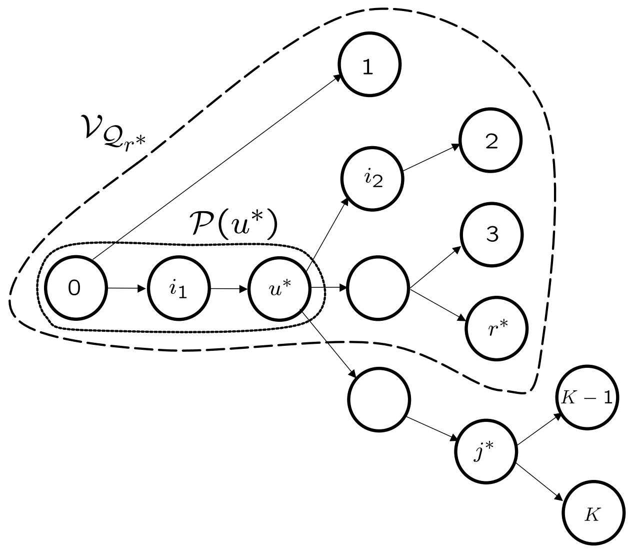

Fig. 1. Notation for Lemmas 1 and 2

(For example, in Figure 1, consider node $i_1 \in \mathcal{P}(u^*)$, then $Q_K(i_1) = \{2, 3, r^*, K-1, K\}$ and $\mathcal{Q}_{r^*}(i_1) = \{2, 3, r^*\}$. Thus $\mathcal{Q}_K \setminus \mathcal{Q}_K(i_1) = \mathcal{Q}_{r^*} \setminus \mathcal{Q}_{r^*}(i_1) = \{1\}$.)

Next, note that for any $i \in \mathcal{P}(u^*)$, it follows from (9) that $d_{0j} \leq \overline{D}_{\mathcal{Q}_K}(i) = d_{0K}$ for any $j \in \mathcal{Q}_K$ and $d_{0j} \leq \overline{D}_{\mathcal{Q}_{r^*}}(i) = d_{0r^*}$ for any $j \in \mathcal{Q}_{r^*}$. Thus for any node $i \in \mathcal{P}(u^*)$, $\widetilde{D}_{\mathcal{Q}_K}(i) = \max\{d_{0j} : j \in \mathcal{Q}_K \setminus \mathcal{Q}_K(i)\}$ and $\widetilde{D}_{\mathcal{Q}_{r^*}}(i) = \max\{d_{0j} : j \in \mathcal{Q}_{r^*} \setminus \mathcal{Q}_{r^*}(i)\}$. It then follows from (10) that

$$\widetilde{D}_{\mathcal{Q}_K}(i) = \widetilde{D}_{\mathcal{Q}_{r^*}}(i) \quad \text{for any } i \in \mathcal{P}(u^*). \tag{11}$$

Since $\mathcal{Q}_{r^*}(i) \subset \mathcal{Q}_K(i)$, we also have

$$M_{\mathcal{Q}_K}(i) \geq M_{\mathcal{Q}_{r^*}}(i) \quad \text{for any } i \in \mathcal{P}(u^*). \tag{12}$$

The lemma follows from (9), (11), (12) and the definition of Δ. $\square$

Lemma 2. *$\Delta_{\mathcal{Q}_K}(i) = \Delta_{\mathcal{Q}_{r^*}}(i)$ for any $i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*)$.*

Proof. We first claim that

$$j^* \notin \mathcal{V}_{\mathcal{Q}_{r^*}}. \tag{13}$$

Suppose that $j^* \in \mathcal{V}_{\mathcal{Q}_{r^*}}$. Then there exists $r_{j^*} \in \mathcal{Q}$ such that $r_{j^*} \leq r^* < K$, i.e., $r_{j^*} \in \mathcal{Q}_K(j^*)$. Note that by definition $r^* \notin \mathcal{Q}_K(j^*)$. Since $K \in \mathcal{Q}_K(j^*)$ and $r_{j^*} \leq r^* < K$, we have a contradiction to (A3). Thus (13) holds.

Next, we show that

$$\mathcal{Q}_{r^*}(i) = \mathcal{Q}_K(i) \quad \text{for any } i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*). \tag{14}$$

Clearly $\mathcal{Q}_{r^*}(i) \subseteq \mathcal{Q}_K(i)$. Now, suppose there exists some $k \in \mathcal{Q}_K(i)$ such that $k > r^*$. Note that $i \in \mathcal{V}_{\mathcal{Q}_{r^*}}$ and $j^* \notin \mathcal{V}_{\mathcal{Q}_{r^*}}$ from (13), thus $j^* \notin \mathcal{P}(i)$. Furthermore we also have $i \notin \mathcal{P}(j^*)$, for if we had $i \in \mathcal{P}(j^*)$ then by definition of u^* we would have $i \in \mathcal{P}(u^*)$. Thus $i \notin \mathcal{V}_{\mathcal{Q}_K}(j^*)$ and so $k \notin \mathcal{V}_{\mathcal{Q}_K}(j^*)$. Thus $d_{0r^*} = \widetilde{D}_{\mathcal{Q}_K}(j^*) = \max\{d_{0j} : j \in \mathcal{Q}_K \setminus \mathcal{Q}_K(j^*) \text{ and } d_{0j} \leq \overline{D}_{\mathcal{Q}_K}(j^*) = d_{0K}\} \geq d_{0k}$, which is a contradiction to $k > r^*$. Thus (14) is true. (The claim is clear in Figure 1. Consider the node $i_2 \in \mathcal{V}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*)$. Here $\mathcal{Q}_{r^*}(i_2) = \mathcal{Q}_K(i_2) = \{2\}$.)

From (14), we have

$$\overline{D}_{\mathcal{Q}_K}(i) = \overline{D}_{\mathcal{Q}_{r^*}}(i) \quad \text{for any } i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*), \tag{15}$$

and

$$M_{\mathcal{Q}_K}(i) = M_{\mathcal{Q}_{r^*}}(i) \quad \text{for any } i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*). \tag{16}$$

From (14) and (15), we have $\widetilde{D}_{\mathcal{Q}_K}(i) = \max\{d_{0j} : j \in \mathcal{Q}_K \setminus \mathcal{Q}_{r^*}(i) \text{ and } d_{0j} \leq \overline{D}_{\mathcal{Q}_{r^*}}(i)\}$. Now, consider the set

$$\begin{aligned} &\{j : \; j \in \mathcal{Q}_K \setminus \mathcal{Q}_{r^*}(i) \text{ and } d_{0j} \leq \overline{D}_{\mathcal{Q}_{r^*}}(i)\} \\ =\; &\{j : \; j \in (\mathcal{Q}_{r^*} \cup \{r^*+1, \ldots, K\}) \setminus \mathcal{Q}_{r^*}(i) \text{ and } d_{0j} \leq \overline{D}_{\mathcal{Q}_{r^*}}(i)\} \\ =\; &\{j : \; j \in \mathcal{Q}_{r^*} \setminus \mathcal{Q}_{r^*}(i) \text{ and } d_{0j} \leq \overline{D}_{\mathcal{Q}_{r^*}}(i)\}, \end{aligned}$$

where the last step follows from the fact that $\overline{D}_{\mathcal{Q}_{r^*}}(i) \leq d_{0r^*}$ and $d_{0j} > d_{0r^*}$ for all $j \in \{r^*+1, \ldots, K\}$. Thus

$$\widetilde{D}_{\mathcal{Q}_K}(i) = \widetilde{D}_{\mathcal{Q}_{r^*}}(i) \quad \text{for any } i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*). \tag{17}$$

The lemma follows from (15), (16), (17) and the definition of Δ. □

We are now ready to state the $(\mathcal{Q}, S_\mathcal{Q})$ inequalities and prove their validity.

Theorem 2. *Given any $\mathcal{Q} \subseteq \mathcal{V}$ satisfying (A1), (A2), and (A3) and any subset $S_\mathcal{Q} \subseteq \mathcal{V}_\mathcal{Q}$, the inequality*

$$\sum_{i \in S_\mathcal{Q}} x_i + \sum_{i \in \overline{S}_\mathcal{Q}} \Delta_\mathcal{Q}(i) y_i \geq M_\mathcal{Q}(0),$$

where $\overline{S}_\mathcal{Q} = \mathcal{V}_\mathcal{Q} \setminus S_\mathcal{Q}$, called a $(\mathcal{Q}, S_\mathcal{Q})$ inequality, is valid for X_{SLS}.

Proof. We show by induction over $k \in \{1, \ldots, Q\}$ that any $(\mathcal{Q}_k, S_{\mathcal{Q}_k})$ inequality is valid for X_{SLS}.

The base case ($k = 1$): Note that $\overline{D}_{\mathcal{Q}_1}(i) = d_{01}$, $\widetilde{D}_{\mathcal{Q}_1}(i) = 0$, and $M_{\mathcal{Q}_1}(i) = d_{i1}$ for all $i \in \mathcal{V}_{\mathcal{Q}_1}$. Given any point $(x, y) \in X_{\text{SLS}}$, the left-hand-side of the $(\mathcal{Q}_1, S_{\mathcal{Q}_1})$ inequality is given by

$$\sum_{i \in S_{\mathcal{Q}_1}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_1}} \min\{d_{01}, d_{i1}\} y_i = \sum_{i \in S_{\mathcal{Q}_1}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_1}} d_{i1} y_i \geq d_{01} = M_{\mathcal{Q}_1}(0).$$

The first equality follows from the fact that $d_{01} \geq d_{i1}$; the inequality follows from the validity of the (ℓ, S) inequality with $\ell = 1$ and $S = S_{\mathcal{Q}_1}$; the last equality follows from the definition of $M_{\mathcal{Q}_1}(0)$.

The inductive step: We assume that for all $k \in \{1, \dots, K-1\}$ (where $K-1 < Q$), given any $S_{\mathcal{Q}_k} \subseteq \mathcal{V}_{\mathcal{Q}_k}$, the $(\mathcal{Q}_k, S_{\mathcal{Q}_k})$ inequality is valid for X_{SLS}. Consider any $S_{\mathcal{Q}_K} \subseteq \mathcal{V}_{\mathcal{Q}_K}$, we show that the $(\mathcal{Q}_K, S_{\mathcal{Q}_K})$ inequality

$$\sum_{i \in S_{\mathcal{Q}_K}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_K}} \Delta_{\mathcal{Q}_K}(i) y_i \geq M_{\mathcal{Q}_K}(0)$$

is also valid for X_{SLS}.

Let $\mathcal{F}_K = \{i \in \mathcal{P}(K) \cap \overline{S}_{\mathcal{Q}_K} : \ \overline{D}_{\mathcal{Q}_K}(i) - \widetilde{D}_{\mathcal{Q}_K}(i) < M_{\mathcal{Q}_K}(i)\}$. Given any solution $(x, y) \in X_{\mathrm{SLS}}$, we consider two cases: (a) there exists $j^* \in \mathcal{F}_K$ such that $y_{j^*} = 1$, and (b) $y_j = 0$ for all $j \in \mathcal{F}_K$.

Case (a): Note that $\overline{D}_{\mathcal{Q}_K}(j^*) - \widetilde{D}_{\mathcal{Q}_K}(j^*) < M_{\mathcal{Q}_K}(j^*)$ implies $\widetilde{D}_{\mathcal{Q}_K}(j^*) > 0$ since $\overline{D}_{\mathcal{Q}_K}(j^*) \geq M_{\mathcal{Q}_K}(j^*)$. Thus there exists $r^* \in \mathcal{Q}$ such that $\widetilde{D}_{\mathcal{Q}_K}(j^*) = d_{0r^*}$. Let $S_{\mathcal{Q}_{r^*}} = S_{\mathcal{Q}_K} \cap \mathcal{V}_{\mathcal{Q}_{r^*}}$ and $\overline{S}_{\mathcal{Q}_{r^*}} = \overline{S}_{\mathcal{Q}_K} \cap \mathcal{V}_{\mathcal{Q}_{r^*}}$. The left-hand-side of the $(\mathcal{Q}_K, S_{\mathcal{Q}_K})$ inequality is then equal to

$$\sum_{i \in S_{\mathcal{Q}_{r^*}}} x_i + \tag{18}$$

$$\sum_{i \in S_{\mathcal{Q}_K} \setminus S_{\mathcal{Q}_{r^*}}} x_i + \tag{19}$$

$$\sum_{i \in \overline{S}_{\mathcal{Q}_{r^*}}} \Delta_{\mathcal{Q}_K}(i) y_i + \tag{20}$$

$$\sum_{i \in \overline{S}_{\mathcal{Q}_K} \setminus \overline{S}_{\mathcal{Q}_{r^*}}} \Delta_{\mathcal{Q}_K}(i) y_i. \tag{21}$$

As before, let $u^* = \mathrm{argmax}\{t(i) : i \in \mathcal{V}_{\mathcal{Q}_{r^*}} \cap \mathcal{P}(K)\}$. Expression (20) can be further disaggregated into

$$\sum_{i \in \overline{S}_{\mathcal{Q}_{r^*}} \cap \mathcal{P}(u^*)} \Delta_{\mathcal{Q}_K}(i) y_i + \tag{22}$$

$$\sum_{i \in \overline{S}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*)} \Delta_{\mathcal{Q}_K}(i) y_i. \tag{23}$$

From Lemma 1, it follows that

$$(22) \geq \sum_{i \in \overline{S}_{\mathcal{Q}_{r^*}} \cap \mathcal{P}(u^*)} \Delta_{\mathcal{Q}_{r^*}}(i) y_i,$$

and from Lemma 2, it follows that

$$(23) = \sum_{i \in \overline{S}_{\mathcal{Q}_{r^*}} \setminus \mathcal{P}(u^*)} \Delta_{\mathcal{Q}_{r^*}} y_i.$$

From the validity of the $(\mathcal{Q}_{r^*}, S_{\mathcal{Q}_{r^*}})$ inequality, we then have

$$(18) + (22) + (23) \geq M_{\mathcal{Q}_{r^*}}(0) = d_{0r^*}.$$

Now consider the expression (21). Since $j^* \in \overline{S}_{\mathcal{Q}_K} \setminus \overline{S}_{\mathcal{Q}_{r^*}}$ and all coefficients are non-negative, we have that

$$(21) \geq \overline{D}_{\mathcal{Q}_K}(j^*) - \widetilde{D}_{\mathcal{Q}_K}(j^*) = d_{0K} - d_{0r^*}.$$

Thus

$$(18) + (22) + (23) + (21) \geq d_{0K},$$

which implies

$$(18) + (19) + (22) + (23) + (21) \geq d_{0K} = M_{\mathcal{Q}_K}(0).$$

Therefore the $(\mathcal{Q}_K, S_{\mathcal{Q}_K})$ inequality is valid.

Case (b): The left-hand-side of the $(\mathcal{Q}_K, S_{\mathcal{Q}_K})$ inequality equals

$$\begin{aligned}
&\sum_{i \in S_{\mathcal{Q}_K}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_K}} \Delta_{\mathcal{Q}_K}(i) y_i \\
\geq &\sum_{i \in S_{\mathcal{Q}_K} \cap \mathcal{P}(K)} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_K} \cap \mathcal{P}(K)} \Delta_{\mathcal{Q}_K}(i) y_i \\
= &\sum_{i \in S_{\mathcal{Q}_K} \cap \mathcal{P}(K)} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_K} \cap \mathcal{P}(K)} M_{\mathcal{Q}_K}(i) y_i \\
= &\sum_{i \in S_{\mathcal{Q}_K} \cap \mathcal{P}(K)} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_K} \cap \mathcal{P}(K)} d_{iK} y_i \\
\geq &\; d_{0K} = M_{\mathcal{Q}_{K(0)}},
\end{aligned}$$

where the third expression follows from the fact that $y_j = 0$ for all $j \in \overline{S}_{\mathcal{Q}_K} \cap \mathcal{P}(K)$ such that $\overline{D}_{\mathcal{Q}_K}(j) - \widetilde{D}_{\mathcal{Q}_K}(j) < M_{\mathcal{Q}_K}(j)$, the fourth expression follows from the definition of $M_{\mathcal{Q}_K}(j)$, and the fifth expression follows from the validity of the (ℓ, S) inequality with $\ell = K$ and $S = S_{\mathcal{Q}_K} \cap \mathcal{P}(K)$. Therefore the $(\mathcal{Q}_K, S_{\mathcal{Q}_K})$ inequality is valid. □

We conclude this section with a discussion of properties (A1) and (A2) and an example that illustrates the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities. Suppose property (A1) does not hold for some $\mathcal{Q}$. In particular, suppose there exists one pair of nodes $q_1, q_2 \in \mathcal{Q}$ such that $d_{0q_1} = d_{0q_2}$. Without loss of generality, we index the nodes in $\mathcal{Q}$ such that $q_2 > q_1$. Let $\mathcal{Q}' = \mathcal{Q} \setminus \{q_2\}$. Note that $\mathcal{Q}'$ satisfies (A1). From the fact that

$d_{0q_1} = d_{0q_2}$, it can be easily verified that $\Delta_{\mathcal{Q}'}(i) = \Delta_{\mathcal{Q}}(i)$ for all $i \in \mathcal{V}_{\mathcal{Q}'}$ and $M_{\mathcal{Q}'}(0) = M_{\mathcal{Q}}(0)$. Now, let $S_{\mathcal{Q}'} = S_{\mathcal{Q}} \cap \mathcal{V}_{\mathcal{Q}'}$ and $\overline{S}_{\mathcal{Q}'} = \overline{S}_{\mathcal{Q}} \cap \mathcal{V}_{\mathcal{Q}'}$. Then

$$\begin{aligned}
&\sum_{i \in S_{\mathcal{Q}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y_i \\
&\geq \sum_{i \in S_{\mathcal{Q}'}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}'}} \Delta_{\mathcal{Q}}(i) y_i \\
&= \sum_{i \in S_{\mathcal{Q}'}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}'}} \Delta_{\mathcal{Q}'}(i) y_i \\
&\geq M_{\mathcal{Q}'}(0) = M_{\mathcal{Q}}(0).
\end{aligned}$$

Thus the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is valid. However, this inequality is clearly dominated by the $(\mathcal{Q}', S_{\mathcal{Q}'})$ inequality. Consequently, (A1) is without loss of generality.

Suppose property (A2) does not hold for some $\mathcal{Q}$ and there exists one pair of nodes $q_1, q_2 \in \mathcal{Q}$ such that $q_1 \in \mathcal{P}(q_2)$. Then $\mathcal{V}_{\mathcal{Q}} = \mathcal{V}_{\mathcal{Q} \setminus \{q_1\}}$ and we only need to consider $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities corresponding to $\mathcal{Q} \setminus \{q_1\}$ instead of $\mathcal{Q}$. Consequently, (A2) is without loss of generality.

Example: Consider an instance of (SLSP) with 7 nodes as shown in Figure 2. The problem parameters are shown in the columns labelled $\overline{\alpha}_i$, $\overline{\beta}_i$ and d_i in Table 1. The optimal LP relaxation objective value of (SLSP) is 2654.27 and the corresponding optimal solution (x, y) is shown in the columns labelled x^1 and y^1 in Table 1. We augment the LP relaxation with 5 $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities:

$$\begin{aligned}
10y_0 &\geq 10 \text{ i.e., } \mathcal{Q} = \{0\}, \overline{S}_{\mathcal{Q}} = \{0\} \\
x_0 + x_1 + 5y_2 &\geq 30 \text{ i.e., } \mathcal{Q} = \{1,2\}, \overline{S}_{\mathcal{Q}} = \{2\} \\
x_0 + x_1 + 10y_3 &\geq 35 \text{ i.e., } \mathcal{Q} = \{1,3\}, \overline{S}_{\mathcal{Q}} = \{3\} \\
x_0 + x_2 + x_4 + x_3 + 10y_6 &\geq 45 \text{ i.e., } \mathcal{Q} = \{4,6\}, \overline{S}_{\mathcal{Q}} = \{6\} \\
x_0 + x_2 + x_4 + x_1 + 10y_5 &\geq 45 \text{ i.e., } \mathcal{Q} = \{4,5\}, \overline{S}_{\mathcal{Q}} = \{5\}.
\end{aligned}$$

Then we obtain an integral optimal solution (as shown in columns labelled x^2 and y^2 in Table 1) with the corresponding optimal objective value of 3117.

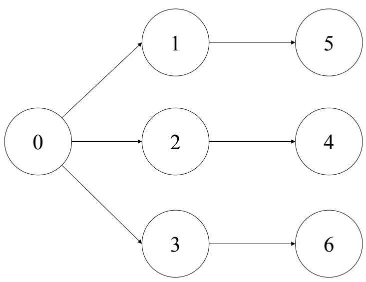

Fig. 2. An example

	$\overline{\alpha}_i$	$\overline{\beta}_i$	d_i	x^1	y^1	x^2	y^2
0	100	1	10	25	0.56	25	1
1	0	300	20	5	0.17	15	1
2	0	6000	15	0	0.00	0	0
3	0	300	25	10	0.29	20	1
4	0	1	10	10	1.00	0	0
5	1	1	15	15	1.00	15	1
6	0	1	10	10	1.00	0	0

Table 1. An example

4. Facets for the Stochastic Lot-Sizing Problem

In this section we give some classes of facets for the stochastic lot-sizing polyhedron. First, we identity some facets from the original inequalities defining X_{SLS}. Next, we provide necessary and sufficient conditions under which a $(\mathcal{Q}, S_\mathcal{Q})$ inequality is facet-defining.

We make the following assumption throughout the remainder of this paper.

(A4) $d_i > 0$ for all $i \in \mathcal{V}$.

Under assumption (A4), the following results can be shown by constructing appropriate sets of affinely independent solutions. Recall that $|\mathcal{V}| = N$.

Proposition 1. *The dimension of* X_{SLS} *is* $2N - 1$.

Proposition 2. *The inequalities*

(i) $x_i \le M_i y_i$ *for* $i \in \mathcal{V} \setminus \{0\}$,
(ii) $y_i \le 1$ *for* $i \in \mathcal{V} \setminus \{0\}$,
(iii) $x_i \ge 0$ *for* $i \in \mathcal{V} \setminus \{0\}$,

are facet-defining for X_{SLS}.

Note that, the inequalities $y_i \ge 0$, $i \in \mathcal{V} \setminus \{0\}$, are not facet-defining. This is because $y_i = 0$ implies $x_i = 0$, and therefore we can have no more than $2N - 2$ affinely independent solutions satisfying $y_i = 0$.

We now establish a set of conditions guaranteeing that a $(\mathcal{Q}, S_\mathcal{Q})$ inequality is facet-defining. Let $\mathcal{F}_\mathcal{Q} = \{i \in \overline{S}_\mathcal{Q} : \overline{D}_\mathcal{Q}(i) - \widetilde{D}_\mathcal{Q}(i) < M_\mathcal{Q}(i)\}$ and $\mathcal{G}_\mathcal{Q} = \overline{S}_\mathcal{Q} \setminus \mathcal{F}_\mathcal{Q}$. Thus, $\mathcal{V}_\mathcal{Q} = \mathcal{F}_\mathcal{Q} \cup \mathcal{G}_\mathcal{Q} \cup S_\mathcal{Q}$. We need the following definitions.

Definition 1. *Given* $\mathcal{Q} \subseteq \mathcal{V}$ *and* $S_\mathcal{Q} \subseteq \mathcal{V}_\mathcal{Q}$, *the* neighborhood *of* $(\mathcal{Q}, S_\mathcal{Q})$ *is*

$$\mathcal{N}(\mathcal{Q}, S_\mathcal{Q}) = \bigcup_{i \in \mathcal{V}_\mathcal{Q} \setminus (\cup_{j \in \overline{S}_\mathcal{Q}} \mathcal{V}_\mathcal{Q}(j))} \mathcal{C}(i) \setminus \mathcal{V}_\mathcal{Q}.$$

For example, in Figure 3, let $\mathcal{Q} = \{1, 2, 3, 4\}$ and $S_\mathcal{Q} = \{0, 3, 5, 9\}$, then $\mathcal{N}(\mathcal{Q}, S_\mathcal{Q})$ contains the two nodes shaded horizontally.

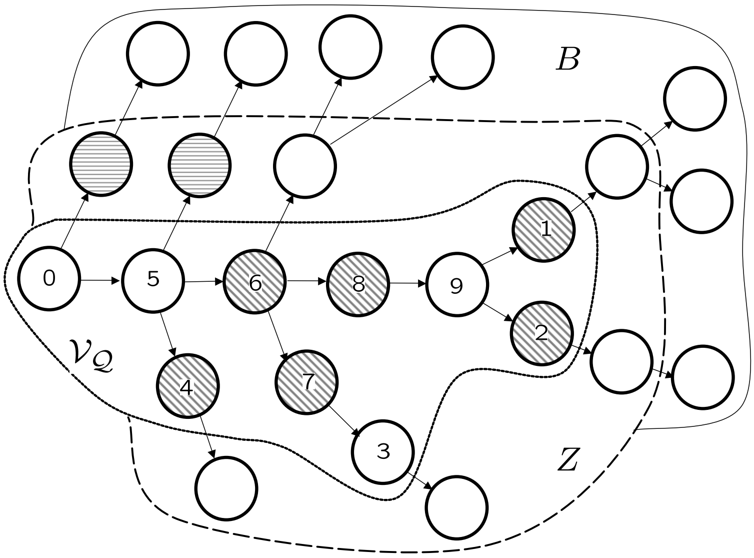

Fig. 3. Partitioning of the node set $\mathcal{V}$ used in Theorem 3

Definition 2. *Given $j \in \mathcal{V}_{\mathcal{Q}}$, let $q_j = \max\{i : \ i \in \mathcal{Q}(j)\}$ and*

$$\mathcal{W}(j) = \bigcup_{i \in \mathcal{Q} \setminus \mathcal{Q}_{q_j}} \operatorname{argmin} \left\{ t(m) : m \in \overline{S}_{\mathcal{Q}} \cap \mathcal{P}(i) \setminus \mathcal{V}_{\mathcal{Q}_{q_j}} \right\}.$$

For example, in Figure 3, if $j = 9$ then $q_j = 2$ and $\mathcal{W}(j) = \{4, 7\}$; and if $j = 6$ then $q_j = 3$ and $\mathcal{W}(j) = \{4\}$.

Theorem 3. *The $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality*

$$\sum_{i \in S_{\mathcal{Q}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y_i \geq M_{\mathcal{Q}}(0)$$

is facet-defining if and only if

(i) $0 \in S_{\mathcal{Q}}$,
(ii) $M_{\mathcal{Q}}(0) \geq \max_{i \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})} \{d_{0i}\}$,
(iii) *For each $j \in \mathcal{V}_{\mathcal{Q}}$,*
  (a) $\mathcal{W}(j) \cap \mathcal{P}(i) \neq \emptyset, \ \forall i \in \mathcal{Q} \setminus \mathcal{Q}_{q_j}$,
  (b) If $j \in \mathcal{F}_{\mathcal{Q}}$, then $\widetilde{D}_{\mathcal{Q}}(j) \geq d_{0a(k)}, \ \forall k \in \mathcal{W}(j)$,
  (c) If $j \in \mathcal{G}_{\mathcal{Q}}$, then $d_{0a(j)} \geq d_{0a(k)}, \ \forall k \in \mathcal{W}(j)$,
  (d) If $j \in S_{\mathcal{Q}}$, then $\overline{D}_{\mathcal{Q}}(j) > d_{0a(k)}, \ \forall k \in \mathcal{W}(j)$,
(iv) *$(\cup_{i \in \mathcal{G}_{\mathcal{Q}}} argmax\{j : j \in \mathcal{Q}(i)\}) \cap \mathcal{L} = \emptyset$.*

Proof. The proof is constructive and the details are given in the Appendix.

Example (continued): Consider the five inequalities added in the example. The first one is not facet-defining since $0 \notin S_{\mathcal{Q}}$. The second one is not facet-defining since it does not satisfy condition (ii). The fourth one is not facet-defining since $\overline{D}_{\mathcal{Q}}(4) = d_{0a(6)}$ and $6 \in \mathcal{W}(4)$, which contradicts condition (d) of (iii). However, the third and fifth inequalities are facet-defining.

Recall that any (ℓ, S) inequality is a $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality with $\mathcal{Q} = \{\ell\}$ and $S_{\mathcal{Q}} = S$. We then have the following corollary to Theorem 3.

Corollary 1. *An (ℓ, S) inequality is facet-defining if and only if ℓ and S are such that $0 \in S$, $d_{0\ell} \geq \max_{i \in N(\ell,S)} d_{0i}$ and $\mathcal{P}(\ell) \setminus S \neq \emptyset$, $\ell \notin \mathcal{L}$ or $\mathcal{P}(\ell) \setminus S = \emptyset$, $\ell \in \mathcal{L}$.*

In this case, the neighborhood is simply $N(\ell, S) = \{j : j \in \mathcal{C}(i) \setminus \mathcal{P}(\ell)$ where $i < \operatorname{argmin}\{t(k) : k \in \overline{S}\}\}$, and condition (iii) is redundant.

5. Separation of $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities

Given the set $\mathcal{Q}$, and a fractional solution (x^*, y^*) of (SLS), let

$$S^*_{\mathcal{Q}} = \{i \in \mathcal{V}_{\mathcal{Q}} : \ x^*_i \leq \Delta_{\mathcal{Q}}(i) y^*_i\}. \tag{24}$$

If $\sum_{i \in S^*_{\mathcal{Q}}} x^*_i + \sum_{i \in \overline{S}^*_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y^*_i < M_{\mathcal{Q}}(0)$, then the $(\mathcal{Q}, S^*_{\mathcal{Q}})$ inequality is violated. On the other hand, if (x^*, y^*) satisfies the $(\mathcal{Q}, S^*_{\mathcal{Q}})$ inequality then there are no violated $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities corresponding to the node set $\mathcal{Q}$, since

$$\min_{S_{\mathcal{Q}} \subseteq \mathcal{V}_{\mathcal{Q}}} \left\{ \sum_{i \in S_{\mathcal{Q}}} x^*_i + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y^*_i \right\} = \sum_{i \in S^*_{\mathcal{Q}}} x^*_i + \sum_{\overline{S}^*_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y^*_i \geq M_{\mathcal{Q}}(0).$$

The difficulty in separating $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities is how to determine $\mathcal{Q}$. The $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities with $|\mathcal{Q}| = Q$ can be separated in $\mathcal{O}(N^{Q+1})$ time. Since there are $\binom{N}{Q}$ ways to choose a node set $\mathcal{Q}$, and for each such $\mathcal{Q}$, we can check for a violated $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality in $\mathcal{O}(N)$ time. Since separation of $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities is probably NP-hard, we check for all of the $|\mathcal{Q}| = 1$ and $|\mathcal{Q}| = 2$ inequalities for violations and then we apply a heuristic (Algorithm 1) to try to find some violated inequalities for larger $|\mathcal{Q}|$.

The basic idea of Algorithm 1 is to add nodes to $\mathcal{Q}$, using a depth-first strategy, such that the right-hand-side of the inequality is not changed while the left-hand-side decreases. The process stops as soon as we find a violated $(\mathcal{Q}, S^*_{\mathcal{Q}})$ inequality. If no violated inequality is found after exhausting the depth-first search, we re-start the search with a new node.

6. Computational Experiments

In this section, we report on the computational effectiveness of the proposed $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities on randomly generated instances of single-item, uncapacitated, stochastic lot-sizing problems.

Algorithm 1 Heuristic separation of $\{\mathcal{Q}, S_{\mathcal{Q}}\}$ inequalities with $|\mathcal{Q}| \geq 3$

Input: a fractional solution (x^*, y^*).
for $\ell \in \mathcal{V}$ **do**
  Step 0. Set $\mathcal{Q} = \{\ell\}$ and $i = \ell$.
  Step 1. If $|\mathcal{Q}| \geq 3$, go to Step 2. Otherwise, go to Step 3.
  Step 2. Compute $S^*_{\mathcal{Q}}$ as in (24). If the $(\mathcal{Q}, S^*_{\mathcal{Q}})$ inequality is violated **stop**.
  Step 3. For some node $j \in \mathcal{V}(a(i)) \setminus \mathcal{V}(i)$, let $\mathcal{Q}' = \mathcal{Q} \cup \{j\}$. If a node $k = \text{argmax}\{d_{0j} : \ j \in \mathcal{V}(a(i)) \setminus \mathcal{V}(i),\ d_{0j} < d_{0i}$ and $\sum_{i \in S^*_{\mathcal{Q}'}} x^*_i + \sum_{i \in \overline{S}^*_{\mathcal{Q}'}} \Delta_{\mathcal{Q}'}(i) y^*_i < \sum_{i \in S^*_{\mathcal{Q}}} x^*_i + \sum_{i \in \overline{S}^*_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y^*_i\}$ exists, go to Step 5. Otherwise, go to Step 4.
  Step 4. If $i \neq 0$, set $i \leftarrow a(i)$ and **go to** Step 3. If $i = 0$ **end for**.
  Step 5. Set $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{k\}$ and $i \leftarrow k$ and **go to** Step 1.
end for

6.1. Implementation

We implemented a branch-and-cut scheme in which complete separation of $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities is done for $|\mathcal{Q}| = 1$ and $|\mathcal{Q}| = 2$ followed by Algorithm 1. We add all violated $|\mathcal{Q}| = 1$ inequalities if some are found and repeat until no more are found. We do the same for $|\mathcal{Q}| = 2$ inequalities. When no more of these are found, we apply Algorithm 1 and add inequalities one-at-a-time until no further violation is found.

Our implementation was carried out in C using the callable libraries of CPLEX 8.1. Default CPLEX options were used throughout. All computations were carried out on a 2.4GHz Intel Xeon/Linux workstation with 2GB RAM with one hour time limit per run.

6.2. Test problem generation

A number of instances of (SLS) were generated corresponding to different structures of the underlying scenario trees, different ratios of the production cost to the inventory holding cost, and different ratios of the setup cost to the inventory holding cost.

We assumed that the underlying scenario tree is balanced with T stages and K branches per stage. We considered 6 different tree structures with $K = 2$ and $T \in \{10, 11\}$; $K = 3$ and $T \in \{6, 7\}$; $K = 4$ and $T \in \{5, 6\}$. We considered three different levels of production to holding cost ratio $\alpha/h \in \{50, 100, 200\}$, and three different levels of setup to holding cost ratio $\beta/h \in \{1750, 3500, 7000\}$.

For each of the 54 combinations of the tree structure, α/h and β/h, we generated three random instances as follows. For each node i of the tree, the holding cost $h_i \sim U[0.01, 0.05]$, i.e., a uniform random number in the interval $[0.01, 0.05]$; $\alpha_i \sim U[0.8(\alpha/h)\bar{h}, 1.2(\alpha/h)\bar{h}]$ where $\bar{h} = 0.03$ is the average holding cost; $\beta_i \sim U[0.8(\beta/h)\bar{h}, 1.2(\beta/h)\bar{h}]$; and $d_i \sim U[10, 100]$. Finally, each of the K children of a node was assigned equal probability.

6.3. Results

Tables 2, 3, and 4 report on the effectiveness of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities in tightening the LP relaxation gap for the instances corresponding to $K = 2, 3$

and 4, respectively. The column labelled LP Gap % reports the relative LP relaxation gap of the original formulation (SLS) with respect to the best feasible solution found with our branch and cut scheme. The columns labelled $|\mathcal{Q}| = 1$, $|\mathcal{Q}| = 2$ and General $\mathcal{Q}$ correspond to the results from separating all $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities for $|\mathcal{Q}| = 1$ and then those for $|\mathcal{Q}| = 2$, and then for heuristically separating some of the general $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities from the LP relaxation of (SLS), respectively. For each combination of T, β/h and α/h, there are two rows corresponding to the columns labelled $|\mathcal{Q}| = 1$, $|\mathcal{Q}| = 2$ and General $\mathcal{Q}$. The first row reports the LP relaxation gap after adding the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities, and the second row reports the number of $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities added. Note that all reported numbers are averages over three instances. Significant tightening of the LP relaxation is achieved via the proposed $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities. In some cases, the LP relaxation gap is reduced from over 20% to 0.4%. Furthermore, in most cases, the LP relaxation gap is small after adding the inequalities corresponding to $|\mathcal{Q}| = 1$ and $|\mathcal{Q}| = 2$.

The results from our branch and cut scheme are reported in Tables 5, 6, and 7 for the instances corresponding to $K = 2, 3$ and 4, respectively. For each combination of T, β/h and α/h, there are two rows. The first row reports on the performance of the default CPLEX MIP solver and the second row reports on the performance of our branch and cut scheme. We give the number of cutting planes added by the default CPLEX MIP solver and by our branch and cut scheme respectively, the relative optimality gap upon termination, the number of nodes explored (apart from the root node), and the total CPU time. The reported data is averaged over three instances. The numbers in square brackets indicate the number of instances *not* solved to default CPLEX optimality tolerance within the allotted time limit of one hour. The default CPLEX MIP solver adds several types of cuts including flow covers, Gomory fractional cuts and mixed integer rounding cuts. Our branch and cut algorithm adds $(\mathcal{Q}, S_{\mathcal{Q}})$ cuts at each node after the CPLEX default cuts have been added. For the total CPU time, we report the average CPU time for instances that are solved to default CPLEX optimality tolerance within the allotted time limit of one hour. Otherwise, we use "$_{***}$" to represent the case that *no* instance can be solved to default CPLEX optimality tolerance within the allotted time. The efficiency of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities within our branch and cut is clearly observed. Our branch and cut algorithm proves optimality for all instances for $K = 2$, has only 11 and 25 instances unsolved to optimality for $K = 3$ and $K = 4$, respectively. In contrast, the unsolved instances corresponding to default CPLEX are 6, 43 and 52, respectively. For cases where neither algorithm could prove optimality, our algorithm yielded much smaller optimality gaps. Moreover, our cuts dramatically reduced the number of nodes in the tree and, although we added many more cuts, the running times were smaller as well. Because we add so many $(\mathcal{Q}, S_{\mathcal{Q}})$ inequalities, we thought that the running times might be reduced substantially by deleting cuts that were no longer tight. However, experiments using cut management did not yield significant improvement.

Table 2. Results for the root node ($K = 2$)

T	β/h	α/h	LP Gap %	$\lvert\mathcal{Q}\rvert = 1$	$\lvert\mathcal{Q}\rvert = 2$	General $\mathcal{Q}$
10	1750	50	7.19	0.04	0.01	0.01
				3473	1185	18
10	1750	100	6.60	0.04	0.00	0.00
				3492	1238	19
10	1750	200	5.28	0.04	0.00	0.00
				3451	1124	0
10	3500	50	13.06	0.11	0.01	0.01
				3424	2513	51
10	3500	100	12.10	0.10	0.01	0.01
				3374	2630	80
10	3500	200	9.87	0.08	0.00	0.00
				3433	1868	12
10	7000	50	22.13	0.19	0.02	0.01
				3183	4267	98
10	7000	100	20.81	0.26	0.02	0.01
				3420	3679	84
10	7000	200	17.35	0.35	0.09	0.02
				3238	4718	310
11	1750	50	2.75	0.02	0.01	0.01
				7953	2769	29
11	1750	100	2.61	0.02	0.00	0.00
				7958	2331	12
11	1750	200	2.26	0.01	0.00	0.00
				7880	2233	7
11	3500	50	5.25	0.06	0.02	0.01
				7691	6675	291
11	3500	100	4.99	0.04	0.00	0.00
				7769	5177	125
11	3500	200	4.36	0.03	0.00	0.00
				7911	3204	24
11	7000	50	9.57	0.16	0.02	0.02
				7179	12042	280
11	7000	100	9.21	0.16	0.02	0.02
				7437	9968	223
11	7000	200	8.17	0.11	0.01	0.01
				7656	7452	71

Table 3. Results for the root node ($K = 3$)

T	β/h	α/h	LP Gap %	$\lvert\mathcal{Q}\rvert = 1$	$\lvert\mathcal{Q}\rvert = 2$	General $\mathcal{Q}$
6	1750	50	10.03	0.62	0.04	0.03
				1560	3243	98
6	1750	100	8.26	0.65	0.06	0.04
				1479	4139	144
6	1750	200	5.36	0.54	0.02	0.01
				1438	5784	33
6	3500	50	16.29	1.29	0.28	0.19
				1464	6553	311
6	3500	100	13.76	1.24	0.21	0.17
				1442	6939	120
6	3500	200	9.39	0.95	0.06	0.05
				1436	7412	78
6	7000	50	23.52	1.97	0.38	0.27
				1365	10041	334
6	7000	100	20.93	2.18	0.40	0.31
				1422	10044	335
6	7000	200	15.59	1.81	0.24	0.17
				1405	12248	183
7	1750	50	4.90	0.28	0.04	0.03
				5706	9580	423
7	1750	100	4.38	0.33	0.03	0.02
				5524	12058	298
7	1750	200	3.32	0.28	0.01	0.01
				5341	15223	77
7	3500	50	8.51	0.55	0.08	0.06
				5434	19017	894
7	3500	100	7.75	0.55	0.06	0.05
				5384	20521	466
7	3500	200	6.12	0.52	0.04	0.03
				5335	21474	361
7	7000	50	14.04	0.75	0.16	0.13
				5147	26233	588
7	7000	100	13.03	0.82	0.16	0.13
				5184	28916	590
7	7000	200	10.64	0.85	0.13	0.10
				5197	29711	592

Table 4. Results for the root node ($K = 4$)

T	β/h	α/h	LP Gap %	$\lvert\mathcal{Q}\rvert = 1$	$\lvert\mathcal{Q}\rvert = 2$	General $\mathcal{Q}$
5	1750	50	8.80	1.35	0.21	0.17
				1905	7381	133
5	1750	100	7.42	1.25	0.15	0.08
				1894	7347	213
5	1750	200	4.66	1.47	0.09	0.08
				1651	18741	61
5	3500	50	13.12	1.68	0.27	0.21
				1852	10956	215
5	3500	100	11.40	1.88	0.29	0.20
				1842	12182	369
5	3500	200	7.52	2.33	0.30	0.22
				1619	21298	321
5	7000	50	14.06	1.53	0.33	0.24
				1781	13067	1838
5	7000	100	17.32	3.36	0.75	0.60
				1679	18449	341
5	7000	200	12.10	3.28	0.71	0.52
				1546	32367	477
6	1750	50	4.25	0.53	0.07	0.05
				9779	28553	797
6	1750	100	3.73	0.66	0.08	0.06
				9310	53983	904
6	1750	200	2.92	0.69	0.05	0.04
				8561	70253	336
6	3500	50	7.17	0.88	0.17	0.12
				9380	65631	1438
6	3500	100	6.41	1.05	0.20	0.16
				8979	75479	1318
6	3500	200	5.24	1.12	0.15	0.12
				8487	74747	645
6	7000	50	11.20	1.32	0.35	0.27
				8589	89049	1658
6	7000	100	10.31	1.55	0.45	0.39
				8339	93640	1160
6	7000	200	8.84	1.62	0.42	0.35
				8383	98949	1358

Table 5. Results for branch and cut ($K = 2$)

T	β/h	α/h	No. of cuts	Optimality gap %	Nodes	CPU secs
10	1750	50	519	0.00	1239	4.4
			4676	0.00	0	0.7
10	1750	100	505	0.00	103	1.6
			4749	0.00	0	0.6
10	1750	200	464	0.00	4	0.7
			4575	0.00	0	0.5
10	3500	50	612	0.00	131850	220.2
			5996	0.00	0	3.0
10	3500	100	598	0.00	39828	70.8
			6129	0.00	0	5.4
10	3500	200	513	0.00	343	2.4
			5313	0.00	0	1.8
10	7000	50	671	0.00	1336827	2619.7
			7737	0.00	0	13.9
10	7000	100	682	0.00	915006	1715.7
			7213	0.00	0	5.0
10	7000	200	597	0.00	13124	26.0
			8407	0.00	0	23.5
11	1750	50	882	0.00	30	2.5
			10751	0.00	0	1.7
11	1750	100	859	0.00	3	1.9
			10301	0.00	0	1.6
11	1750	200	780	0.00	3	1.2
			10120	0.00	0	1.6
11	3500	50	1065	0.00	644407	820.2
			14946	0.00	0	63.5
11	3500	100	994	0.00	9807	42.9
			13071	0.00	0	3.3
11	3500	200	852	0.00	889	9.2
			11139	0.00	0	2.5
11	7000	50	1126	0.03[3]	826644	***
			20784	0.00	0	189.0
11	7000	100	1112	0.03[3]	907471	***
			17796	0.00	0	35.9
11	7000	200	1084	0.00	414122	1496.7
			15179	0.00	0	15.5

Table 6. Results for branch and cut ($K = 3$)

T	β/h	α/h	No. of cuts	Optimality gap %	Nodes	CPU secs
6	1750	50	523	0.01[1]	1010894	60.1
			4957	0.00	0	3.0
6	1750	100	551	0.00	157889	244.9
			5896	0.00	4	9.2
6	1750	200	489	0.00	4913	9.1
			7259	0.00	0	1.5
6	3500	50	575	0.12[3]	2703911	***
			9507	0.00	373	91.7
6	3500	100	573	0.14[3]	2787691	***
			9618	0.00	438	131.9
6	3500	200	540	0.00	253920	387.6
			9091	0.00	20	11.2
6	7000	50	507	0.23[3]	2879642	***
			13746	0.00	9409	2207.5
6	7000	100	528	0.39[3]	3154270	***
			14552	0.05[2]	8356	867.3
6	7000	200	609	0.57[3]	2777630	***
			15072	0.02[2]	5533	90.1
7	1750	50	1236	0.09[3]	1148262	***
			15971	0.00	0	31.7
7	1750	100	1220	0.07[3]	1181449	***
			18187	0.00	13	85.1
7	1750	200	1117	0.02[3]	967725	***
			20653	0.00	0	19.3
7	3500	50	1306	0.21[3]	1076628	***
			28354	0.00	2751	3218.1
7	3500	100	1300	0.17[3]	1089148	***
			27531	0.00	286	724.6
7	3500	200	1209	0.10[3]	1059317	***
			27589	0.00	0	143.4
7	7000	50	1255	0.31[3]	1045952	***
			35932	0.02[1]	2172	3078.9
7	7000	100	1340	0.29[3]	1004477	***
			37756	0.02[3]	2000	***
7	7000	200	1332	0.27[3]	1085362	***
			38215	0.02[3]	1768	***

Table 7. Results for branch and cut ($K = 4$)

T	β/h	α/h	No. of cuts	Optimality gap %	Nodes	CPU secs
5	1750	50	670	0.12[3]	2185170	***
			10158	0.00	251	59.1
5	1750	100	660	0.03[3]	1925658	***
			9585	0.00	47	24.4
5	1750	200	575	0.09[2]	1858810	1506.3
			20931	0.00	24	98.2
5	3500	50	694	0.10[3]	1997388	***
			13399	0.00	1794	356.7
5	3500	100	716	0.15[3]	2257218	***
			14643	0.00	208	99.7
5	3500	200	673	0.21[3]	2174847	***
			24571	0.00	480	636.9
5	7000	50	642	0.04[2]	1175516	213.9
			18065	0.00	806	1275.2
5	7000	100	858	0.37[3]	1570320	***
			25026	0.10[2]	2057	3451.2
5	7000	200	620	0.33[3]	2009171	***
			36770	0.07[2]	600	993.6
6	1750	50	2071	0.22[3]	658145	***
			40204	0.00	155	817.5
6	1750	100	2043	0.24[3]	643715	***
			67106	0.01[3]	483	***
6	1750	200	1810	0.17[3]	708248	***
			80495	0.00	198	2003.2
6	3500	50	1984	0.42[3]	633599	***
			79711	0.05[3]	425	***
6	3500	100	1987	0.47[3]	619146	***
			88734	0.07[3]	143	***
6	3500	200	1973	0.37[3]	630579	***
			85886	0.04[3]	112	***
6	7000	50	1771	0.67[3]	611857	***
			102151	0.14[3]	46	***
6	7000	100	2048	0.72[3]	617064	***
			105606	0.24[3]	0	***
6	7000	200	2022	0.57[3]	634604	***
			112756	0.24[3]	0	***

Appendix

Theorem 3. *The* $(\mathcal{Q}, S_{\mathcal{Q}})$ *inequality*

$$\sum_{i \in S_{\mathcal{Q}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y_i \geq M_{\mathcal{Q}}(0)$$

is facet-defining if and only if

(i) $0 \in S_{\mathcal{Q}}$,
(ii) $M_{\mathcal{Q}}(0) \geq \max_{i \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})}\{d_{0i}\}$,
(iii) *For each* $j \in \mathcal{V}_{\mathcal{Q}}$,
  (a) $\mathcal{W}(j) \cap \mathcal{P}(i) \neq \emptyset, \ \forall i \in \mathcal{Q} \setminus \mathcal{Q}_{q_j}$,
  (b) *If* $j \in \mathcal{F}_{\mathcal{Q}}$, *then* $\widetilde{D}_{\mathcal{Q}}(j) \geq d_{0a(k)}, \ \forall k \in \mathcal{W}(j)$,
  (c) *If* $j \in \mathcal{G}_{\mathcal{Q}}$, *then* $d_{0a(j)} \geq d_{0a(k)}, \ \forall k \in \mathcal{W}(j)$,
  (d) *If* $j \in S_{\mathcal{Q}}$, *then* $\overline{D}_{\mathcal{Q}}(j) > d_{0a(k)}, \ \forall k \in \mathcal{W}(j)$,
(iv) $(\cup_{i \in \mathcal{G}_{\mathcal{Q}}} \mathit{argmax}\{j : j \in \mathcal{Q}(i)\}) \cap \mathcal{L} = \emptyset$.

Proof of sufficiency.
We first describe the construction of $2N-1$ vectors that are in X_{SLS}, and satisfy the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality. Then we show that the vectors are linearly independent.

Given the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality, we partition $\mathcal{V}$ into disjoint sets $\mathcal{V} = \{0\} \cup A \cup Z \cup B$, where $A = \mathcal{V}_{\mathcal{Q}} \setminus \{0\}$, $Z = \{j : j \in \mathcal{V} \setminus \mathcal{V}_{\mathcal{Q}} \text{ and } a(j) \in \mathcal{V}_{\mathcal{Q}}\}$ and $B = \mathcal{V} \setminus (\mathcal{V}_{\mathcal{Q}} \cup Z)$. Note that we have $\mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}}) \subseteq Z$. Nodes in the set $\mathcal{V} \setminus \mathcal{V}_{\mathcal{Q}}$ correspond to a forest, and Z represents the set of root nodes of the subtrees in this forest. This partitioning is illustrated in Figure 3. Here $\mathcal{Q} = \{1, 2, 3, 4\}$, $\mathcal{V}_{\mathcal{Q}} = \{0, 1, 2, \ldots, 9\}$, $S_{\mathcal{Q}} = \{0, 3, 5, 9\}$, $\overline{S}_{\mathcal{Q}} = \{1, 2, 4, 6, 7, 8\}$ (shaded diagonally), and $A = \{1, 2, \ldots, 9\}$. The two horizontally shaded nodes in Z represent $\mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$.

Construction: We create one vector u^0 for the root node $\{0\}$ and two vectors u^j and v^j for each node $j \in \mathcal{V} \setminus \{0\}$.

We let

$$u^0 = M_{\mathcal{Q}}(0) e^{x_0} + e^{y_0} + \sum_{i \in Z} (M_i e^{x_i} + e^{y_i}),$$

where e^{x_i} and e^{y_i} are unit vectors in $\mathbb{R}^{2N}$ corresponding to the coordinates x_i and y_i, respectively.

$j \in B$: We let

$$\begin{aligned} u^j &= u^0 + e^{y_j}, \quad \text{and} \\ v^j &= u^0 + M_j e^{x_j} + e^{y_j}. \end{aligned}$$

$j \in A$:

If $j \in S_{\mathcal{Q}}$, we let

$$\begin{aligned} u^j &= u^0 + (\overline{D}_{\mathcal{Q}}(j) - \varepsilon - M_{\mathcal{Q}}(0))e^{x_0} \\ &\quad + \varepsilon e^{x_j} + e^{y_j} \\ &\quad + \textstyle\sum_{i \in \mathcal{W}(j)} (M_{\mathcal{Q}}(i) e^{x_i} + e^{y_i}), \\ &\quad \text{where } \varepsilon \text{ is a sufficiently small positive number, and} \\ v^j &= u^0 + e^{y_j}. \end{aligned}$$

If $j \in \overline{S}_{\mathcal{Q}}$, we let

$$\begin{aligned} u^j &= u^0 + (\overline{D}_{\mathcal{Q}}(j) - \Delta_{\mathcal{Q}}(j) - M_{\mathcal{Q}}(0))e^{x_0} \\ &\quad + \Delta_{\mathcal{Q}}(j) e^{x_j} + e^{y_j} \\ &\quad + \textstyle\sum_{i \in \mathcal{W}(j)} (M_{\mathcal{Q}}(i) e^{x_i} + e^{y_i}) \quad \text{and} \\ v^j &= u^j + \varepsilon e^{x_j}. \end{aligned}$$

$j \in Z$:
If $j \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$, we let

$$\begin{aligned} u^j &= u^0 - M_j e^{x_j} - e^{y_j} + \textstyle\sum_{i \in B} (M_i e^{x_i} + e^{y_i}) \quad \text{and} \\ v^j &= u^j + e^{y_j}. \end{aligned}$$

If $j \in Z \setminus \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$, define $k_j = \text{argmin}\{t(i) : \ i \in \overline{S}_{\mathcal{Q}} \cap \mathcal{P}(j)\}$. Note that $k_j \in \overline{S}_{\mathcal{Q}}$ by definition. We let

$$\begin{aligned} u^j &= u^{k_j} + (M_{k_j} - \Delta_{\mathcal{Q}}(k_j))e^{x_{k_j}} - M_j e^{x_j} - e^{y_j} \quad \text{and} \\ v^j &= u^j + e^{y_j}. \end{aligned}$$

Feasibility: It is obvious that $u^0 \in X_{SLS}$. Consequently, the vectors $\{u^j, v^j\}_{j \in B}$ and $\{v^j\}_{j \in S_{\mathcal{Q}}}$ are also feasible.

Now we verify the feasibility of u^j for $j \in \overline{S}_{\mathcal{Q}}$. Given $j \in \overline{S}_{\mathcal{Q}}$, u^j satisfies $0 \le x_i \le M_i y_i$ and $y_i \in \{0, 1\}$ for all $i \in \mathcal{V}$ since $x_0 < M_{\mathcal{Q}}(0) \le M_0$, $\Delta_{\mathcal{Q}}(i) \le M_{\mathcal{Q}}(i) \le M_i$ and $M_{\mathcal{Q}}(k) \le M_k$ $\forall k \in \mathcal{W}(j)$. Therefore, we just need to check that u^j satisfies constraint (2) for all $i \in \mathcal{V} = \{0\} \cup A \cup Z \cup B$.

Clearly u^j satisfies constraint (2) for $i = 0$. Also, note that if u^j satisfies constraint (2) for $i \in \{0\} \cup A$, then it satisfies constraint (2) for $i \in Z \cup B$ since $x_i = M_i$ and $y_i = 1$ for all $i \in Z$, and the nodes in Z include an ancestor of each node in B. Therefore, we just need to show that u^j satisfies constraint (2) for $i \in A = S_{\mathcal{Q}} \cup \overline{S}_{\mathcal{Q}}$.

Note that u^j yields

$$\begin{aligned} x_0 &= \overline{D}_{\mathcal{Q}}(j) - \Delta_{\mathcal{Q}}(j) \\ &\ge \overline{D}_{\mathcal{Q}}(j) - M_{\mathcal{Q}}(j) \\ &= d_{0a(j)}, \end{aligned} \tag{25}$$

where the second line follows from the definition of $\Delta_{\mathcal{Q}}(j)$ and the third line follows from the definition of $\overline{D}_{\mathcal{Q}}(j)$ and $M_{\mathcal{Q}}(j)$. It then follows that u^j satisfies constraint (2) for all $i \in \mathcal{P}(a(j))$.

Next, note that u^j yields

$$\begin{aligned} x_0 &= \overline{D}_{\mathcal{Q}}(j) - \Delta_{\mathcal{Q}}(j) \\ &\geq \overline{D}_{\mathcal{Q}}(j) - (\overline{D}_{\mathcal{Q}}(j) - \widetilde{D}_{\mathcal{Q}}(j)) \\ &= \widetilde{D}_{\mathcal{Q}}(j), \end{aligned} \tag{26}$$

where the second line follows from the definition of $\Delta_{\mathcal{Q}}(j)$. If $\widetilde{D}_{\mathcal{Q}}(j) > 0$, then we know that there exists $r_j \in Q$ such that $\widetilde{D}_{\mathcal{Q}}(j) = d_{0r_j}$. Thus (26) implies that u^j satisfies constraint (2) for all $i \in \mathcal{V}_{\mathcal{Q}_{r_j}}$.

Also, note that u^j yields

$$x_0 + x_j = \overline{D}_{\mathcal{Q}}(j). \tag{27}$$

Since $0 \in \mathcal{P}(i)$ and $j \in \mathcal{P}(i)$ for all $i \in \mathcal{V}_Q(j)$, (27) implies that u^j satisfies (2) for all $i \in \mathcal{V}_Q(j)$.

Next, considering (b) and (c) of condition (iii), (25) and (26) imply that u^j satisfies

$$x_0 \geq d_{0a(k)} \quad \forall k \in \mathcal{W}(j). \tag{28}$$

Then u^j satisfies (2) for all $i \in \mathcal{P}(a(k)) \quad \forall k \in \mathcal{W}(j)$.

Finally, note that

$$\{0\} \cup A = \mathcal{V}_{\mathcal{Q}} = \mathcal{P}(j) \cup \mathcal{V}_{\mathcal{Q}_{r_j}} \cup \mathcal{V}_Q(j) \cup \left(\bigcup_{k \in \mathcal{W}(j)} \mathcal{P}(a(k)) \right) \cup \left(\bigcup_{k \in \mathcal{W}(j)} \mathcal{V}_Q(k) \right).$$

So it only remains to check that u^j satisfies (2) for all $i \in \bigcup_{k \in \mathcal{W}(j)} \mathcal{V}_Q(k)$. Given any $k \in \mathcal{W}(j)$, note that u^j satisfies

$$\begin{aligned} x_0 + x_k &\geq d_{0a(k)} + M_{\mathcal{Q}}(k) \\ &= \overline{D}_{\mathcal{Q}}(k), \end{aligned} \tag{29}$$

where the first line follows from (28) and the second line follows from the definition of $\overline{D}_{\mathcal{Q}}(k)$. Since, for all $i \in \mathcal{V}(k)$ we have $0 \in \mathcal{P}(i)$, $k \in \mathcal{P}(i)$ and $d_{0i} \leq \overline{D}_{\mathcal{Q}}(k)$, it follows that u^j satisfies constraint (2) for all $i \in \mathcal{V}_Q(k)$ for any $k \in \mathcal{W}(j)$.

v^j for $j \in \overline{S}_{\mathcal{Q}}$ is feasible because v^j satisfies constraint (2) since $v^j \geq u^j$ and condition (iv) ensures that v^j satisfies $0 \leq x_i \leq M_i y_i$ and $y_i \in \{0, 1\}$.

The feasibility of u^j for $j \in S_{\mathcal{Q}}$ can be established using analogous arguments as long as $\varepsilon \leq \Delta_{\mathcal{Q}}(j)$ and $\overline{D}_{\mathcal{Q}}(j) - \varepsilon \geq d_{0a(k)} \quad \forall k \in \mathcal{W}(j)$.

We now verify the feasibility of u^j for $j \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$. As before, we only need to verify that u^j satisfies constraint (2) for all $i \in \mathcal{V}$. Since the construction of u^j only affects nodes $i \in \mathcal{V}(j)$, from the feasibility of u^0, constraint (2) is satisfied for all $i \in \mathcal{V} \setminus \mathcal{V}(j)$. Given any node $i \in \mathcal{V}(j)$, note that u^j satisfies

$$\begin{aligned} \textstyle\sum_{k \in \mathcal{P}(i)} x_k &= M_{\mathcal{Q}}(0) + \textstyle\sum_{k \in \mathcal{P}(i) \setminus \mathcal{P}(j)} M_k \\ &\geq d_{0j} + \textstyle\sum_{k \in \mathcal{P}(i) \setminus \mathcal{P}(j)} d_k \\ &= d_{0i}, \end{aligned} \tag{30}$$

where the first line follows from the construction of u^j and the second line follows from condition (ii). Thus u^j satisfies (2) for all $i \in \mathcal{V}$.

We now verify the feasibility of u^j for $j \in Z \setminus \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$. Since the construction of u^j only affects nodes $i \in \mathcal{V}(k_j)$, from the feasibility of u^{k_j} (recall that $k_j \in \overline{S}_{\mathcal{Q}}$), constraint (2) is satisfied for all $i \in \mathcal{V} \setminus \mathcal{V}(k_j)$. Given any node $i \in \mathcal{V}(k_j)$, note that u^j satisfies

$$\begin{aligned} x_0 + x_{k_j} &\geq d_{0a(k_j)} + M_{k_j} \\ &\geq d_{0i}, \end{aligned} \tag{31}$$

where the first line follows from (28) and the construction of u^j, and the second line follows the definition of M_{k_j} and the fact that $k_j \in \mathcal{P}(i)$. Thus u^j satisfies constraint (2) for all $i \in \mathcal{V}$.

Finally, v^j for $j \in Z$ is feasible since $v^j \geq u^j$.

Tightness of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality: Here we prove the claim that the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is tight or active at each of the solutions vectors u^0 and $\{u^j, v^j\}_{j \in \mathcal{V} \setminus \{0\}}$. This claim is true for u^0, $\{u^j, v^j\}_{j \in B}$, $\{v^j\}_{j \in S_{\mathcal{Q}}}$ and $\{u^j, v^j\}_{j \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})}$. Furthermore, for any $j \in Z \setminus \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$, we have that u^j and v^j satisfy the claim as long as u^{k_j} satisfies the claim since $k_j \in \overline{S}_{\mathcal{Q}}$. Similarly the solutions $\{v^j\}_{j \in \overline{S}_{\mathcal{Q}}}$ satisfy the claim as long as $\{u^j\}_{j \in \overline{S}_{\mathcal{Q}}}$ satisfy the claim. Therefore, we just need to prove the claim for $\{u^j\}_{j \in S_{\mathcal{Q}} \cup \overline{S}_{\mathcal{Q}}}$. Here we prove the claim for $\{u^j\}_{j \in \overline{S}_{\mathcal{Q}}}$. The proof for $\{u^j\}_{j \in S_{\mathcal{Q}}}$ is nearly identical, see Guan [11] for details.

Since $j \in \overline{S}_{\mathcal{Q}}$ and $\mathcal{W}(j) \subseteq \overline{S}_{\mathcal{Q}}$, we have that u^j satisfies

$$x_i^j = \begin{cases} \overline{D}_{\mathcal{Q}}(j) - \Delta_{\mathcal{Q}}(j) & \text{if } i = 0 \\ 0 & \text{if } i \in S_{\mathcal{Q}} \setminus \{0\}, \end{cases}$$

and,

$$y_i^j = \begin{cases} 1 \text{ if } i \in \{j\} \cup \mathcal{W}(j) \\ 0 \text{ if } i \in \overline{S}_{\mathcal{Q}} \setminus (\{j\} \cup \mathcal{W}(j)). \end{cases}$$

Thus u^j satisfies

$$\sum_{i \in S_{\mathcal{Q}}} x_i^j + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y_i^j = \overline{D}_{\mathcal{Q}}(j) + \sum_{i \in \mathcal{W}(j)} \Delta_{\mathcal{Q}}(i). \tag{32}$$

It remains to show that the right-hand side of the above expression is equal to $M_{\mathcal{Q}}(0)$.

If $\mathcal{W}(j) = \emptyset$ then $\overline{D}_{\mathcal{Q}}(j) = M_{\mathcal{Q}}(0)$ by definition of $\mathcal{W}(j)$. If $\mathcal{W}(j) \neq \emptyset$, note that for any $i \in \mathcal{W}(j)$,

$$\overline{D}_{\mathcal{Q}}(i) - \widetilde{D}_{\mathcal{Q}}(i) \leq \overline{D}_{\mathcal{Q}}(i) - \overline{D}_{\mathcal{Q}}(j) \leq \overline{D}_{\mathcal{Q}}(i) - d_{0a(i)} = M_{\mathcal{Q}}(i),$$

where the first inequality follows from the fact that $\widetilde{D}_{\mathcal{Q}}(i) \geq d_{0q_j} = \overline{D}_{\mathcal{Q}}(j)$ since $i \notin \mathcal{V}_{\mathcal{Q}_{q_j}}$, and the second inequality follows from the fact that $\overline{D}_{\mathcal{Q}}(j) \geq \widetilde{D}_{\mathcal{Q}}(j) \geq$

$d_{0a(i)}$ from case (b) of condition (iii) or $\overline{D}_{\mathcal{Q}}(j) \geq d_{0a(j)} \geq d_{0a(i)}$. Thus, for any node $i \in \mathcal{W}(j)$,

$$\Delta_{\mathcal{Q}}(i) = \overline{D}_{\mathcal{Q}}(i) - \widetilde{D}_{\mathcal{Q}}(i). \tag{33}$$

By Property (A2), we index the nodes in $\mathcal{W}(j)$ as $i_1, i_2, \ldots, i_W$ such that $\overline{D}_{\mathcal{Q}}(i_1) < \overline{D}_{\mathcal{Q}}(i_2) < \ldots < \overline{D}_{\mathcal{Q}}(i_W)$. From this indexing scheme, the definition of $\overline{D}_{\mathcal{Q}}$, $\widetilde{D}_{\mathcal{Q}}$, and $\mathcal{W}(j)$, it follows that $\widetilde{D}_{\mathcal{Q}}(i_1) = \overline{D}_{\mathcal{Q}}(j)$, $\overline{D}_{\mathcal{Q}}(i_W) = M_{\mathcal{Q}}(0)$, and

$$\overline{D}_{\mathcal{Q}}(i_k) = \widetilde{D}_{\mathcal{Q}}(i_{k+1}) \qquad k = 1, 2, \ldots, W-1.$$

Thus

$$\overline{D}_{\mathcal{Q}}(j) + \sum_{i \in \mathcal{W}(j)} \Delta_{\mathcal{Q}}(i) = M_Q(0),$$

and the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is tight for u^j, $j \in \overline{S}_{\mathcal{Q}}$.

Linear Independence: Given the $2N-1$ vectors u^0 and $\{u^j, v^j\}_{j \in \mathcal{V} \setminus \{0\}}$, we perform a sequence of linear combinations to obtain the following $(2N - |\mathcal{V}_{\mathcal{Q}}| - 1)$ unit vectors.

$j \in B$:

$$\begin{aligned} e^{x_j} &= \tfrac{1}{M_j}(v^j - u^j), \quad \text{and} \\ e^{y_j} &= u^j - u^0. \end{aligned}$$

$j \in A$:

If $j \in S_{\mathcal{Q}}$:

$$e^{y_j} = v^j - u^0.$$

If $j \in \overline{S}_{\mathcal{Q}}$:

$$e^{x_j} = \tfrac{1}{\varepsilon}(v^j - u^j).$$

$j \in Z$:

If $j \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$:

$$\begin{aligned} e^{y_j} &= v^j - u^j, \quad \text{and} \\ e^{x_j} &= \tfrac{1}{M_j}(u^0 - u^j - e^{y_j} + \textstyle\sum_{i \in B}(M_i e^{x_i} + e^{y_i})). \end{aligned}$$

If $j \in Z \setminus \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$, let $k_j = \operatorname{argmin}\{t(i) : \ i \in \overline{S}_{\mathcal{Q}} \cap \mathcal{P}(j)\}$.

$$\begin{aligned} e^{y_j} &= v^j - u^j, \quad \text{and} \\ e^{x_j} &= \tfrac{1}{M_j}(u^{k_j} + (M_{k_j} - \Delta_{\mathcal{Q}}(k_j))e^{x_{k_j}} - u^j - e^{y_j}). \end{aligned}$$

An additional sequence of linear combinations gives the following additional $|\mathcal{V}_{\mathcal{Q}}|$ vectors.

$\overline{u}^0 = u^0 - \sum_{i \in Z}(M_i e^{x_i} + e^{y_i})$.

$j \in S_{\mathcal{Q}} \setminus \{0\}$,

$$\begin{aligned} \overline{u}^j &= u^j - \textstyle\sum_{i \in Z}(M_i e^{x_i} + e^{y_i}) - \sum_{i \in \mathcal{W}(j)} M_{\mathcal{Q}}(i) e^{x_i} - e^{y_j} \\ &= (\overline{D}_{\mathcal{Q}}(j) - \varepsilon)e^{x_0} + e^{y_0} + \textstyle\sum_{i \in \mathcal{W}(j)} e^{y_i} + \varepsilon e^{x_j}. \end{aligned}$$

$j \in \overline{S}_{\mathcal{Q}}$,

$$
\begin{aligned}
\overline{v}^j &= v^j - \textstyle\sum_{i\in Z}(M_i e^{x_i} + e^{y_i}) \\
&\quad - (\Delta_{\mathcal{Q}}(j) + \varepsilon)e^{x_j} - \textstyle\sum_{i\in \mathcal{W}(j)} M_{\mathcal{Q}}(i)e^{x_i} \\
&= (\overline{D}_{\mathcal{Q}}(j) - \Delta_{\mathcal{Q}}(j))e^{x_0} + e^{y_0} + \textstyle\sum_{i\in\mathcal{W}(j)} e^{y_i} + e^{y_j}.
\end{aligned}
$$

We now construct a matrix $\mathcal{M}$ whose rows are the $(2N-1)$ vectors $\overline{u}^0$, $\{e^{x_j}\}_{j\in B}$, $\{e^{y_j}\}_{j\in B}$, $\{\overline{u}^j\}_{j\in S_{\mathcal{Q}}\setminus\{0\}}$, $\{e^{y_j}\}_{j\in S_{\mathcal{Q}}\setminus\{0\}}$, $\{e^{x_j}\}_{j\in\overline{S}_{\mathcal{Q}}}$, $\{\overline{v}^j\}_{j\in\overline{S}_{\mathcal{Q}}}$, $\{e^{x_j}\}_{j\in\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})}$, $\{e^{y_j}\}_{j\in\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})}$, $\{e^{x_j}\}_{j\in Z\setminus\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})}$, and $\{e^{y_j}\}_{j\in Z\setminus\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})}$. The resulting matrix $\mathcal{M}$ has the following form:

	$\{0\}$		B		$S_{\mathcal{Q}}\setminus\{0\}$		$\overline{S}_{\mathcal{Q}}$		$\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})$		$Z\setminus\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})$	
	x_0	y_0	x	y	x	y	x	y	x	y	x	y
$\{0\}$	$M_Q(0)$	1										
B			I									
B				I								
$S_{\mathcal{Q}}\setminus\{0\}$	E	1			εI			F				
$S_{\mathcal{Q}}\setminus\{0\}$						I						
$\overline{S}_{\mathcal{Q}}$							I					
$\overline{S}_{\mathcal{Q}}$	G	1						H				
$\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})$									I			
$\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})$										I		
$Z\setminus\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})$											I	
$Z\setminus\mathcal{N}(\mathcal{Q},S_{\mathcal{Q}})$												I

In the matrix $\mathcal{M}$, the submatrices E and F arise from the nonzero elements of the vectors $\{\overline{u}^j\}_{j\in S_{\mathcal{Q}}\setminus\{0\}}$, and the submatrices G and H arise from the nonzero elements of the vectors $\{\overline{v}^j\}_{j\in\overline{S}_{\mathcal{Q}}}$. Consider the $|\overline{S}_{\mathcal{Q}}|\times|\overline{S}_{\mathcal{Q}}|$ submatrix H. This matrix has a column corresponding to each $j\in\overline{S}_{\mathcal{Q}}$. We arrange the columns of H such that the column corresponding to $i\in\overline{S}_{\mathcal{Q}}$ is before the column corresponding to $j\in\overline{S}_{\mathcal{Q}}$ if $\overline{D}_{\mathcal{Q}}(i)<\overline{D}_{\mathcal{Q}}(j)$ or $t(i)<t(j)$ if $\overline{D}_{\mathcal{Q}}(i)=\overline{D}_{\mathcal{Q}}(j)$. Note that this arrangement is uniquely defined by assumption (A1) on the set $\mathcal{Q}$. This arrangement guarantees that, for any $j\in\overline{S}_{\mathcal{Q}}$, the column corresponding to $i\in\mathcal{W}(j)$ is before the column corresponding to j. Consequently, the matrix H is lower-triangular and then it follows that the matrix $\mathcal{M}$ has rank $2N-1$. This is observed by exchanging rows labelled $S_{\mathcal{Q}}\setminus\{0\}$ and $\overline{S}_{\mathcal{Q}}$, and exchanging columns labelled x in $S_{\mathcal{Q}}\setminus\{0\}$ and y in $\overline{S}_{\mathcal{Q}}$. Since $\mathcal{M}$ was obtained by a sequence of elementary row operations on the $(2N-1)\times 2N$ matrix whose rows are the vectors u^0 and $\{u^j,v^j\}_{j\in\mathcal{V}\setminus\{0\}}$, it follows that these vectors are affinely independent. $\square$

Lemma 3. *Consider a feasible solution (x,y) satisfying the $(\mathcal{Q},S_{\mathcal{Q}})$ inequality at equality. Let $j^*\in\mathcal{V}_{\mathcal{Q}}$ be such that $y_{j^*}=1$, and let $q_{j^*}=argmax\{i : i\in\mathcal{Q}(j^*)\}$. Then, for all $q\in(\mathcal{Q}\setminus\mathcal{Q}_{q_{j^*}})\cup\{q_{j^*}\}$, there exists exactly one node $j_q\in\mathcal{F}_{\mathcal{Q}}\cap\mathcal{P}(q)$ such that $y_{j_q}=1$ and*

(i) $x_i=y_i=0\ \ \forall i\in\overline{S}_{\mathcal{Q}}\cap\mathcal{P}(a(j_q))$,
(ii) $x_i=0\ \forall i\in\mathcal{P}(a(j_q))\setminus\mathcal{V}_{\mathcal{Q}_{r_{j_q}}}$ *where* $r_{j_q}=\{i\in\mathcal{Q}: d_{0i}=\widetilde{D}_{\mathcal{Q}}(j_q)\}$,
(iii) $x_i=0\ \forall i\in S_{\mathcal{Q}}\cap\mathcal{V}_{\mathcal{Q}}(j_q)$ *and* $y_i=0\ \forall i\in\overline{S}_{\mathcal{Q}}\cap\mathcal{V}_{\mathcal{Q}}(j_q)$.

(iv) $\sum_{i \in S_{\mathcal{Q}_{r_{j_q}}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_{j_q}}}} \Delta_{\mathcal{Q}}(i) y_i = \sum_{i \in S_{\mathcal{Q}_{r_{j_q}}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_{j_q}}}} \Delta_{\mathcal{Q}_{r_{j_q}}}(i) y_i = d_{0 r_{j_q}}$.

Proof. For any $q \in \mathcal{Q}$, define $w(q) = \operatorname{argmin}\{t(i) : i \in \bar{S}_{\mathcal{Q}} \cap \mathcal{P}(q) \text{ and } y_i = 1\}$.

First consider $q = Q$. For brevity, let $w = w(Q)$.

Case (a): If w does not exist, then $\sum_{i \in \mathcal{P}(Q)} x_i \geq M_{\mathcal{Q}}(0)$ and $i \notin \overline{S}_{\mathcal{Q}}$ $\forall i \in \mathcal{P}(Q)$. Thus, $j^* \notin \mathcal{P}(Q)$ since $j^* \in \overline{S}_{\mathcal{Q}}$ and the left-hand side of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is at least

$$\sum_{i \in \mathcal{P}(Q)} x_i + \overline{D}_{\mathcal{Q}}(j^*) - \widetilde{D}_{\mathcal{Q}}(j^*) > M_{\mathcal{Q}}(0),$$

which contradicts the assumption that the feasible solution satisfies the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality.

Case (b): If $w \in \mathcal{G}_{\mathcal{Q}}$, then $\sum_{i \in \mathcal{P}(a(w)) \cap S_{\mathcal{Q}}} x_i + M_{\mathcal{Q}}(w) \geq d_{0a(w)} + M_{\mathcal{Q}}(w) = M_{\mathcal{Q}}(0)$ since $x_i = y_i = 0$ $\forall i \in \mathcal{P}(a(w)) \cap \overline{S}_{\mathcal{Q}}$ by the definition of w. Also, $j^* \neq w$ because $w \in \mathcal{G}_{\mathcal{Q}}$ and $j^* \in \mathcal{F}_{\mathcal{Q}}$, then the left-hand side of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is at least

$$\sum_{i \in \mathcal{P}(a(w)) \cap S_{\mathcal{Q}}} x_i + M_{\mathcal{Q}}(w) + \overline{D}_{\mathcal{Q}}(j^*) - \widetilde{D}_{\mathcal{Q}}(j^*) > M_{\mathcal{Q}}(0),$$

which again gives a contradiction.

Case (c): If $w \in \mathcal{F}_{\mathcal{Q}}$, let $r_w = \{i \in \mathcal{Q} : d_{0i} = \widetilde{D}_{\mathcal{Q}}(w)\}$. Then by Lemmas 1 and 2, we have

$$\sum_{i \in S_{\mathcal{Q}_{r_w}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_w}}} \Delta_{\mathcal{Q}}(i) y_i \geq \sum_{i \in S_{\mathcal{Q}_{r_w}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_w}}} \Delta_{\mathcal{Q}_{r_w}}(i) y_i \geq d_{0 r_w} = \widetilde{D}_{\mathcal{Q}}(w).$$

Then we have the left-hand side of the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is

$$\begin{aligned} &\geq \sum_{i \in S_{\mathcal{Q}_{r_w}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_w}}} \Delta_{\mathcal{Q}}(i) y_i + \overline{D}_{\mathcal{Q}}(w) - \widetilde{D}_{\mathcal{Q}}(w) && (34) \\ &\geq \widetilde{D}_{\mathcal{Q}}(w) + \overline{D}_{\mathcal{Q}}(w) - \widetilde{D}_{\mathcal{Q}}(w) && (35) \\ &= \overline{D}_{\mathcal{Q}}(w) = M_{\mathcal{Q}}(0) && (36) \end{aligned}$$

Therefore, when the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality holds at equality, we have the following four properties:

(a) $x_i = y_i = 0 \;\; \forall i \in \overline{S}_{\mathcal{Q}} \cap \mathcal{P}(a(w))$,
(b) $x_i = 0 \; \forall i \in \mathcal{P}(a(w)) \setminus \mathcal{V}_{\mathcal{Q}_{r_w}}$,
(c) $x_i = 0 \; \forall i \in S_{\mathcal{Q}} \cap \mathcal{V}_{\mathcal{Q}}(w)$ and $y_i = 0 \; \forall i \in \overline{S}_{\mathcal{Q}} \cap \mathcal{V}_{\mathcal{Q}}(w)$,
(d) $\sum_{i \in S_{\mathcal{Q}_{r_w}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_w}}} \Delta_{\mathcal{Q}}(i) y_i = \sum_{i \in S_{\mathcal{Q}_{r_w}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_w}}} \Delta_{\mathcal{Q}_{r_w}}(i) y_i = d_{0 r_w}$,

where (a) follows from the definition of w, (b) and (c) follow from the tightness of the inequality (34) and (d) follows from the tightness of the inequality (35). Thus, by letting $j_Q = w$, we have proved the claim for $q = Q$.

Now, for any $q \in \{Q-1, \ldots, r_w + 1\}$, we have that $w(q) = w = j_Q$. Thus the claim holds for all such q. Let us define a set $G(j^*) = \{w\} = \{j_Q\}$.

Now consider the case when $q = r_w$. Recall that $\mathcal{Q}_{r_w} = \{1, 2, \ldots, r_w\}$. From property (d),

$$\sum_{i \in S_{\mathcal{Q}_{r_w}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_w}}} \Delta_{\mathcal{Q}_{r_w}}(i) y_i = d_{0 r_w}.$$

Thus the $(\mathcal{Q}_{r_w}, S_{\mathcal{Q}_{r_w}})$ inequality is tight. By proceeding recursively in the above manner, we can show properties (a)-(d) for $\mathcal{Q}_{r_w}$. Note that this recursion terminates when $w = j^*$. Since, otherwise, there must exist a w selected at some step such that $w \in \mathcal{P}(j^*)$, which contradicts property (c) since $y_{j^*} \neq 0$. At each recursive step, we update $G(j^*) = G(j^*) \cup \{w\}$ except the termination step. Since properties (a)-(d) hold at each recursive step and at termination with $w = j^*$, the claim is proven. $\square$

Proof of necessity.
We consider in turn the conditions (i)-(iv) and show that if any condition is removed, the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality is not facet-defining.

Condition (i): The proof is by contradiction. Suppose $0 \in \overline{S}_{\mathcal{Q}}$. Since $y_0 = 1$ and $\Delta_{\mathcal{Q}}(0) = M_{\mathcal{Q}}(0)$, then we have $x_i = 0 \ \forall i \in S_{\mathcal{Q}} \setminus \{0\}$ and $y_i = 0 \ \forall i \in \overline{S}_{\mathcal{Q}}$ in order to satisfy the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality. Thus, $\dim(X_{SLSF}) \leq 2\mathcal{N} - 2 - |S_{\mathcal{Q}} \setminus \{0\}| - |\overline{S}_{\mathcal{Q}}| < 2\mathcal{N} - 2$, where X_{SLSF} is the set of feasible solutions satisfying the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality.

Condition (ii): The proof is by contradiction. Suppose there is a node $j^* \in \mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$ such that $M_{\mathcal{Q}}(0) < d_{0j^*}$. Let $w = \{i \in \mathcal{V}_{\mathcal{Q}} : j^* \in \mathcal{C}(i)\}$. Then

$$M_{\mathcal{Q}}(0) = \sum_{i \in S_{\mathcal{Q}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y_i \geq \sum_{i \in \mathcal{P}(w)} x_i$$

since $i \in S_{\mathcal{Q}} \quad \forall i \in \mathcal{P}(w)$ by the definition of $\mathcal{N}(\mathcal{Q}, S_{\mathcal{Q}})$. Then, $\sum_{i \in \mathcal{P}(w)} x_i \leq M_{\mathcal{Q}}(0) < d_{0j^*}$. Thus, we have $y_{j^*} = 1$ for all feasible solutions satisfying the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality and $\dim(X_{SLSF}) < 2\mathcal{N} - 2$.

Condition (iii): The proof of (a) is by contradiction. Suppose $q^* = \operatorname{argmax}\{i \in \mathcal{Q} : \mathcal{W}(j) \cap \mathcal{P}(i) = \emptyset\}$. Then we have $\sum_{i \in S_{\mathcal{Q} \setminus \mathcal{Q}_{q^*-1}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q} \setminus \mathcal{Q}_{q^*-1}}} \Delta_{\mathcal{Q} \setminus \mathcal{Q}_{q^*-1}}(i) y_i \geq M_{\mathcal{Q}}(0)$ corresponding to leaf node set $\mathcal{Q} \setminus \mathcal{Q}_{q^*-1}$ since $i \in S_{\mathcal{Q}} \ \forall i \in \mathcal{P}(q^*)$. Thus, $x_i = 0 \ \forall i \in S_{\mathcal{Q}_{q^*}} \setminus \mathcal{P}(q^*)$ and $y_i = 0 \ \forall i \in \overline{S}_{\mathcal{Q}_{q^*}} \setminus \mathcal{P}(q^*)$, which implies $\dim(X_{SLSF}) < 2\mathcal{N} - 2$.

The proofs of (b), (c) and (d) are similar. We only prove case (b), see Guan [11] for proofs of the other two cases. Suppose $y_{j^*} = 1$ for some feasible solution satisfying the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality, we will prove that

$\widetilde{D}_{\mathcal{Q}}(j^*) \geq d_{0a(k)}$, $\forall k \in \mathcal{W}(j^*)$, which implies that if $\exists k \in \mathcal{W}(j^*)$ such that $\widetilde{D}_{\mathcal{Q}}(j^*) < d_{0a(k)}$, then $y_{j^*} = 0$ for any feasible solution satisfying the $(\mathcal{Q}, S_{\mathcal{Q}})$ inequality at equality and $\dim(X_{SLSF}) < 2N - 2$.

Now suppose $y_{j^*} = 1$ for some feasible solution satisfying the inequality at equality. Let $u_j = \operatorname{argmax}\{t(i) : i \in \mathcal{P}(j) \cap \mathcal{P}(j^*)\}$ $\forall j \in G(j^*)$ and $u_{j^*} = \operatorname{argmax}\{t(i) : i \in \mathcal{P}(r_{j^*}) \cap \mathcal{P}(j^*)\}$, where the set $G(j^*)$ is as constructed in Lemma 3. From property (iv) in Lemma 3,

$$\widetilde{D}_{\mathcal{Q}}(j^*) = \sum_{i \in S_{\mathcal{Q}_{r_{j^*}}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_{j^*}}}} \Delta_{\mathcal{Q}_{r_{j^*}}}(i) y_i \tag{37}$$

$$\geq \sum_{i \in S_{\mathcal{Q}_{r_{j^*}}} \cap \mathcal{P}(u_{j^*})} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}_{r_{j^*}}} \cap \mathcal{P}(u_{j^*})} \Delta_{\mathcal{Q}_{r_{j^*}}}(i) y_i \tag{38}$$

$$= \sum_{i \in \mathcal{P}(u_{j^*})} x_i \tag{39}$$

$$= \sum_{i \in \mathcal{P}(a(j^*))} x_i, \tag{40}$$

where (39) follows from property (i) of Lemma 3 and (40) follows from property (ii) of Lemma 3 as $j_q = j^*$. Thus

$$\widetilde{D}_{\mathcal{Q}}(j^*) \geq \sum_{i \in \mathcal{P}(a(j^*))} x_i \geq \sum_{i \in \mathcal{P}(u_j)} x_i \geq \sum_{i \in \mathcal{P}(a(j))} x_i \geq d_{0a(j)} \; \forall j \in G(j^*), \tag{41}$$

where the third inequality follows from property (ii) of Lemma 3.

Finally, from the definition of $\mathcal{W}(j^*)$, we have $\mathcal{W}(j^*) \cap \mathcal{P}(q) \in \mathcal{P}(G(j^*) \cap \mathcal{P}(q))$ $\forall q \in \mathcal{Q} \setminus \mathcal{Q}_{q_{j^*}}$. Then, $\widetilde{D}_{\mathcal{Q}}(j^*) \geq d_{0a(k)}$ $\forall k \in \mathcal{W}(j^*)$.

Condition (iv): The proof is by contradiction. Suppose, for some $j \in \mathcal{G}_{\mathcal{Q}}$, there exists a $\overline{q} \in \mathcal{L} \cap \mathcal{Q}$ such that $\overline{q} = \operatorname{argmax}\{q : q \in \mathcal{Q}(j)\}$. Now consider the values of x_j and y_j for any feasible solution satisfying the inequality at equality. If $y_j = 0$, then $x_j = 0$. If $y_j = 1$, then from the recursion in the proof of (c) in condition (iii), we have $\sum_{i \in \mathcal{P}(a(j))} x_i = M_{\mathcal{Q}_{\overline{q}}}(0) - M_{\mathcal{Q}_{\overline{q}}}(j)$, which implies that $x_j \geq M_{\mathcal{Q}_{\overline{q}}}(j) = M_{\mathcal{Q}}(j) = M_j$ in order to keep feasibility since $x_i = 0$ $\forall i \in \mathcal{V}_{\mathcal{Q}}(j)$, which implies $x_j = M_j$. Thus, we have $x_j = M_j y_j$, which is independent of $y_0 = 1$ and $\sum_{i \in S_{\mathcal{Q}}} x_i + \sum_{i \in \overline{S}_{\mathcal{Q}}} \Delta_{\mathcal{Q}}(i) y_i = M_{\mathcal{Q}}(0)$ so that $\dim(X_{SLSF}) < 2N - 2$. □

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