

Arbitrage pricing of American contingent claims in incomplete markets – a convex optimization approach

Teemu Pennanen Alan King

February 18, 2004

Abstract

Convex optimization provides a natural framework for pricing and hedging financial instruments in incomplete market models. Duality theory of convex optimization has been shown to yield elementary proofs of well-known martingale-expressions for prices of European contingent claims. This paper extends the analysis to American contingent claims in incomplete markets. The pricing problems of the seller and the buyer of an American contingent claim are first expressed as convex optimization problems, after which martingale-expressions for the buyer's and seller's prices are obtained by inspecting the dual optimization problems. Besides its simplicity, one of the main advantages of the present approach is that it is computational. Indeed, many algorithms are available for pricing problems as soon as they are set up as convex optimization problems. Also, portfolio constraints and transaction costs can be immediately incorporated to the definitions of the buyer's and seller's prices and into computational approaches based on optimization.

1 Introduction

An *American contingent claim* (ACC) associated with a real-valued stochastic process $X = (X_k)_{k=0}^K$ is a security whose owner can at any stage $k = 0, \dots, K$ choose to take X_k euros, after which the security becomes worthless. ACCs cover a wide variety of financial options: an American call option on a security S with strike K corresponds to $X = S - K$. American put is obtained by reversing the sign of X . Bermudan call option is obtained from the corresponding American call by setting $X_k = 0$ for k in a predetermined subset of $\{0, \dots, K\}$. Bermudan put is obtained similarly from the American put. European call option with maturity k' is a Bermudan option where $X_k = 0$ for $k \neq k'$. European put is obtained by reversing the sign of X . Defining $X_k = \max_{l=0, \dots, k} (S_l - K)$ leads similarly to Russian and look-back options etc. One could also define X as a given function of multiple underlyings.

In arbitrage pricing of financial options, one tries to find a price so that neither buying nor selling the option at this price leads to arbitrage, i.e. a possibility to start a self-financing trading strategy with zero initial investment and nonnegative terminal wealth that is positive with a positive probability. Here, self-financing means that the only cash going into the investment portfolio during the trading horizon is that coming from buying (selling) the option and receiving (paying out) the cash-flows the option gives rights to. Arbitrage pricing of ACCs was initiated by Bensoussan [1] and Karatzas [5] for complete market models in continuous time; see the survey of Myeni [9]. Arbitrage pricing of ACCs under portfolio constraints have been studied by Karatzas and Kou [6] and by Buckdahn and Hu [2] who also considered jump diffusion models for stock prices. To our knowledge, only Föllmer and Schied [4, Chapters 6 and 7] have studied arbitrage pricing of ACCs in general incomplete market models in discrete time.

In this paper, we use techniques of convex optimization in arbitrage pricing of ACCs in incomplete discrete time models. This yields elementary proofs of many of the sophisticated results in [4]. Föllmer and Schied [4, Definition 6.31] defined a certain interval of prices (which they called “arbitrage-free prices”) in terms of martingale measures and stopping times after which they showed, using upper Snell envelopes and a uniform Doob decomposition, that these prices actually correspond to prices that do not lead to arbitrage opportunities by buying or selling the ACC. We proceed in the opposite, more natural order: we define arbitrage-free prices directly in terms of arbitrage opportunities, and then we use convex duality to characterize these prices in terms of martingale measures and stopping times. Besides the simplicity, our approach has certain advantages compared to the approach of [4]. First, we need not assume positivity of the process X . Second, our definition of arbitrage-free prices can be immediately adapted to more realistic market models where there may be transaction costs, portfolio constraints or other market imperfections. Third, formulating the lower and upper bounds as optimal values of convex optimization problems, allows for the use of efficient computational approaches developed for such problems. This paper can be seen as a continuation of King [7], where European contingent claims were covered and of King, Koivu and Pennanen [8], where the framework of [7] was extended by incorporating market traded ECCs as hedging instruments.

The rest of this paper is organized as follows. Section 2 describes the market model and characterizes the arbitrage-free prices of European contingent claims. In Section 3, we first express the lower end-point of the interval of arbitrage-free prices of an ACC as an optimal value of a convex optimization problem. We then use convex duality to express it also in terms of martingale measures and stopping times. In Section 4, we do the same for the upper end-point. In Section 5, we characterize the set of arbitrage free prices for ACCs.

2 Preliminaries

The market consists of $J + 1$ tradable securities that are traded in a finite number of *decision stages* $k = 0, \dots, K$. The price of security j at stage k will be denoted by S_k^j . The price process $S = (S_k^0, \dots, S_k^J)_{k=0}^K$ will be modeled as a random variable in a probability space (Ξ, \mathcal{F}, P) , and it is assumed that investors have no influence on the prices. The information available to investors at each stage $k = 0, \dots, K$ will be modeled by a sequence $\mathcal{F}_0 \subset \dots \subset \mathcal{F}_K$ of subfields of \mathcal{F} . By this it is meant that, at stage k , the investors do not know which element $\xi \in \Xi$ will be realized, but only in which element of \mathcal{F}_k it belongs to. In particular, the price process S is *adapted* to $(\mathcal{F}_k)_{k=0}^K$, which means that, for each $k = 0, \dots, K$, S_k only depends on which element of \mathcal{F}_k has been realized at stage k . It is helpful to think of the samples ξ as scenarios $\xi = (\xi_0, \dots, \xi_K)$ of random data where the value of ξ_k is observed just before trading at stage k . The process ξ could be taken to contain any information that is regarded relevant for a given decision problem. We assume that ξ_0 is known to the investors, so that \mathcal{F}_0 is the trivial σ -field.

Throughout this paper, we assume that the sample space Ξ is *finite*. This considerably simplifies the analysis and it allows a natural description of the market model in terms of a *scenario tree*: atoms of \mathcal{F}_k , denoted by \mathcal{N}_k , are the *nodes* of the scenario tree at stage k . The set \mathcal{N}_0 consists of a single node, the *root* of the tree, which will be labeled 0. The collection of all nodes will be denoted by $\mathcal{N} \equiv \bigcup_{k=0}^K \mathcal{N}_k$. Since $(\mathcal{F}_k)_{k=0}^K$ is a filtration, we have that for $k = 1, \dots, K$, each $n \in \mathcal{N}_k$ is contained in a unique element $a(n) \in \mathcal{N}_{k-1}$. This is where the tree-structure comes from. For $k = 0, \dots, K - 1$, we denote the set of *child nodes* of a node $n \in \mathcal{N}_k$ by $\mathcal{C}(n) = \{m \in \mathcal{N}_{k+1} \mid a(m) = n\}$. The value of the price vector S_k in a node $n \in \mathcal{N}_k$ will be denoted by S_n . Note that we do not assume that the tree of prices is recombining. This is essential in incomplete markets where trading strategies are in general path dependent; see the discussion in [3, Section IIIA].

The probability measure P attaches a weight $p_n > 0$ to each $n \in \mathcal{N}_K$ so that $\sum_{n \in \mathcal{N}_K} p_n = 1$. The probability of each $n \in \mathcal{N} \setminus \mathcal{N}_K$ is determined recursively by $p_n = \sum_{m \in \mathcal{C}(n)} p_m$. The expected value

under P of S_k is given by the finite sum

$$E^P S_k := \sum_{n \in \mathcal{N}_k} p_n S_n.$$

A probability measure $Q = \{q_n\}_{n \in \mathcal{N}_K}$ is called a *martingale measure for a process Y* if

$$E^Q[Y_{k+1} | \mathcal{F}_k] = Y_k \quad Q\text{-a.s.} \quad k = 0, \dots, K-1$$

In our setting, this can be written as

$$\sum_{m \in \mathcal{C}(n)} \frac{q_m}{q_n} Y_m = Y_n \quad \forall n \in \mathcal{N} \setminus \mathcal{N}_k, \text{ when } q_n > 0.$$

A measure Q is said to be *equivalent* to P if $q_n > 0$ exactly when $p_n > 0$.

A *trading strategy* is an \mathbb{R}^{J+1} -valued $(\mathcal{F}_k)_{k=0}^K$ -adapted process $\theta = (\theta_k^0, \dots, \theta_k^J)_{k=0}^K$, where the value of θ_k^j is interpreted as the number of units of security j held during period $(k, k+1]$. The value of a portfolio $\theta_k = (\theta_k^0, \dots, \theta_k^J)$ at stage k is simply the scalar product

$$S_k \cdot \theta_k := \sum_{j=0}^J S_k^j \theta_k^j.$$

An *arbitrage opportunity* is the possibility to find a self-financing trading strategy which starts from zero initial wealth and produces terminal wealth which is almost surely nonnegative and positive with a nonzero probability. In mathematical terms, this means that there exists a trading strategy θ that satisfies

$$\begin{aligned} S_0 \cdot \theta_0 &= 0, \\ S_k \cdot (\theta_k - \theta_{k-1}) &= 0 \quad P\text{-a.s.} \quad k = 1, \dots, K, \\ S_K \cdot \theta_K &\geq 0 \quad P\text{-a.s.}, \\ E^P S_K \cdot \theta_K &> 0. \end{aligned}$$

We will assume throughout that the price of one of the securities, say S^0 , is always strictly positive, so that we can define the *discount process* $\beta_k = S_0^0/S_k^0$. Then, by the *fundamental theorem of asset pricing*, (FTAP) the absence of arbitrage is equivalent to the existence of a martingale measure Q for the discounted price process $\beta_k S_k$, such that Q is equivalent to P ; see [7] for a simple proof in the present market model.

A *European contingent claim* (ECC) is a security that gives its owner a stochastic $(\mathcal{F}_k)_{k=0}^K$ -adapted cash-flow $F = (F_k)_{k=0}^K$. The ECC is called a *derivative* of S if F_k is a function of S_0, \dots, S_k . This definition extends [4, Definition 5.19], where a ECC has *nonnegative* payouts that are nonzero only at stage K . In addition to the examples in [4] (which include put and call versions of European, Asian, barrier and look-back options) our definition covers futures contracts and fixed income instruments like government bonds. Note that both definitions cover derivatives of multiple underlyings and of nontradable instruments.

A number V is said to be an *arbitrage-free price of F* if buying or selling F at this price does not lead to arbitrage opportunities. Any number below the optimum value of the optimization problem

$$\begin{aligned} &\underset{V, \theta}{\text{maximize}} && V \\ &\text{subject to} && S_0 \cdot \theta_0 = F_0 - V, \\ & && S_k \cdot (\theta_k - \theta_{k-1}) = F_k, \quad P\text{-a.s.} \quad k = 1, \dots, K, \\ & && S_K \cdot \theta_K \geq 0, \quad P\text{-a.s.} \\ & && \theta \text{ is } (\mathcal{F}_k)\text{-adapted} \end{aligned} \tag{B}$$

is clearly not an arbitrage-free price of F . On the other hand, buying the option for a price which is above the optimum value does not lead to arbitrage opportunity. Similarly, any number above the optimum value of

$$\begin{aligned}
& \underset{V, \theta}{\text{minimize}} && V \\
& \text{subject to} && S_0 \cdot \theta_0 = V - F_0, \\
& && S_k \cdot (\theta_k - \theta_{k-1}) = -F_k, \quad P\text{-a.s. } k = 1, \dots, K, \\
& && S_K \cdot \theta_K \geq 0, \quad P\text{-a.s.} \\
& && \theta \text{ is } (\mathcal{F}_k)\text{-adapted}
\end{aligned} \tag{S}$$

is not an arbitrage-free price of F , while selling the option for a price which is below the optimum value does not lead to an arbitrage opportunity. The set of arbitrage-free prices of F (if any) is thus an interval whose end points are the optimum values of (B) and (S). These optimum values will be called *buyer's* and *seller's prices*, respectively, of F .

The following shows, in particular, that in an arbitrage-free market, the buyer's price is less than or equal to the seller's price. A simple proof based on LP-duality can be found in King [7, Section 3].

Theorem 1 *In an arbitrage-free market, the buyer's and seller's prices can be expressed as*

$$\inf_{Q \in \mathcal{M}} E^Q \sum_{k=0}^K \beta_k F_k \quad \text{and} \quad \sup_{Q \in \mathcal{M}} E^Q \sum_{k=0}^K \beta_k F_k,$$

respectively, where \mathcal{M} denotes the set of martingale measures for the discounted price process βS .

By the FTAP, the set \mathcal{M} is nonempty whenever the market is arbitrage-free. In fact, in an arbitrage-free market, the set of equivalent martingale measures is exactly the relative interior $\text{ri } \mathcal{M}$ of \mathcal{M} ; see [10, Theorem 6.5].

It is not a priori clear whether the buyer's and seller's prices themselves are arbitrage-free prices or not. Indeed, it may happen, for example, that buying the option at buyer's price and following the buyer's trading strategy leads to a strictly positive terminal wealth with a positive probability. The following clarifies the situation. We will say that a ECC is *replicable* if there exists a trading strategy θ which together with some V satisfies the constraints of (S) and $S_K \cdot \theta_K = 0$ holds P -almost surely.

Theorem 2 *In an arbitrage free market, the following are equivalent.*

1. F is replicable;
2. the buyer's and seller's prices of F are equal;
3. the buyer's price is arbitrage-free;
4. the seller's price is arbitrage-free.

Proof. Assume first that F is replicable and let $\bar{\theta}$ be the associated trading strategy. If $\tilde{\theta}$ is any optimal solution to (S), then the strategy $\theta := \tilde{\theta} - \bar{\theta}$ satisfies

$$\begin{aligned}
S_0 \cdot \theta_0 &\leq 0, \\
S_k \cdot (\theta_k - \theta_{k-1}) &= 0 \quad P\text{-a.s. } k = 1, \dots, K, \\
S_K \cdot \theta_K &\geq 0 \quad P\text{-a.s.}
\end{aligned}$$

Since there is no arbitrage, it must hold that $S_0 \cdot \tilde{\theta}_0 = S_0 \cdot \bar{\theta}_0$. Similarly, if $\tilde{\theta}$ is optimal for the buyer, then the strategy $\theta := \tilde{\theta} + \bar{\theta}$ satisfies the above system and we must have that $S_0 \cdot \tilde{\theta}_0 = S_0 \cdot \bar{\theta}_0$. Thus, 2 holds.

If 2 holds, then one can implement both the buyer's and seller's strategies with zero initial investment. Because both these strategies guarantee nonnegative terminal wealth and since the market is arbitrage-free, both strategies must end up with zero terminal wealth P -almost surely. So 3 and 4 hold.

If 4 holds, every solution of (S) must end up with zero terminal wealth P -almost surely. This means that every solution of (S) replicates F , so 1 holds. If 3 holds, the same argument applies to the negative of the buyer's strategy. \square

The market model will be said to be *complete* if every ECC is replicable. The above theorems imply that, in an arbitrage-free market, completeness is equivalent to \mathcal{M} being a singleton, in which case $\text{ri } \mathcal{M} = \mathcal{M}$.

3 Buyer's price

Throughout, we will assume that $X = (X_k)_{k=0}^K$ is a real-valued $(\mathcal{F}_k)_{k=0}^K$ -adapted stochastic process. We will characterize the lower end-point of the interval of arbitrage-free prices of the ACC associated with X through the optimization problem

$$\begin{aligned}
& \underset{V, \theta, e}{\text{maximize}} && V \\
& \text{subject to} && S_0 \cdot \theta_0 = X_0 e_0 - V, \\
& && S_k \cdot (\theta_k - \theta_{k-1}) = X_k e_k && P\text{-a.s. } k = 1, \dots, K, \\
& && S_K \cdot \theta_K \geq 0 && P\text{-a.s.}, \\
& && \sum_{k=0}^K e_k \leq 1 && P\text{-a.s.}, \\
& && e_k \in \{0, 1\} && P\text{-a.s. } k = 0, \dots, K, \\
& && \theta, e \text{ are } (\mathcal{F}_k)\text{-adapted,}
\end{aligned} \tag{BP}$$

where e_k denotes the amount of the ACC exercised at stage k . The constraints on e mean that the option is exercised in at most one stage. It is clear that any number below the optimum value of (BP) is not an arbitrage-free price of X . On the other hand, buying the ACC for a price which is above the optimum value does not lead to arbitrage opportunity. The optimum value will be called the *buyer's price* of X .

Instead of the variables e above, it is more common to describe exercise strategies for an ACC through *stopping times* which are functions $\tau : \Xi \rightarrow \{0, \dots, K\} \cup \{+\infty\}$ such that $\{\xi \in \Xi \mid \tau(\xi) = k\} \in \mathcal{F}_k$, for each $k = 0, \dots, K$. The relation $e_k = 1 \iff \tau = k$ defines a one-to-one correspondence between stopping times and processes $e = (e_0, \dots, e_K) \in E$, where

$$E = \{e \mid e \text{ is } (\mathcal{F}_k)_{k=0}^K\text{-adapted, } \sum_{k=0}^K e_k \leq 1 \text{ and } e_k \in \{0, 1\} \text{ } P\text{-a.s.}\}.$$

The advantage of the above formulation is that (BP) is not far from being a convex optimization problem. Indeed, if we replace the second last constraint by $e \geq 0$, the problem becomes convex. We will refer to this relaxed problem as (BPR). Problem (BPR) can be interpreted as a pricing problem for a batch of ACCs when one can exercise only some of the claims and keep the rest for a possible later exercise. This corresponds to the standard of treating positions in the underlying assets as continuous rather than discrete variables. The following result shows that relaxing the integrality requirement is not really interesting to the holder of the option, in the sense that, if one owns more than one option, the possibility of exercising them at different stages is not worth a positive initial endowment.

Theorem 3 *The set of solutions to (BPR) always contains a solution with $e_k(\xi) \in \{0, 1\}$ P -a.s. for all $k = 0, \dots, K$. In particular, the optimum value of (BPR) equals that of (BP).*

Proof. Problem (BPR) can be written in the tree-notation as

$$\begin{aligned}
& \underset{V, \theta, e}{\text{maximize}} && V \\
& \text{subject to} && S_0 \cdot \theta_0 = X_0 e_0 - V, \\
& && S_n \cdot (\theta_n - \theta_{a(n)}) = X_n e_n \quad n \in \mathcal{N}_k, \quad k = 1, \dots, K, \\
& && S_n \cdot \theta_n \geq 0 \quad n \in \mathcal{N}_K, \\
& && \sum_{m \in \mathcal{A}(n)} e_m \leq 1 \quad n \in \mathcal{N}_K, \\
& && e_n \geq 0 \quad n \in \mathcal{N},
\end{aligned} \tag{1}$$

where $\mathcal{A}(n)$ denotes the path history of a node $n \in \mathcal{N}$ including n itself. Defining the set of descendant nodes of a node $n \in \mathcal{N}$ by $\mathcal{D}(n) = \{m \in \mathcal{N} \mid n \in \mathcal{A}(m)\}$, the Lagrangian can be written as

$$\begin{aligned}
l(\theta, e, \xi, y, z) &= V - y_0[S_0 \cdot \theta_0 - X_0 e_0 + V] - \sum_{k=1}^K \sum_{n \in \mathcal{N}_k} y_n [S_n \cdot (\theta_n - \theta_{a(n)}) - X_n e_n] \\
&+ \sum_{n \in \mathcal{N}_K} x_n S_n \cdot \theta_n - \sum_{n \in \mathcal{N}_K} z_n \left[\sum_{m \in \mathcal{A}(n)} e_m - 1 \right] \\
&= [1 - y_0]V + \sum_{n \in \mathcal{N}_K} [x_n - y_n] S_n \cdot \theta_n + \sum_{k=0}^{K-1} \sum_{n \in \mathcal{N}_k} \left[\sum_{m \in \mathcal{C}(n)} y_m S_m - y_n S_n \right] \cdot \theta_n \\
&+ \sum_{n \in \mathcal{N}} [y_n X_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_K} z_m] e_n + \sum_{n \in \mathcal{N}_K} z_n.
\end{aligned}$$

Since at least one of the assets is always nonnegative, it follows that the supremum of l with respect to θ_n equals $+\infty$ unless $x_n = y_n$. We thus get the dual problem

$$\begin{aligned}
& \underset{y, z}{\text{minimize}} && \sum_{n \in \mathcal{N}_k} z_n \\
& \text{subject to} && y_0 = 1, \\
& && \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n \quad n \in \mathcal{N}_k, \quad k = 0, \dots, K-1, \\
& && y_n X_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_K} z_m \leq 0 \quad n \in \mathcal{N}, \\
& && y_n, z_n \geq 0 \quad n \in \mathcal{N}_K.
\end{aligned} \tag{2}$$

Now, if at an optimal solution, $e_{n'} > 0$ in a node $n' \in \mathcal{N}$, complementary slackness implies that constraint (2) must hold as an equality for n' . It follows that we can remove constraint (2) for $n \in \mathcal{D}(n') \setminus n'$ without affecting the optimum value, and then, by complementary slackness again, (BPR) must have a solution with $e_n = 0$ for all $n \in \mathcal{D}(n') \setminus n'$. It is also clear that we can set $e_{n'} = 1$ (if not already so) without reducing the objective value. Thus, (BPR) has an optimal solution with $e \in \{0, 1\}^{\mathcal{N}}$. \square

The main implication of the above result is that we can express the buyer's price of X as the optimum value of a *convex* minimization problem, in fact, a linear program. Besides the wide range of computational possibilities opened up by this, one can now use convex duality to get expressions for the buyer's price in terms of martingale measures, stopping times. The set of stopping times will be denoted by \mathcal{T} . We also define the set

$$\tilde{E} = \{e \mid e \text{ is } (\mathcal{F}_k)_{k=0}^K\text{-adapted, } \sum_{k=0}^K e_k \leq 1 \text{ and } e \geq 0 \text{ } P\text{-a.s.}\}.$$

Theorem 4 *In an arbitrage-free market, the buyer's price of X can be expressed as*

$$\max_{\tau \in \mathcal{T}} \min_{Q \in \mathcal{M}} E^Q \beta_\tau X_\tau = \min_{Q \in \mathcal{M}} \max_{\tau \in \mathcal{T}} E^Q \beta_\tau X_\tau.$$

Proof. Keeping e fixed in (BP) and minimizing with respect to θ , gives the buyer's price of a ECC with payoff $(X_k e_k)_{k=0}^K$. According to Theorem 1, this equals

$$\min_{Q \in \mathcal{M}} E^Q \sum_{k=0}^K \beta_k X_k e_k,$$

so the buyer's price can be written as

$$\max_{e \in E} \min_{Q \in \mathcal{M}} E^Q \sum_{k=0}^K \beta_k X_k e_k. \quad (3)$$

The correspondence between stopping times and the processes $e \in E$ gives now the first expression. By Theorem 3, we can replace E by \tilde{E} without changing the value of (3), and then, since \tilde{E} and \mathcal{M} are bounded convex sets, [10, Corollary 37.6.1] guarantees that the order of max and min can be reversed without affecting the value in (3). But then, for each fixed $Q \in \mathcal{M}$, the objective is linear in e , so the max over $e \in \tilde{E}$ is achieved at an extreme point of \tilde{E} . It is not hard to verify that e is integer at these points, which yields the second expression. \square

4 Seller's price

Consider the optimization problem

$$\begin{aligned} & \underset{V, \theta}{\text{minimize}} && V \\ & \text{subject to} && S_0 \cdot \theta_0 = V, \\ & && S_k \cdot (\theta_k - \theta_{k-1}) = 0 && P\text{-a.s., } k = 1, \dots, K, \\ & && S_K \cdot \theta_K \geq 0 && P\text{-a.s.,} \\ & && S_k \cdot \theta_k \geq X_k && P\text{-a.s., } k = 0, \dots, K, \\ & && \theta \text{ is } (\mathcal{F}_k)\text{-adapted.} \end{aligned} \quad (\text{SP})$$

The solution of (SP) can be used to hedge against all possible exercise strategies the holder of the option might choose (not just the rational ones). Indeed, if the seller receives an initial wealth V and then follows the optimal θ until the exercise occurs at some stage k , she can draw the amount X_k from her portfolio and still be left with a nonnegative wealth that allows her to guarantee nonnegative terminal wealth (by investing everything in S^0 , for example). Thus, if X could be sold for more than the optimal value of (SP) there would be arbitrage. On the other hand, selling X for a price less than the optimum value does not lead to arbitrage opportunities. The optimum value of (SP) will be called the *seller's price* of X .

Problem (SP) is a linear program, so it is amenable to numerical solution by standard solvers. Moreover, convex duality yields the following expression for the seller's price.

Theorem 5 *In an arbitrage-free market, the seller's price of the ACC can be expressed as*

$$\max_{\tau \in \mathcal{T}} \max_{Q \in \mathcal{M}} E^Q \beta_\tau X_\tau = \max_{Q \in \mathcal{M}} \max_{\tau \in \mathcal{T}} E^Q \beta_\tau X_\tau.$$

Proof. In the tree-notation, (SP) reads

$$\begin{aligned}
& \underset{V, \theta}{\text{minimize}} && V \\
& \text{subject to} && S_0 \cdot \theta_0 = V, \\
& && S_n \cdot (\theta_n - \theta_{a(n)}) = 0 \quad n \in \mathcal{N}_k, \quad k = 1, \dots, K, \\
& && S_n \cdot \theta_n \geq 0 \quad n \in \mathcal{N}, \\
& && S_n \cdot \theta_n \geq X_n \quad n \in \mathcal{N}_k, \quad k = 0, \dots, K,
\end{aligned} \tag{4}$$

for which the Lagrangian can be written as

$$\begin{aligned}
l(\theta, e, \xi, y, z) &= V + y_0[S_0 \cdot \theta_0 - V] + \sum_{k=1}^K \sum_{n \in \mathcal{N}_k} y_n S_n \cdot (\theta_n - \theta_{a(n)}) \\
&\quad - \sum_{n \in \mathcal{N}_K} x_n S_n \cdot \theta_n + \sum_{k=0}^K \sum_{n \in \mathcal{N}_k} z_n (X_n - S_n \cdot \theta_n) \\
&= [1 - y_0]V + \sum_{k=0}^{K-1} \sum_{n \in \mathcal{N}_k} [(y_n - z_n)S_n - \sum_{m \in \mathcal{C}(n)} y_m S_m] \cdot \theta_n \\
&\quad + \sum_{n \in \mathcal{N}_K} [y_n - z_n - x_n]S_n \cdot \theta_n + \sum_{k=0}^K \sum_{n \in \mathcal{N}_k} z_n X_n,
\end{aligned}$$

so we get the dual

$$\begin{aligned}
& \underset{y, z}{\text{maximize}} && \sum_{k=0}^K \sum_{n \in \mathcal{N}_k} z_n X_n \\
& \text{subject to} && y_0 = 1, \\
& && \sum_{m \in \mathcal{C}(n)} y_m S_m = (y_n - z_n)S_n \quad n \in \mathcal{N}_k, \quad k = 0, \dots, K-1, \\
& && y_n - z_n \geq 0 \quad n \in \mathcal{N}_K, \\
& && z \geq 0.
\end{aligned}$$

By the FTAP, there exists a strictly positive vector q such that $\sum_{m \in \mathcal{C}(n)} q_m S_m = q_n S_n$. This can be used to construct a feasible dual solution which satisfies the inequality constraints as strict inequalities, and then by [10, Theorem 6.5], the relative interior of the feasible set is obtained simply by replacing the inequalities by strict inequalities. The positivity of S_0 and the second constraint then imply that $y_n - z_n > 0$ for all $n \in \mathcal{N}$ for every y and z in the relative interior. It follows that the seller's price equals the optimum value of

$$\begin{aligned}
& \underset{y, z}{\text{maximize}} && \sum_{k=0}^K \sum_{n \in \mathcal{N}_k} z_n X_n \\
& \text{subject to} && y_0 = 1, \\
& && \sum_{m \in \mathcal{C}(n)} \frac{y_m}{y_n - z_n} S_m = S_n \quad n \in \mathcal{N}_k, \quad k = 0, \dots, K-1, \\
& && y > z > 0.
\end{aligned}$$

The second constraint means that there exists a $Q \in \text{ri } \mathcal{M}$ such that

$$\frac{y_m}{y_n - z_n} = \frac{q_m \beta_m}{q_n \beta_n} \iff \frac{y_m}{q_m \beta_m} = \frac{y_n}{q_n \beta_n} - \frac{z_n}{q_n \beta_n}.$$

Making the change of variables

$$f_n = \frac{y_n}{q_n \beta_n} \quad \text{and} \quad e_n = \frac{z_n}{q_n \beta_n},$$

we can express the seller's price as

$$\begin{aligned} & \underset{f, e, Q}{\text{maximize}} && \sum_{k=0}^K \sum_{n \in \mathcal{N}_k} q_n \beta_n e_n X_n \\ & \text{subject to} && f_0 = 1, \\ & && f_n = f_{a(n)} - e_{a(n)} \quad n \in \mathcal{N}_k, \quad k = 1, \dots, K, \\ & && f > e > 0, \\ & && Q \in \text{ri } \mathcal{M}. \end{aligned}$$

The constraints on f and e just mean that $e > 0$ and $\sum_{m \in \mathcal{A}(n)} e_n < 1$ for all $n \in \mathcal{N}_K$. Thus, we can write the seller's price as

$$\sup_{e \in \tilde{E}} \sup_{Q \in \mathcal{M}} E^Q \sum_{k=0}^K \beta_k X_k e_k = \sup_{Q \in \mathcal{M}} \sup_{e \in \tilde{E}} E^Q \sum_{k=0}^K \beta_k X_k e_k.$$

Since the objective is linear in e , the optimum value is attained in at an extreme point of \tilde{E} at which e is integer. This yields the desired expressions. \square

5 The arbitrage-free interval

Theorems 4 and 5 imply that, in an arbitrage-free market, buyer's price is at most the seller's price, and in a complete market, the prices are equal. We also know that any value outside the interval given by the two prices leads to arbitrage while any value strictly inside does not. Whether the buyer's and seller's prices themselves are arbitrage-free is still an open question. Like in the case of ECCs, this is related to "replicability" of the claim. We will say that X is *replicable* if there exists a trading strategy θ and a stopping time τ such that θ together with some V satisfies the constraints of (SP) and $S_\tau \cdot \theta_\tau = X_\tau$ holds P -almost surely.

Theorem 6 *In an arbitrage free market, the following are equivalent.*

1. X is replicable;
2. the buyer's and seller's prices of X are equal;
3. the buyer's and seller's prices are both arbitrage-free;
4. the seller's price is arbitrage-free.

Proof. Assume first that X is replicable and let $\bar{\theta}$ and $\bar{\tau}$ be the associated trading strategy and stopping time. If $\hat{\theta}$ is any optimal solution to (SP), then the strategy $\theta := \hat{\theta} - \bar{\theta}$ satisfies

$$\begin{aligned} S_0 \cdot \theta_0 &\leq 0, \\ S_k \cdot (\theta_k - \theta_{k-1}) &= 0 \quad P\text{-a.s.} \quad k = 1, \dots, K, \\ S_{\bar{\tau}} \cdot \theta_{\bar{\tau}} &\geq 0 \quad P\text{-a.s.} \end{aligned}$$

Since there is no arbitrage, it must hold that $S_0 \cdot \tilde{\theta}_0 = S_0 \cdot \bar{\theta}_0$. As for the buyer, let e be the exercise strategy corresponding to $\bar{\tau}$, and define the trading strategy θ by $\hat{\theta}_t = -\bar{\theta}_t$ if $t < \bar{\tau}$ and $\hat{\theta}_t = 0$ if $t \geq \bar{\tau}$. This gives a feasible solution to (BP) with objective value $S_0 \cdot \bar{\theta}_0$. Thus, the buyer's and seller's prices must be equal.

If the buyer's and seller's prices are equal, then one can implement both the buyer's and seller's strategies with zero initial investment. Because both these strategies guarantee nonnegative terminal wealth and since the market is arbitrage-free, both strategies must end up with zero terminal wealth P -almost surely. So the buyer's and seller's prices are both arbitrage free.

If the seller's price is arbitrage-free, there must exist, for every seller's trading strategy $\bar{\theta}$, a stopping time $\bar{\tau}$ so that $S_{\bar{\tau}} \cdot \bar{\theta}_{\bar{\tau}} = X_{\bar{\tau}}$ holds P -almost surely, since otherwise, there would be arbitrage. Thus, X is replicable. \square

By the equivalence of 2 and 4, the seller's price is not arbitrage-free when buyer's and seller's prices are not equal. The following example shows that the same cannot be said about the buyer's price.

Example 7 Consider the one-period model ($K = 1$) where $\mathcal{N}_1 = \{1, 2\}$, $S_n = 1$ for every $n \in \mathcal{N}$ and $X_0 = c$, $X_1 = 0$, $X_2 = 1$. The buyer's and seller's problems are then easily solved by inspection. The buyer's price equals $\max\{c, 0\}$ whereas the seller's price is $\max\{c, 1\}$.

For $c \geq 1$, the buyer's and seller's prices are both arbitrage free by Theorem 6.

For $c \in (0, 1)$, buyer has the unique optimal strategy $e_0 = 1$, $e_1 = 0$, $e_2 = 0$ and $\theta_0 = -c$, $\theta_1 = 0$, $\theta_2 = 0$. Here the terminal wealth is P -a.s. zero, so the buyer's price is arbitrage-free.

For $c \leq 0$, the buyer has e.g. the following optimal strategy $e_0 = 0$, $e_1 = 0$, $e_2 = 1$ and $\theta_0 = 0$, $\theta_1 = 0$, $\theta_2 = 1$. Here the terminal wealth is not P -a.s. zero, so the buyer's price is not arbitrage-free.

6 More realistic market models

In real markets, there hardly exist any securities that can be traded in unbounded amounts or without transaction costs. Having defined the arbitrage interval in terms of real world concepts, instead of such imaginary concepts as martingale measures, immediately suggests how to adapt the definition to more complicated situations. As an example, consider a situation where portfolios of the traders are required to lie in a set $C \subset \mathbb{R}^{J+1}$. Short selling restrictions etc. can be expressed in the form of such restrictions; see e.g. [6]. In such a situation, the buyer's and sellers prices are naturally defined as optimum values of

$$\begin{aligned}
& \underset{V, \theta, e}{\text{maximize}} && V \\
& \text{subject to} && S_0 \cdot \theta_0 = X_0 e_0 - V, \\
& && S_k \cdot (\theta_k - \theta_{k-1}) = X_k e_k && P\text{-a.s. } k = 1, \dots, K, \\
& && S_K \cdot \theta_K \geq 0 && P\text{-a.s.}, \\
& && \theta_k \in C && P\text{-a.s. } k = 0, \dots, K, && (\text{BP}_C) \\
& && \sum_{k=0}^K e_k \leq 1 && P\text{-a.s.}, \\
& && e_k \in \{0, 1\} && P\text{-a.s. } k = 0, \dots, K, \\
& && \theta, e \text{ are } (\mathcal{F}_k)\text{-adapted,}
\end{aligned}$$

and

$$\begin{array}{ll}
\underset{V, \theta}{\text{minimize}} & V \\
\text{subject to} & S_0 \cdot \theta_0 = V, \\
& S_k \cdot (\theta_k - \theta_{k-1}) = 0 \quad P\text{-a.s., } k = 1, \dots, K, \\
& S_K \cdot \theta_K \geq 0 \quad P\text{-a.s.}, \\
& \theta_k \in C \quad P\text{-a.s. } k = 0, \dots, K, \\
& S_k \cdot \theta_k \geq X_k \quad P\text{-a.s., } k = 0, \dots, K, \\
& \theta \text{ is } (\mathcal{F}_k)\text{-adapted,}
\end{array} \tag{SP_C}$$

respectively. Adding constraints to an optimization problem makes the optimum value worse if anything, so portfolio constraints typically lower the buyer's price and increase the seller's price, thus making the arbitrage-free interval wider.

One could also add transaction constraints of the form $\theta_k - \theta_{k-1} \in D$. Through introduction of purchase and sales variables, it is also easy to model transaction costs; see e.g. [3]. In these more complex models, it is no longer possible, in general, to write the buyer's and seller's prices in terms of martingale measures and stopping times. From practical point of view, however, this doesn't really matter since the martingale expressions are not very useful for computation of the actual values of the buyer's and seller's prices. Indeed, it is more natural to base computational procedures on the original, primal, formulations of the buyer's and seller's prices; see [3] for the case of ECCs.

References

- [1] A. Bensoussan. On the theory of option pricing. *Acta Appl. Math.*, 2(2):139–158, 1984.
- [2] R. Buckdahn and Y. Hu. Pricing of American contingent claims with jump stock price and constrained portfolios. *Math. Oper. Res.*, 23(1):177–203, 1998.
- [3] N. C. P. Edirisinghe, Vasant Naik, and R. Uppal. Optimal replication of options with transactions costs and trading restrictions. *Journal of Financial and Quantitative Analysis*, 28:117–138, 1993.
- [4] H. Föllmer and A. Schied. *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2002. An introduction in discrete time.
- [5] I. Karatzas. On the pricing of American options. *Appl. Math. Optim.*, 17(1):37–60, 1988.
- [6] I. Karatzas and S. G. Kou. Hedging American contingent claims with constrained portfolios. *Finance Stoch.*, 2(3):215–258, 1998.
- [7] A. J. King. Duality and martingales: a stochastic programming perspective on contingent claims. *Math. Program.*, 91(3, Ser. B):543–562, 2002. ISMP 2000, Part 1 (Atlanta, GA).
- [8] A. J. King, M. Koivu, and T. Pennanen. Calibrated option bounds. *Stochastic Programming E-Print Series*, 2003.
- [9] R. Myeni. The pricing of the american option. *The Annals of Applied Probability*, 2:1–23, 1992.
- [10] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.