STRESS TESTING FOR VaR AND CVaR

Jitka DUPAČOVÁ and Jan POLÍVKA
Dept. of Probability and Math. Statistics, Charles University, Prague, Czech Republic

ABSTRACT:

Practical use of the contamination technique in stress testing for risk measures Value at Risk (VaR) and Conditional Value at Risk (CVaR) and for optimization problems with these risk criteria is discussed. Whereas for CVaR its application is straightforward, the presence of the simple chance constraint in the definition of VaR requires that various distributional and structural properties are fulfilled, namely, for the unperturbed problem. These requirements rule out direct applications of the contamination technique in the case of discrete distributions, which includes the empirical VaR. On the other hand, in the case of a normal distribution and parametric VaR one may exploit stability results valid for quadratic programs.

Key words: Scenario-based stochastic programs, validation of results, stress testing, contamination bounds, VaR, CVaR.

AMS subject classification: 90C15, 91B28

1 Stress testing

Stress testing is a term used in financial practice without any generally accepted definition. It appears in the context of quantification of losses or risks that may appear under special, mostly extremal circumstances [19]. Such circumstances are described by certain scenarios which may come from historical experience (a crisis observed in the past)— *historical stress test,* or may be judged to be possible in the future given changes of macroeconomic, socioeconomic or political factors—*prospective stress test,* etc. The performance of the obtained optimal decision is then evaluated along these, possibly dynamic scenarios or the model is solved with an alternative input. The stress testing approaches differ among the institutions and also due to the nature of the tested problem and the way in which the stress scenarios have been selected. In this paper, we focus on stress testing of two risk measures, VaR and CVaR, giving the "test" a more precise meaning. This is made possible by exploitation of parametric sensitivity results and the contamination technique.

2 Basic formulas

Let $\mathcal{X} \subset R^n$ be a nonempty, closed set of feasible decisions x,
$\omega \in \Omega \subset R^m$ be a random vector with probability measure P on Ω which does not depend on x.
Denote further

- $g(x, \omega)$ the random loss defined on $\mathcal{X} \times \Omega$,
- $P\{\omega : g(x, \omega) \leq v\} := G(x, P; v)$ the distribution function of the loss connected with a *fixed* decision $x \in \mathcal{X}$,
- $\alpha \in (0, 1)$ the selected confidence level.

Value at Risk (VaR) was introduced and recommended as a generally applicable risk measure to quantify, monitor and limit financial risks, to identify losses which occur with an acceptably small probability. There exist several slightly different formal definitions of VaR which coincide for continuous probability distributions. Here, we shall also deal with VaR for discrete distributions and we shall use the definition from [27]:
The **Value at Risk** at the confidence level α is defined by

$$\mathrm{VaR}_\alpha(x, P) = \min\{v \in R : G(x, P; v) \geq \alpha\} \tag{1}$$

and the "upper" Value at Risk is

$$\mathrm{VaR}^+_\alpha(x, P) = \inf\{v \in R : G(x, P; v) > \alpha\}.$$

Hence, a random loss greater than VaR_α occurs with probability equal (or less than) $1 - \alpha$. This interpretation is well understood in the financial practice. However, VaR_α does not quantify the loss, it is a qualitative risk measure, and it lacks in general the subadditivity property. (An exception are elliptic distributions G, cf. [12], of which the normal distribution is a special case.) Various specific features and weak points of the recommended VaR methodology are summarized and discussed, e.g. in collection [5] or in Chapter 10 of [24]. To settle these problems new risk measures have been introduced, see e.g. [1]. We shall exploit results of [27] to discuss one of them, the Conditional Value at Risk, which may be linked with integrated chance constraints, cf. [18], with constraints involving conditional expectations [23] and with the absolute Lotrenz curve at the point α, cf. [20].

According to [27], CVaR_α, the **Conditional Value at Risk** at the confidence level α, is defined as the mean of the α-tail distribution of $g(x, \omega)$ which in turn is defined by

$$\begin{array}{lcll} G_\alpha(x,P;v) = & 0 & \text{for } v < \mathrm{VaR}_\alpha(x,P) & \\ G_\alpha(x,P;v) = & \frac{G(x,P;v)-\alpha}{1-\alpha} & \text{for } v \geq \mathrm{VaR}_\alpha(x,P). & \quad (2) \end{array}$$

We shall assume in the sequel that $g(x,\omega)$ is a continuous function of x for all $\omega \in \Omega$ and $E_P|g(x,\omega)| < \infty \, \forall x \in \mathcal{X}$. For $v \in R$ define

$$\Phi_\alpha(x,v,P) := v + \frac{1}{1-\alpha} E_P(g(x,\omega)-v)^+. \tag{3}$$

The fundamental minimization formula by Rockafellar and Uryasev [27] helps to evaluate CVaR for general loss distributions and to analyze its stability including stress testing.

Theorem 1 [27]. *As a function of* v, $\Phi_\alpha(x,v,P)$ *is finite and convex (hence continuous) with*

$$\min_v \Phi_\alpha(x,v,P) = \mathrm{CVaR}_\alpha(x,P) \tag{4}$$

and

$$\arg\min_v \Phi_\alpha(x,v,P) = [\mathrm{VaR}_\alpha(x,P), \mathrm{VaR}^+_\alpha(x,P)], \tag{5}$$

a nonempty compact interval (possibly one point only).

The auxiliary function $\Phi_\alpha(x,v,P)$ is evidently linear in P and convex in v. Moreover, if $g(x,\omega)$ is a *convex* function of x, $\Phi_\alpha(x,v,P)$ is convex jointly in (v,x). In addition, $\mathrm{CVaR}_\alpha(x,P)$ is continuous with respect to α, cf. [27].
If P is a discrete probability distribution concentrated on $\omega^1,\dots,\omega^S$, with probabilities $p_s > 0$, $s = 1,\dots,S$, and x a *fixed* element of $\mathcal{X}$, then the optimization problem (4) has the form

$$\min_v \left\{ v + \frac{1}{1-\alpha} \sum_s p_s (g(x,\omega^s)-v)^+ \right\} \tag{6}$$

and can be further rewritten as

$$\min_{v,y_1,\dots,y_S} \left\{ v + \frac{1}{1-\alpha} \sum_s p_s y_s : y_s \geq 0,\; y_s + v \geq g(x,\omega^s) \forall s \right\}.$$

There are various papers discussing properties of VaR, CVaR and relations between CVaR and VaR, see e.g. [5, 21]. We shall focus on contamination-based stress testing for these two risk measures.

3 Stress testing for CVaR

For a *fixed* vector x we consider now a stress test of $\mathrm{CVaR}_\alpha(x,P)$, i.e., of the optimal value of (4). Let Q be the stress probability distribution. We apply the contamination technique and proceed as explained, e.g. in [9, 10]. This means that we model perturbation of P via its contamination by another fixed probability distribution Q, using the family of contaminated probability distributions

$$P_\lambda = (1-\lambda)P + \lambda Q, \quad \lambda \in [0,1]$$

and thus reducing the sensitivity analysis to the case of one *scalar* parameter λ. The idea to model perturbations of P by means of one scalar parameter is a natural one and for sufficiently small values of λ the contaminated probability distribution P_λ belongs to a topological neighborhood of P (with respect to standard probability metrics).

The corresponding objective function $\Phi_\alpha(x,v,\lambda) := \Phi_\alpha(x,v,P_\lambda)$ is linear in λ and convex in v. Its optimal value $\mathrm{CVaR}_\alpha(x,\lambda) := \mathrm{CVaR}_\alpha(x,P_\lambda)$ is concave in λ on [0,1] which guarantees its continuity and existence of directional derivatives in the *open* interval $(0,1)$, whereas continuity at the point $\lambda = 0$ is a property related with stability results for the optimization problem in question. In principle, one needs a nonempty, bounded set of optimal solutions of the initial problem, a condition which is fulfilled in our case, see (5).

The derivative of the optimal value $\mathrm{CVaR}_\alpha(x,\lambda)$ at $\lambda = 0^+$ equals

$$\frac{d}{d\lambda}\mathrm{CVaR}_\alpha(x,0^+) = \min_v \Phi_\alpha(x,v,Q) - \mathrm{CVaR}_\alpha(x,P), \tag{7}$$

with minimization carried over the set (5) of optimal solutions of (4) formulated and solved for the probability distribution P. An upper bound for the derivative is obtained when minimization over (5) is replaced by evaluation of $\Phi_\alpha(x,v,Q)$ at an arbitrary optimal solution, an element of (5).

The contamination bounds for $\mathrm{CVaR}_\alpha(x,\lambda)$ follow from concavity of $\mathrm{CVaR}_\alpha(x,\lambda)$ with respect to λ:

$$(1-\lambda)\mathrm{CVaR}_\alpha(x,0) + \lambda\mathrm{CVaR}_\alpha(x,1) \le \mathrm{CVaR}_\alpha(x,\lambda) \le \tag{8}$$

$$\mathrm{CVaR}_\alpha(x,0) + \lambda\frac{d}{d\lambda}\mathrm{CVaR}_\alpha(x,0^+),\ 0 \le \lambda \le 1.$$

The contaminated probability distribution P_λ may be also understood as a result of contaminating Q by P and an alternative upper bound may be constructed in a similar way.

3.1 Consider first an application of the contamination bounds to **stress testing of the scenario-based form (6) of CVaR.** Let P be a discrete probability distribution concentrated on $\omega^1, \dots, \omega^S$ with probabilities p_s, $s = 1, \dots, S$, x a *fixed* element of $\mathcal{X}$ and Q a discrete probability distribution carried by S' stress or out-of-sample scenarios ω^s, $s = S+1, \dots, S+S'$, with probabilities $p_s, s = S+1, \dots, S+S'$. Both $\mathrm{CVaR}_\alpha(x, P)$ and $\mathrm{CVaR}_\alpha(x, Q)$ values can be obtained by solving the corresponding linear programs (6). Denote $v^*(x, P)$ an optimal solution of (6) for fixed $x \in \mathcal{X}$ and for distribution P.

Bounds for CVaR_α for the contaminated probability distribution P_λ carried by the initial scenarios ω^s, $s = 1, \dots, S$ with probabilities $(1-\lambda)p_s$, $s = 1, \dots, S$, and by the stress scenarios ω^s, $s = S+1, \dots, S+S'$, with probabilities λp_s, $s = S+1, \dots, S+S'$, have the form

$$(1-\lambda)\mathrm{CVaR}_\alpha(x, P) + \lambda \mathrm{CVaR}_\alpha(x, Q) \leq \mathrm{CVaR}_\alpha(x, P_\lambda) \leq \tag{9}$$

$$(1-\lambda)\mathrm{CVaR}_\alpha(x, P) + \lambda \Phi_\alpha(x, v^*(x, P), Q) = \Phi_\alpha(x, v^*(x, P), P_\lambda)$$

and are valid for all $\lambda \in [0, 1]$; compare with (7)–(8).

In the special case of a degenerate probability distribution Q carried only by one scenario, ω^*, $\mathrm{CVaR}_\alpha(x, Q) = g(x, \omega^*)$ and the value $\Phi_\alpha(x, v^*(x, P), Q) = v^*(x, P) + \frac{1}{1-\alpha}(g(x, \omega^*) - v^*(x, P))^+$. The difference between the upper and lower bound equals

$$\lambda[\Phi_\alpha(x, v^*(x, P), Q) - \mathrm{CVaR}_\alpha(x, Q)] =$$

$$\lambda[v^*(x, P) + \frac{1}{1-\alpha}(g(x, \omega^*) - v^*(x, P))^+ - g(x, \omega^*)].$$

In typical applications, the "stress test" is reduced to evaluating the performance of the already obtained optimal solution along the new scenarios, i.e., to evaluation of $\Phi_\alpha(x, v^*(x, P), Q)$, or to obtaining the optimal value such as $\mathrm{CVaR}_\alpha(x, Q)$ for Q carried by the stress scenarios. The contamination approach presented here exploits both these criteria jointly to *quantify* the influence of stress scenarios, taking into account also the probability of their occurrence and thus it provides a genuine stress test.

3.2 To derive **sensitivity properties of optimal solutions** of (4) assume that the optimal solution is *unique*, $v^*(x, P)$; hence, it equals $\mathrm{VaR}_\alpha(x, P)$. This simplifies also the form of the derivative of $\mathrm{CVaR}_\alpha(x, \lambda)$ in (7) to $\Phi_\alpha(x, \mathrm{VaR}_\alpha(x, P), Q) - \mathrm{CVaR}_\alpha(x, P)$. The general results concerning properties of optimal solutions for contaminated distributions, see e.g. [7, 8, 31], require in addition certain differentiability properties of the objective function (3) in (4). To this end we assume that the probability distribution function

$G(x, P; v)$ is continuous, with a positive, continuous density $p(x, P; v)$ on a neighborhood of the unique optimal solution $v^*(x, P) = \mathrm{VaR}_\alpha(x, P)$.
For a fixed $x \in \mathcal{X}$ we denote $g(x, \omega) = \eta$. Except for $v = \eta$, the derivative $\frac{d}{dv}(\eta - v)^+$ exists and

$$\frac{d}{dv}(\eta - v)^+ = -\frac{1}{2}\left(1 + \frac{\eta - v}{|\eta - v|}\right).$$

Thanks to the assumed properties of the distribution function $G(x, P; v)$,

$$E\frac{d}{dv}(\eta - v)^+ = -P(\eta > v) = -1 + G(x, P; v),$$

$$\frac{d}{dv}\Phi_\alpha(x, v, P) = 1 + \frac{G(x, P; v) - 1}{1 - \alpha}$$

and the optimality condition $\frac{d}{dv}\Phi_\alpha(x, v, P) = 0$ provides, as expected,

$$\mathrm{VaR}_\alpha(x, P) = v^*(x, P) = G(x, P)^{-1}(\alpha).$$

The second order derivative $\frac{d^2}{dv^2}\Phi_\alpha(x, v, P) = \frac{p(x,P;v)}{1-\alpha} > 0$ on a neighborhood of $v^*(x, P)$. A direct application of the implicit function theorem to the system

$$\frac{d}{dv}\Phi_\alpha(x, v, P_\lambda) = 0$$

implies existence and uniqueness of optimal solution $v^*(x, \lambda) := v^*(x, P_\lambda)$ of the contaminated problem (4) for $\lambda > 0$ small enough, and the form of its derivative

$$\frac{d}{d\lambda}v^*(x, P_\lambda) = \frac{d}{d\lambda}\mathrm{VaR}_\alpha(x, P_\lambda)$$

which is equal to

$$-\frac{G(x, Q; v^*(x, P)) - \alpha}{p(x, P; v^*(x, P))}$$

for $\lambda = 0^+$. Here, $G(x, Q; v)$ denotes the distribution function of loss under probability distribution Q. Related results for absolutely continuous probability distributions P, Q can be found, e.g., in [25].

3.3 As the next step, let us discuss briefly **optimization problems with the $\mathbf{CVaR}_\alpha(x, P)$ objective function**

$$\text{minimize } \mathrm{CVaR}_\alpha(x, P)$$

on a closed, nonempty set $\mathcal{X} \in R^n$. Using (4), the problem is

$$\min_{x,v} \Phi_\alpha(x, v, P),\ x \in \mathcal{X}. \tag{10}$$

For $\mathcal{X}$ convex, independent of P, and for loss functions $g(\bullet, \omega)$ convex for all ω, $\Phi_\alpha(x, v, P)$ is convex in (x, v) and standard stability results apply. Moreover, if P is the discrete probability distribution considered in 3.1, $g(\bullet, \omega)$ a linear function of x, say, $g(\bullet, \omega) = x^\top \omega$, and $\mathcal{X}$ convex polyhedral, we get a linear program

$$\min_{v, y_1, \ldots, y_S, x} \{v + \frac{1}{1-\alpha} \sum_s p_s y_s : y_s \geq 0,\, x^\top \omega^s - v - y_s \leq 0\, \forall s,\, x \in \mathcal{X}\}. \quad (11)$$

Let $(v_C^*(P),\, x_C^*(P))$ be an optimal solution of (10) and denote $\varphi_C(P)$ the optimal value. To obtain contamination bounds for the optimal value of (10) with P contaminated by a stress probability distribution Q it is sufficient to assume a compact set $\mathcal{X}$, e.g., $\mathcal{X} = \{x \in R^n : \sum_i x_i = 1,\, x_i \geq 0\, \forall i\}$. For a detailed discussion see [6]. The bounds follow the usual pattern, compare with (9):

$$(1-\lambda)\varphi_C(P) + \lambda \varphi_C(Q) \leq \varphi_C(P_\lambda) \leq (1-\lambda)\varphi_C(P) + \lambda \Phi_\alpha(x_C^*(P), v_C^*(P), Q). \quad (12)$$

To apply them one has to evaluate $\Phi_\alpha(x_C^*(P), v_C^*(P), Q)$ and to solve (10) with P replaced by the stress distribution Q.

3.4 An illustrative example. The instruments used in the portfolio management problem (11) are total return stock and bond indices given in the following table.

Asset	**Acronym**	**Description**
MSCI Gross Return index US, USD	1	stock index
MSCI Gross Return index UK, USD	2	stock index
MSCI Gross Return index Germany, USD	3	stock index
MSCI Gross Return index Japan, USD	4	stock index
US government bond index (1-3 y mat), USD	5	
US government bond index (7-10 y mat), USD	6	
UK government bond index (1-3 y mat), GPB	7	
UK government bond index (7-10 y mat), GPB	8	
Germany government bond index (1-5 y mat), EUR	9	
Germany government bond index (7+ y mat), EUR	10	
Japan government bond index (1-3 y mat), JPY	11	
Japan government bond index (7 - 10 y mat), JPY	12	

Table 1. Portfolio assets (MSCI and JP Morgan indexes)

The portfolio limits were set in all cases to $x_i \leq 0.3$, hence,

$$\mathcal{X} = \left\{ x \in R^n : \sum_i x_i = 1,\, 0 \leq x_i \leq 0.3\, \forall i \right\}.$$

Assume that the probability distribution P is the distribution of losses under "normal" conditions whereas the probability distribution Q refers to the situation when adverse conditions prevail on the world market. Both P and Q are distributions of monthly percentage losses to assets $i = 1, \ldots, 12$ which were converted to the home currency (EUR) using the exchange rate mid. We do not consider transaction costs.

The following, historical simulation resembling approach, was taken to construct discrete distributions P and Q. For the asset $i = 1$ (US asset market returns) the percentage returns (not losses) in home currency were computed. We took the empirical 25% quantile to be the cut-off value for all returns of asset 1. The returns below the cut-off value (and all corresponding returns of other assets at the same date) are attributed to a period of adverse conditions prevailing on the market and hence this data set serves as the input for approximation of the distribution Q. The rest of the data sample was used for fitting the distribution P.

The two discrete probability distributions P, Q approximating the true continuous distribution of assets percentage losses in home currency were constructed using the method [15]. We prescribed that both discrete approximations P, Q were carried by 5184 equiprobable scenarios. The empirical means, variances, covariances, skewnesses and kurtosises computed separately from the two data samples enter the scenario fitting procedure for P and Q.

After solving the two CVaR minimization problems with $\alpha = 0.99$, contamination bounds (12) sharpened to

$$(1-\lambda)\varphi_C(P) + \lambda\varphi_C(Q) \leq \varphi_C(P_\lambda) \leq \tag{13}$$

$$\min\{(1-\lambda)\varphi_C(P)+\lambda\Phi_\alpha(x_C^*(P), v_C^*(P), Q), \lambda\varphi_C(Q)+(1-\lambda)\Phi_\alpha(x_C^*(Q), v_C^*(Q), P)\}$$

were constructed. The results of contamination are presented in graph and table bellow. The VaR values $v_C^*(P)$, $v_C^*(Q)$ for distributions P, Q calculated for the optimal portfolios $x_C^*(P)$, $x_C^*(Q)$ are obtained as a byproduct.

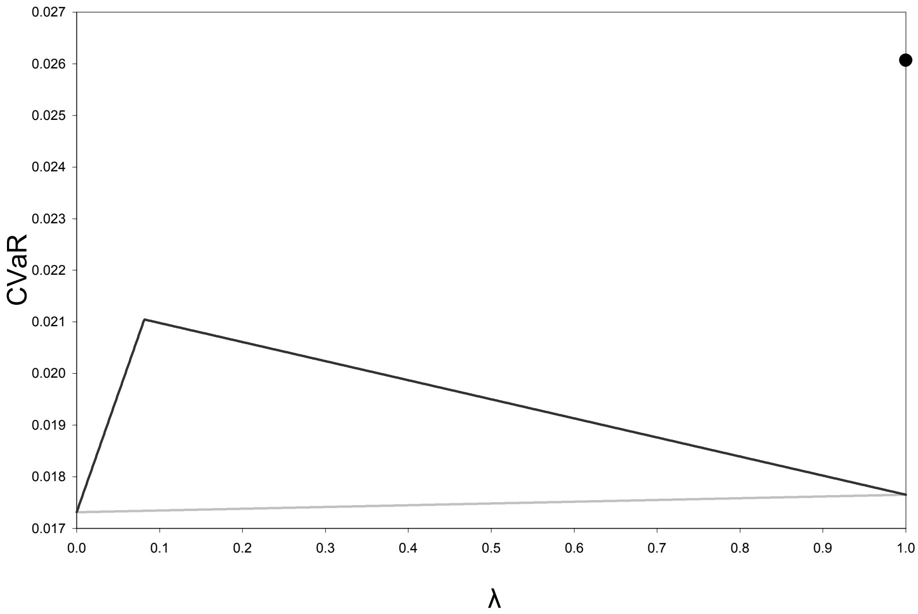

Figure 1: Contamination bounds for CVaR optimization problem without constrain on return.

Quantity	Value
$\varphi_C(P)$	0.01731
$\varphi_C(Q)$	0.01765
$\Phi_\alpha(x_C^*(P), v^*(P), Q)$	0.06309
$\Phi_\alpha(x_C^*(Q), v^*(Q), P)$	0.02135
$x_1^*(P)$	0.12880
$x_7^*(P)$	0.20030
$x_9^*(P)$	0.30000
$x_{10}^*(P)$	0.26470
$x_{11}^*(P)$	0.10620
$v_C^*(P)$	0.01365
$x_5^*(Q)$	0.10000
$x_7^*(Q)$	0.30000
$x_9^*(Q)$	0.30000
$x_{10}^*(Q)$	0.30000
$v_C^*(Q)$	0.01588
$\text{CVaR}(x_C^*(P), Q)$	0.02607

Table 2. Quantities used in contamination bounds (13) and non-zero components of optimal solutions $x_C^*(P)$ and $x_C^*(Q)$; $\alpha = 0.99$.

Some observations:

- The two minimal CVaR values $\varphi_C(P)$, $\varphi_C(Q)$ are not very different. This is the result of optimal restructuring the portfolio in the adverse market situation; see the changed composition of the optimal portfolios. The CVaR value for probability distribution Q at the optimal portfolio $x_C^*(P)$, i.e., without restructuring the portfolio, is much higher.

- The value $\Phi_\alpha(x_C^*(P), v_C^*(P), Q)$ is relatively large and this determines the steep slope of the left upper bound.

- The contamination bounds, see Figure 1, are relatively loose. The maximal difference between the upper and lower bounds occurs approximately at $\lambda = 0.1$. For $\lambda = 0.5$, i.e. for the distribution carried by the pooled sample of 10368 equiprobable scenarios, the minimal CVaR value lies in $[0.0175, 0.0195]$.

3.5 Finally, consider **stress testing for the mean-CVaR problems**, i.e., for bi-criteria problems in which one aims simultaneously on minimization of $\text{CVaR}_\alpha(x, P)$ and maximization of an expected return criterion $E_P r(x, \omega)$ on $\mathcal{X}$, see e.g. [2, 17, 21, 26, 32].

To obtain an efficient solution, one minimizes on $\mathcal{X}$ a parametrized objective function

$$\mathrm{CVaR}_\alpha(x, P) - \rho E_P r(x, \omega) \tag{14}$$

with parameter $\rho > 0$ or assigns a parametric bound on one of the criteria and solves, e.g.,

$$\min \mathrm{CVaR}_\alpha(x, P) \text{ on the set } \{x \in \mathcal{X} : E_P r(x, \omega) \geq r\}. \tag{15}$$

The optimal solution and the corresponding values of the two criteria, CVaR_α and expected return, depend on the chosen parameter values. To get the efficient frontier, (14) and (15) may be solved by parametric programming techniques with scalar parameters ρ or r, respectively. For $g(x, \omega) = x^\top \omega = -r(x, \omega)$, for polyhedral set $\mathcal{X}$ and a discrete probability distribution P, both (14) and (15) are then parametric linear programs with one scalar parameter, see e.g. [29]. By solving (15), the efficient frontier is directly obtained. To get the efficient frontier in the case of (14), values of $E_P r(x, \omega)$ and $\mathrm{CVaR}_\alpha(x, P)$ have to be computed at the optimal solution obtained for a specific value of ρ. Hence, (15) is favored for the straightforward possibility to interpret the trade-off between the two criteria, whereas the form of (14) is suitable for developing sensitivity and stability results, including stress testing.

Contamination of the probability distribution P introduces an additional parameter λ into (14), (15) and the two problems loose in general the easily solvable form of parametric linear programs: nonlinearity with respect to ρ and λ appears in the objective function of (14) and both the objective function and the set of feasible solutions of (15) depend on parameters. It is still possible to obtain directional derivatives of the optimal value function for the corresponding contaminated problem, however, the optimal value function is no longer concave, hence, the crucial property for construction of contamination bounds is lost. The same applies also to problem formulations with several CVaR constraints, each with a different confidence level α, called "risk-shaping"; cf. [27].

Nevertheless, contamination bounds may be obtained for the special form of the return function $r(x, \omega) = -x^\top \omega$ and for a certain class of probability distributions. Problem (15) is

$$\text{minimize } \mathrm{CVaR}_\alpha(x, P)$$

on the set

$$\mathcal{X}(P, r) = \{x \in \mathcal{X} : -x^\top E_P \omega \geq r\}. \tag{16}$$

Let $\varphi_r(P)$ denote the optimal value and $\mathcal{X}_r^*(P)$ the set of optimal solutions and assume that $\mathcal{X}_r^*(P) \neq \emptyset$, is bounded.

Suppose in addition that the expected values are equal, $E_P\omega = E_Q\omega = \bar{\omega}$. Such an assumption is in agreement with scenario generation methods based on moment fitting, e.g. [15, 16], and has been used also in stability studies of [17]. Then the expected return constraint is $-x^\top\bar{\omega} \geq r$ both for the initial probability distribution P, the contaminating distribution Q and for P_λ, $\lambda \in [0,1]$, and it does not depend on λ. The optimal value function $\varphi_r(P_\lambda) = \varphi_r(\lambda)$ is concave and the contamination bounds have a similar form as (12) and (13). They are obtained for (10) with the set of feasible decisions $\mathcal{X}$ replaced by $\mathcal{X}(P,r) = \{x \in \mathcal{X} : -x^\top\bar{\omega} \geq r\}$. Moreover, there are parametric programming techniques (e.g. [14]) applicable to the contaminated problem (16), i.e., to minimization of $\mathrm{CVaR}_\alpha(x, P_\lambda)$ on the set $\mathcal{X}(P,r) = \{x \in \mathcal{X} : -x^\top\bar{\omega} \geq r\}$.
Notice that for $E_P\omega = E_Q\omega = \bar{\omega}$, problem (14) simplifies, too, and the objective function is *linear* in the two parameters ρ, λ.

When the expected loss differs under P and Q, the optimal value $\varphi_C(P_\lambda)$ is a natural lower bound for $\varphi_r(P_\lambda)$, hence by (12),

$$\varphi_r(P_\lambda) \geq (1-\lambda)\varphi_C(P) + \lambda\varphi_C(Q). \tag{17}$$

To construct an upper bound for $\varphi_r(P_\lambda)$ we add the additional constraint $-x^\top E_Q\omega \geq r$ to $\mathcal{X}(P,r)$. The set of feasible solutions $\mathcal{X}(P,r) \cap \mathcal{X}(Q,r) \subset \mathcal{X}(P_\lambda, r)$ is polyhedral and does not depend on λ. If $\mathcal{X}(P,r) \cap \mathcal{X}(Q,r) \neq \emptyset$ we obtain a *concave* upper bound

$$U_r(\lambda) := \min_{x \in \mathcal{X}(P,r) \cap \mathcal{X}(Q,r)} \mathrm{CVaR}_\alpha(x, P_\lambda) \geq \varphi_r(P_\lambda)$$

which may be bounded from above by the corresponding upper contamination bound. The derivative at the point $\lambda = 0^+$ is of the familiar form— $\min \Phi_\alpha(x, v, Q) - U_r(0)$ with minimization carried over the set of optimal solutions of (10) for $\mathcal{X}$ replaced by $\mathcal{X}(P,r) \cap \mathcal{X}(Q,r)$; denote $\hat{x}_r(P), \hat{v}_r(P)$ one of them:

$$(1-\lambda)U_r(0) + \Phi_\alpha(\hat{x}_r(P), \hat{v}_r(P), Q) \geq U_r(\lambda) \geq \varphi_r(P_\lambda). \tag{18}$$

Naturally, the resulting bounds (17), (18) may be quite loose.

4 Stress testing for VaR

Up to the nonuniqueness of definitions, $\mathrm{VaR}_\alpha(x, P)$ is the same as the α-*quantile of the loss distribution* $G(x, P; v)$. One can also treat $\mathrm{VaR}_\alpha(x, P)$ as the optimal value of the stochastic program (1) with one probabilistic constraint. Such approach enables to exploit the existing stability results for stochastic programs of that form, cf. [28], which are valid under special distributional and regularity assumptions.

Normal distribution of losses is one of the manageable cases and, initially, *parametric* VaR was developed to quantify risks connected with normally distributed losses $g(x, \omega)$, whose distribution at a fixed point x is fully determined by its expectation $\mu(x)$ and variance $\sigma^2(x)$:

absolute $\mathrm{VaR}_\alpha(x) = \mu(x) + \sigma(x) \cdot u_\alpha$, and *relative* $\mathrm{VaR}_\alpha(x) = \sigma(x) \cdot u_\alpha$

where u_α is the α-quantile of the standard normal $\mathcal{N}(0, 1)$ distribution.
For an arbitrary $\alpha > 0.5$, minimization of the relative VaR_α reduces to minimization of the standard deviation (volatility) of the portfolio losses and minimization of the absolute VaR_α is minimization of the weighted sum of the standard deviation and the expectation.

4.1 Optimization problem with the relative $\mathrm{VaR}_\alpha(x, P)$ objective function. Choose $\alpha > 0.5$ and assume that losses are of the form

$$g(x, \omega) = x^\top \omega,$$

$\mathcal{X}$ is a nonempty, convex polyhedral set, $0 \notin \mathcal{X}$, ω is normally distributed with the mean vector μ and a positive definite variance matrix Σ.

The problem is to select portfolio composition $x \in \mathcal{X}$ such that VaR_α is minimal, i.e. to minimize the convex quadratic function $x^\top \Sigma x$ on the set $\mathcal{X}$. In this case, for all values of $\alpha > 0.5$ there is *the same, unique* optimal solution $x^*(\Sigma)$, the composition of the portfolio, which depends on the input variance matrix Σ that was obtained by an estimation procedure and is subject to an estimation error. The same optimal solution comes by minimization of $\mathrm{CVaR}_\alpha(x, P)$, see [26].
Asymptotic statistics and a detailed analysis of optimal solutions of parametric quadratic programs may help to derive asymptotic results concerning the "estimated" optimal portfolio composition obtained for an asymptotically normal estimate $\tilde{\Sigma}$ of Σ.
Here we follow a suggestion of [19] and rewrite the variance matrix as $\Sigma = DCD$ with the diagonal matrix D of "volatilities" (standard deviations of the marginal distributions) and the correlation matrix C. Changes in the covariances may be then modeled by "stressing" the correlation matrix C by a positive semidefinite *stress correlation matrix* $\hat{C}$

$$C(\gamma) = (1 - \gamma)C + \gamma \hat{C} \tag{19}$$

with $\gamma \in [0,1]$ a parameter. This type of perturbation of the initial quadratic program allows us to apply stability results of [3] to the perturbed problem

$$\min_{x \in \mathcal{X}} x^\top DC(\gamma)Dx,\ \gamma \in [0,1] : \tag{20}$$

- The optimal value $\varphi_V(\gamma)$ of (20) is concave and continuous in $\gamma \in [0,1]$;
- The optimal solution $x^*(\gamma)$ is a continuous vector in the range of γ where $C(\gamma)$ is positive definite;
- The directional derivative of $\varphi_V(\gamma)$

$$\varphi_V'(0^+) = x^{*\top}(0)D\hat{C}Dx^*(0) - \varphi_V(0).$$

Contamination bounds constructed as in Section 3 quantify the effect of the considered change of the input data.

4.2 Stress testing of the relative VaR with respect to an additional scenario ω^*. In this case, the contaminating distribution Q is degenerate, $Q = \delta_{\{\omega^*\}}$. Using [30] , we have

$$\frac{d}{d\lambda}\mathrm{VaR}_\alpha(x, P_\lambda)|_{\lambda=0^+} = \frac{\alpha - I\{g(x,\omega^*) \le \mathrm{VaR}_\alpha(x,P)\}}{\phi(\mathrm{VaR}_\alpha(x,P))}; \tag{21}$$

in the above formula, x is fixed, ϕ denotes the density of the normal distribution $\mathcal{N}(\mu(x), \Sigma(x))$ of $g(x,\omega)$ and I is the indicator function.

Assume in addition that $g(x,\omega) = x^\top\omega$. Using the results of Sections 4.1 and 3.2 for the normal distribution $P \sim \mathcal{N}(\mu, \Sigma)$ and degenerate distribution $Q = \delta_{\{\omega^*\}}$, we have the unique optimal portfolio $x^*(\Sigma)$ for P and both $\mathrm{VaR}_\alpha(x, P_\lambda)$ and its derivative with respect to λ are continuous for $\lambda \ge 0$ small enough. This can be used to derive **sensitivity properties of the minimal relative VaR value**

$$\varphi(\lambda) := \min_{x \in \mathcal{X}} \mathrm{VaR}_\alpha(x, P_\lambda)$$

in the case of $\mathcal{X} \neq \emptyset$, compact and small $\lambda > 0$, i.e., when testing the influence of a rare stress scenario. According to [4], we have

$$\varphi'(0^+) = \frac{d}{d\lambda}\mathrm{VaR}_\alpha(x^*(\Sigma), P_\lambda)|_{\lambda=0^+} = \frac{\alpha - I\{g(x^*(\Sigma),\omega^*) \le \mathrm{VaR}_\alpha(x^*(\Sigma),P)\}}{\phi(\mathrm{VaR}_\alpha(x^*(\Sigma),P)};$$

compare with (21). Then, the minimal VaR_α value for the stressed distribution P_λ is approximated by

$$\min_{x \in \mathcal{X}} \mathrm{VaR}_\alpha(x, P_\lambda) \cong \mathrm{VaR}_\alpha(x^*(\Sigma), P) + \lambda\varphi'(0^+)$$

for $\lambda > 0$ small enough.

4.3 Nonparametric VaR. For general probability distributions evaluation of VaR_α of a fixed portfolio x is mostly based on a nonparametric approach which is distribution free and applicable also for complicated financial instruments. One exploits a finite number, S, of scenarios so that for each fixed $x \in \mathcal{X}$, the underlying probability distribution P is replaced by a discrete distribution P_S carried by these scenarios and the probability distribution of the loss $g(x, \omega)$ is discrete with jumps at $g(x, \omega^s)\, \forall s$.
For a fixed x, let us order $g(x, \omega^s)$ as

$$g^{[1]} < \cdots < g^{[S]} \tag{22}$$

with probability of $g^{[s]}$ equal $p^{[s]} > 0\, \forall s$. Let s_{α,P_S} be the unique index such that

$$\sum_{s=1}^{s_{\alpha,P_S}} p^{[s]} \geq \alpha > \sum_{s=1}^{s_{\alpha,P_S}-1} p^{[s]}. \tag{23}$$

Then $\text{VaR}_\alpha(x, P_S) = g^{[s_{\alpha,P_S}]}$.

Consistency of sample quantiles is valid under mild assumptions regarding smoothness of the distribution function G, and one may even prove their asymptotic normality, cf. [30]. For example, if there is a positive continuous density $p(x, P; v)$ of $G(x, P; v)$ on a neighborhood of $\text{VaR}_\alpha(x, P)$ and P_S denotes an associated empirical distribution then $\text{VaR}_\alpha(x, P_S)$ is asymptotically normal,

$$\text{VaR}_\alpha(x, P_S) \sim \mathcal{N}\left(\text{VaR}_\alpha(x, P), \frac{\alpha(1-\alpha)}{p^2(x, P; \text{VaR}_\alpha(x, P))S}\right).$$

Estimating $\text{VaR}_\alpha(x, P)$ by the *nonparametric* $\text{VaR}_\alpha(x, P_S)$ calls for a large number of scenarios, especially for α close to 1; see [24] for extensive numerical results. Moreover, it is evident from (23) that even for fixed x inclusion of an additional scenario may cause an abrupt change of VaR_α.

Sensitivity results for VaR_α similar to (21) are obtained if the (unique) optimal solution of the CVaR_α problem (4) is differentiable, see 3.2. Another possibility is to derive them by a direct sensitivity analysis of the simple chance-constrained stochastic program (1). In both cases, additional assumptions concerning the probability distribution P are required, such as its continuity properties listed in 3.2. There is more freedom as to the choice of the contaminating distribution Q. We refer to [6, 28] for details.

4.4 Stress testing of nonparametric VaR computed for a discrete probability distribution P carried by a finite number of scenarios ω^s, $s = 1, \ldots, S$, is more involved. To obtain an upper bound for $\mathrm{VaR}_\alpha(x, P_\lambda)$ for a fixed portfolio x one may use the contamination based upper bound for $\mathrm{CVaR}_\alpha(x, P_\lambda)$ in (9). Formula (23) in the definition of the empirical VaR_α implies that for $\alpha < \sum_{s=1}^{s_{\alpha,P}} p^{[s]}$, the value of VaR_α is robust with respect to small changes of probabilities $p^{[s]}$. This indicates a possibility to cover the interval $[0, 1]$ by a finite number of non-overlapping intervals $[0, \lambda_1], (\lambda_1, \lambda_2], \dots, (\lambda_{\bar{\imath}}, 1]$ and to construct bounds for $\mathrm{VaR}_\alpha(x, P_\lambda)$ separately for each of them.

We shall illustrate the approach on the case of one additional "stress" scenario ω^* with

$$g^{[1]} < \dots < g^{[s_{\omega^*}-1]} < g(x, \omega^*) < g^{[s_{\omega^*}]} < \dots < g^{[S]}, \tag{24}$$

and with probabilities

$$(1-\lambda)p^{[1]}, \dots, (1-\lambda)p^{[s_{\omega^*}-1]}, \lambda, (1-\lambda)p^{[s_{\omega^*}]}, \dots, (1-\lambda)p^{[S]}.$$

Suppose that the stress scenario satisfies $g^{[s_{\alpha,P}]} < g^{[s_{\omega^*}-1]}$. We shall see that in the case of $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} > \alpha$ we get $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}]} = \mathrm{VaR}_\alpha(x, P)$ for sufficiently small $\lambda \geq 0$. On the other hand, if $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} = \alpha$ then $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ for sufficiently small $\lambda > 0$.

The α-quantile of the contaminated distribution fulfils

$$\sum_{s=1}^{s_{\alpha,P_\lambda}} (1-\lambda)p^{[s]} \geq \alpha \text{ and } \sum_{s=1}^{s_{\alpha,P_\lambda}-1} (1-\lambda)p^{[s]} < \alpha. \tag{25}$$

For $\lambda = 0$ these inequalities are identical with (23). They remain valid with s_{α,P_λ} replaced by the original $s_{\alpha,P}$ for

$$\lambda \leq 1 - \frac{\alpha}{\sum_{s=1}^{s_{\alpha,P}} p^{[s]}} \text{ and } 1 - \frac{\alpha}{\sum_{s=1}^{s_{\alpha,P}-1} p^{[s]}} < \lambda.$$

The first inequality provides an upper bound λ_1 and the second one is fulfilled for all $\lambda \geq 0$.

For $\lambda > \lambda_1$, $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ and by solving (25) for $s_{\alpha,P_\lambda} = s_{\alpha,P} + 1$ with respect to λ we get an upper bound λ_2 of the interval on which $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ holds true. Notice that $\lambda_1 = 0$ if $\sum_{s=1}^{s_{\alpha,P_S}} p^{[s]} = \alpha$ and in this case, $\lambda_2 > 0$.

Similarly for $\lambda > \lambda_i$ with $i < s_{\omega^*} - s_{\alpha,P}$ we get an upper bound λ_{i+1} of the interval, for which $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+i]}$. This procedure stops when $i = \bar{\imath} := s_{\omega^*} - s_{\alpha,P}$. In this case, (25) is modified to

$$\sum_{s=1}^{s_{\alpha,P}+\bar{\imath}-1} (1-\lambda)p^{[s]} + \lambda \cdot 1 \geq \alpha$$

valid for all $\lambda \geq 0$; hence, $\mathrm{VaR}_\alpha(x, P_\lambda) = g(x, \omega^*)$ for $\lambda_{\bar{\imath}} < \lambda \leq 1$.

To summarize: For contamination by one scenario, setting

$$\begin{aligned}\lambda_0 &= 0 \\ \lambda_i &= 1 - \frac{\alpha}{\sum_{s=1}^{s_{\alpha,P}+i-1} p^{[s]}} \text{ for } i = 1, \ldots, s_{\omega^*} - s_{\alpha,P} \\ \lambda_i &= 1 \text{ for } i > s_{\omega^*} - s_{\alpha,P}\end{aligned}$$

we obtain

Theorem 2 [22]. *For* $g^{[s_{\alpha,P}]} < g^{[s_{\omega^*}-1]}$, $\lambda \in (\lambda_i, \lambda_{i+1}]$, $i = 0, 1, \ldots, s_{\omega^*} - s_{\alpha,P} - 1$,

(a) $\mathrm{VaR}_\alpha(x, P_\lambda) \leq \mathrm{VaR}_\alpha(x, Q)$,

(b) $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+i]}$,

(c) if $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} > \alpha$ or if $i \geq 2$ and $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} = \alpha$ then $\mathrm{VaR}_\alpha(x, P_{\lambda_i}) = \mathrm{VaR}_\alpha(x, P_\lambda) < \mathrm{VaR}_\alpha(x, P_{\lambda_{i+1}})$;
if $\sum_{s=1}^{s_{\alpha,P}} p^{[s]} = \alpha$ then $\mathrm{VaR}_\alpha(x, P_{\lambda_1}) = g^{[s_{\alpha,P}]}$ and $\mathrm{VaR}_\alpha(x, P_\lambda) = g^{[s_{\alpha,P}+1]}$ for $\lambda \in (\lambda_1, \lambda_2]$,

(d) $\mathrm{VaR}_\alpha(x, P_\lambda) = g(x, \omega^*) = \mathrm{VaR}_\alpha(x, Q)$, for $\lambda > \lambda_{\bar{\imath}}$, $\bar{\imath} = s_{\omega^*} - s_{\alpha,P}$.

This procedure can be extended to stress testing with respect to another discrete probability distribution Q, carried by scenarios $\omega_1^*, \ldots, \omega_{S'}^*$ with probabilities $q^{[1]}, \ldots, q^{[S']}$ and associated losses $g(x, \omega_1^*) < \ldots < g(x, \omega_{S'}^*)$. Now, we have to know how the support of P is related to the support of Q, e.g. that the following ordering holds.

$$g^{[1]} < \ldots < g^{[s_{\alpha,P}]} < \ldots < g^{[s_{\omega_1^*}-1]} < g(x, \omega_1^*) < g^{[s_{\omega_1^*}]} < \ldots < g^{[s_{\omega_2^*}-1]}$$

$$< g(x, \omega_2^*) < g^{[s_{\omega_2^*}]} < \ldots < g^{[s_{\omega_{S'}^*}-1]} < g(x, \omega_{S'}^*) < g^{[s_{\omega_{S'}^*}]} < \ldots < g^{[S]}.$$

Covering of the interval $[0, 1]$ depends on probabilities $q^{[s]}$, namely, on the difference of their partial cumulative sums and α. For the obtained λ_i values, parallel statements to (a)–(c) of Theorem 2 can be derived, cf. [22].

4.5 Except for the case of the normal distribution considered in Sections 4.1 and 4.2, **minimization** of $\mathrm{VaR}_\alpha(x, P)$ with respect to x is in general a nonconvex, even discontinuous problem, which may have several local minima. It can be written as

$$\min\{v \,:\, P\{\omega : g(x, \omega) \leq v\} \geq \alpha,\, x \in \mathcal{X},\, v \in R\}. \tag{26}$$

Stability of the minimal $\mathrm{VaR}_\alpha(P)$ value $v_V^*(P)$ and of the optimal solutions $x_V^*(P)$ with respect to P holds true only under additional, restrictive assumptions; consult [28]. For $g(x,\omega)$ jointly continuous in x,ω and $\mathcal{H}(x,v) := \{\omega : g(x,\omega) \leq v\}$, a verifiable sufficient condition is $P(\mathcal{H}(x_V^*(P), v_V^*(P))) > \alpha$, which is fulfilled for instance for (nondegenerate) normal distributions, or $\alpha < \sum_{s=1}^{s_{\alpha,P}} p^{[s]}$ in (23) for the ordered sample of $g(x_V^*(P), \omega^s)$ with discrete distribution P_S; see [6].

To approximate VaR minimization problems one may apply the corresponding problems with CVaR criteria, as suggested and tested numerically in [26]: The $v_C^*(P)$ part of the optimal solution of (11) is then the value of $\mathrm{VaR}_\alpha(x^*(P), P)$ for the optimal (or efficient) $\mathrm{CVaR}_\alpha(x, P)$ portfolio. Further suggestions are to approximate the VaR minimization problems by a sequence of CVaR minimizations, cf. [21], to use a smoothed VaR objective, cf. [13], or to apply the worst-case VaR criterion for the family of probability distributions with given first and second order moments, cf. [11].

5 Conclusions

Application of the contamination technique to CVaR evaluation and optimization is straightforward, and the obtained results provide a genuine stress *quantification*. Stress testing via contamination for the mean-CVaR problems turns out to be more delicate.

The presence of the simple chance constraint in definition of VaR requires that for VaR stress testing via contamination, various distributional and structural properties are fulfilled for the unperturbed problem. These requirements rule out direct applications of the contamination technique in the case of discrete distributions, which includes the empirical VaR. Nevertheless, even in this case, it is possible to construct bounds for VaR of the contaminated distribution. In the case of a normal distribution and parametric VaR one may exploit stability results valid for quadratic programs to stress testing of VaR minimization problems.

Using the contamination technique we derived computable bounds which can be extended to stress testing of other risk criteria and risk optimization problems. The presented approaches provide a deeper insight into stress behavior of VaR and CVaR than the common numerical evaluations based solely on backtesting and out-of-sample analysis.

Acknowledgements. This work was partly supported by research projects *Mathematical Methods in Stochastics* – MSM 113200008 and by the Grant Agency of the Czech Republic (grants 201/02/0621 and 402/02/1015).

References

[1] Acerbi, C., Tasche, D. (2002), On the coherence of Expected Shortfall, *J. of Banking and Finance* **26**, 1487–1503.

[2] Andersson, F., Mausser, H., Rosen, D., Uryasev, S. (2001), Credit risk optimization with Conditional Value-at-Risk criterion, *Math. Program,* **B 89**, 273–291.

[3] Bank, B., Guddat, J., Klatte, D., Kummer, B., Tammer, K. (1982), *Non-Linear Parametric Optimization*, Akademie-Verlag, Berlin.

[4] Danskin, J. M. (1967), *Theory of Max-Min*, Springer, Berlin (*Econometrics and Operations Research* **5**).

[5] Dempster, M. A. H., ed. (2002), *Risk Management: Value at Risk and Beyond*, Cambridge Univ. Press.

[6] Dobiáš, P. (2003), *Stability in Stochastic Programming with Applications to Risk Management*, PhD Thesis, Charles University in Prague, Faculty of Mathematics and Physics.

[7] Dupačová, J. (1986), Stability in stochastic programming with recourse – contaminated distributions, *Math. Progr. Study* **27**, 133–144.

[8] Dupačová, J. (1987), Stochastic programming with incomplete information: A survey of results on postoptimization and sensitivity analysis, *Optimization* **18**, 507–532.

[9] Dupačová, J. (1996), Scenario based stochastic programs: Resistance with respect to sample, *Ann. Oper. Res.* **64**, 21–38.

[10] Dupačová, J. (1999), Portfolio optimization via stochastic programming: Methods of output analysis, *MMOR* **50**, 245–270.

[11] El Ghaoui, L., Oks, M., Oustry, F. (2003), Worst-case Value-at-Risk and robust portfolio optimization: A conic programming approach, *Oper. Res.* **51**, 543–556.

[12] Embrechts, P., McNeil, A. J., Straumann, D. (2002), Correlation and dependence in risk management: Properties and Pitfalls, in: [5], pp. 176–223.

[13] Gaivoronski, A.A., Pflug, G. (2004), Value at Risk in portfolio optimization, *J. of Risk.* to appear.

[14] Guddat, J., Guerra Vasquez, F., Tammer, K., Wendler, K. (1985), *Multi-objective and Stochastic Optimization Based on Parametric Optimization*, Akademie-Verlag, Berlin.

[15] Høyland, K., Kaut, M., Wallace, S. W. (2003), A heuristic for moment-matching scenario generation, *Computational Optimization and Applications* **24**, 169–185.

[16] Høyland, K., Wallace, S. W. (2001), Generating scenario trees for multistage decision problems, *Management Science* **47**, 295–307.

[17] Kaut, M., Vladimirou, H., Wallace, S. W., Zenios, S. A. (2003), Stability analysis of a portfolio management model based on the conditional value-at-risk measure, submitted.

[18] Klein Haneweld, W. K. (1986), *Duality in Stochastic Linear and Dynamic Programming*, LNEMS 274, Springer, Berlin.

[19] Kupiec, P. (2002), Stress testing in a Value at Risk framework, in: [5], pp. 76–99.

[20] Ogryczak, W., Ruszczyński, A. (2002), Dual stochastic dominance and related mean-risk models, *SIAM J. Optimization* **13**, 60–78.

[21] Pflug, G. Ch. (2001), Some remarks on the Value-at-Risk and the Conditional Value-at-Risk, in: S. Uryasev (ed.), *Probabilistic Constrained Optimization, Methodology and Applications*, Kluwer Acad. Publ., Dordrecht, pp. 272–281.

[22] Polívka, J. (2004), *ALM via Stochastic Programming*, PhD Thesis, Charles University in Prague, Faculty of Mathematics and Physics. In preparation.

[23] Prékopa, A. (1973), Contributions to the theory of stochastic programming, *Mathematical Programming* **4**, 202–221.

[24] Rachev, S., Mittnik, S. (2000), *Stable Paretian Models in Finance*, Wiley, Chichester.

[25] Rau-Bredow, H. (2004), Value-at-Risk, Expected Shortfall and Marginal Risk Contribution, in: G. Szegö (ed.), *Risk Measures for the 21st Century*, Wiley Finance, Chichester, pp.61–68.

[26] Rockafellar, R. T., Uryasev, S. (2000), Optimization of conditional value-at-risk, *J. of Risk* **2**, 21–41.

[27] Rockafellar, R. T., Uryasev, S. (2001), Conditional value-at-risk for general loss distributions, *J. of Banking and Finance* **26**, 1443–1471.

[28] Römisch, W. (2003), Stability of stochastic programming problems, Chapter 8 in: A. Ruszczynski, A. Shapiro (eds.), *Handbook on Stochastic Programming*, Elsevier, Amsterdam, pp. 483–554.

[29] Ruszczyński, A., Vanderbei, R. J. (2003), Frontiers of stochastically nondominated portfolios, *Econometrica* **71** 1287–1297.

[30] Serfling, R. J. (1980), *Approximation Theorems of Mathematical Statistics*, Wiley, New York.

[31] Shapiro, A. (1990), On differential stability in stochastic programming, *Math. Progr.* **47**, 107–116.

[32] Topaloglou, N., Vladimirou, H., Zenios, S.A. (2002), CVaR models with selective hedging for international asset allocation, *J. Banking and Finance* **26**, 1535–1561.