SAMPLING-BASED DECOMPOSITION METHODS FOR RISK-AVERSE MULTISTAGE STOCHASTIC PROGRAMS

VINCENT GUIGUES AND WERNER RÖMISCH

Abstract. We define a risk averse nonanticipative feasible policy for multistage stochastic programs and propose a methodology to implement it. The approach is based on dynamic programming equations written for a risk averse formulation of the problem.

This formulation relies on a new class of multiperiod risk functionals called extended polyhedral risk measures. Dual representations of such risk functionals are given and used to derive conditions of coherence. In the one-period case, conditions for convexity and consistency with second order stochastic dominance are also provided. The risk averse dynamic programming equations are specialized considering convex combinations of one-period extended polyhedral risk measures such as spectral risk measures.

To implement the proposed policy, the approximation of the risk averse recourse functions for stochastic linear programs is discussed. In this context, we detail a stochastic dual dynamic programming algorithm which converges to the optimal value of the risk averse problem.

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1. Introduction

Let us consider a T-stage optimization problem of the form

$$\begin{array}{ll} \inf \mathbb{E}[\sum_{t=1}^{T} f_t(x_t, \xi_t)] \\ x_t \in \chi_t(x_{t-1}, \xi_t),\ a.s.,\ x_t\ \mathcal{F}_t\text{-measurable},\ t = 1, \ldots, T, \end{array} \tag{1}$$

where $(\xi_t)_{t=1}^{t}$ is a stochastic process and $\mathcal{F}_t$ is the sigma-algebra $\mathcal{F}_t := \sigma(\xi_j, j \leq t)$. Depending on the assumptions on the nature of uncertainty, two principal methods have been proposed so far for making decisions in this uncertain environment. The first one is *Robust Optimization* which is a worst case oriented approach where the parameters are only known to belong to some given uncertainty sets. The second method is *Stochastic Programming* (SP) on which we focus and which assumes that the uncertain parameters are realizations of random variables. In this setting, multistage stochastic optimization problems set two challenging questions. The first question refers to modeling: how to deal with uncertainty in this context? Once a model is chosen, the second question is to design suitable solution methods.

For the first of these questions, we are interested in defining *nonanticipative* policies. This means that the decision we make at any time step should be a function of the available history $\xi_{[t]}$ of the process at this time step. This is a necessary condition for a *policy* to be implementable since a decision has to be made on the basis of the available information. We will focus on models with recourse. More precisely, introducing a recourse function $\mathcal{Q}_{t+1}$ for time step t and given x_{t-1}, the

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decision x_t is found by solving the problem

$$
\begin{array}{l}
\inf_{x_t} f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) \\
x_t \in \chi_t(x_{t-1}, \xi_t)
\end{array}
\tag{2}
$$

at time step t. In this problem, we have assumed that ξ_t is available at time step t and thus $\xi_{[t]}$ gathers all the realizations of ξ_j up to time step t. The policy depends crucially on the choice of the recourse function $\mathcal{Q}_{t+1}$ used in (2). Given x_0 and the information $\xi_{[1]}$, a non-risk averse model uses the recourse functions defined by

$$
\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}) = \mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf_{x_t} f_t(x_t, \xi_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}) \\ x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right)
\tag{3}
$$

for $t = 1, \ldots, T$, with $\mathcal{Q}_{T+1} \equiv 0$. These dynamic programming equations are associated to the non-risk averse model

$$
\begin{array}{l}
\inf \mathbb{E}[\sum_{t=1}^{T} f_t(x_t(\xi_{[t]}), \xi_t)] \\
x_t(\xi_{[t]}) \in \chi_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \; t = 1, \ldots, T.
\end{array}
\tag{4}
$$

For the second of these questions, most of the efforts so far have been placed on solution methods that approximate the recourse functions (3) in the case of multistage stochastic linear programs. In this paper, we contribute to these two questions as follows.

From the modeling point of view, we define risk averse recourse functions. For this purpose, a common approach (Ruszczyński and Shapiro [RS06a], [RS06b]) is based on a risk-averse nested formulation of the problem using conditional (coherent) risk measures. In this situation, it is in general difficult, even for simple risk measures such as the *Conditional Value-at-Risk* (CVaR) (Rockafellar and Uryasev [RU02]), to determine a risk averse problem (using a risk measure that has a physical interpretation) whose stagewise decomposition is given by these dynamic programming equations. However, such an interpretation is important. This is why we define instead a risk averse problem for (1) that is then decomposed by stages to obtain dynamic programming equations. A similar idea appears in the recent book [Shapiro et. al, [SDR09], Chapter 6, p. 326] where a convex combination of the expectation and of the CVaR of the final wealth is used for a portfolio selection problem. Instead, we control partial costs (the sum of the costs up to the current time step) and use a new class of risk measures that is suitable for decomposing the risk averse problem by stages. This class of multiperiod risk measures called *extended polyhedral risk measures* has three appealing properties. First, the class is large: it contains the polyhedral risk measures (Eichhorn and Römisch [ER05]); in the one-period case some special cases include the Optimized Certainty Equivalent Ben-Tal and Teboulle [BTT07], some spectral risk measures (Acerbi [Ace02]) and the CVaR. More generally, conditions for such functionals to be coherent or convex are provided. Second, as stated above, it allows us to define dynamic programming equations for our risk averse problem. Finally, these equations are suitable for proposing convergent solution methods for a class of stochastic linear programs.

Regarding algorithmic issues, exact decomposition algorithms such as the Nested Decomposition (ND) algorithm have shown their superiority to direct solution methods for obtaining approximations of the recourse functions. Each iteration of these algorithms compute upper and lower bounds on the optimal mean cost. If an optimal solution to the problem exists, the algorithm finds an optimal solution after a finite number of iterations. These exact algorithms build at each iteration and each node of the scenario tree a cut for the recourse functions. These cuts form an outer linearization of these recourse functions.

There are two important variants of the ND algorithm: a variant that adds quadratic proximal terms in the objective functions of the master problems (Ruszczyński [Rus86]) and a variant that uses multi-cuts (Birge and Louveaux [BL88]).

The purpose of the first variant is to discourage the solution to move too far from the best solution found so far and this can significantly accelerate the convergence of the method even if the master problems are quadratic programs with this approach. The proximal term penalties are positive and can be dynamically modified in the course of the algorithm.

In the ND algorithm, for a given node in the scenario tree and a given input state x_{t-1} at t, the subproblems associated to all the realizations in stage $t+1$ are solved to obtain their optimal simplex multipliers. These multipliers are then aggregated to obtain a single cut for each node in each iteration. In the multi-cut variant, there are as many cuts as descendants realizations that are built at each iteration. More information is thus passed from the children nodes to their immediate ancestor by sending disaggregate cuts. The size of the master programs increases but we expect less iterations (see Birge and Louveaux [BL88]).

However, for many applications, the number of scenarios is so large that even these improved variants are difficult to apply since they entail prohibitive computational efforts. In this case, approximation and bounding schemes may be used. A usual approximation scheme consists of computing lower and upper bounds using Jensen and Edmundson-Madansky inequalities (see Birge and Wets[BW89], [BW86], Kall et. al [KRF88]). In general, when the number of random parameters is large, these approximation and bounding schemes are still difficult to apply.

Monte Carlo sampling-based algorithms constitute an interesting alternative in such situations. Higle and Sen [HS96] introduced a stochastic cutting plane method for two-stage stochastic programs and showed its convergence with probability one. Recently, Higle et al. [HRS10] extended this idea to multistage models by applying a stochastic cutting plane method to the dual problem resulting when dualizing nonanticipativity constraints. Their method is, hence, based on scenario decomposition. A different approach for two-stage problems based on Monte Carlo (importance) sampling within the L-shaped method was introduced by Dantzig and Glynn [DG90] and Infanger [Inf92]. For multistage stochastic linear programs whose number of immediate descendant nodes is small but with many stages, Pereira and Pinto [PP91] propose to sample in the forward pass of the ND. This sampling-based variant of the ND is the so-called Stochastic Dual Dynamic Programming algorithm on which we focus our attention. We detail a *Stochastic Dual Dynamic Programming* (SDDP) algorithm (Pereira and Pinto [PP91]) to approximate our risk averse recourse functions, to be used in (2) in place of $\mathcal{Q}_{t+1}$. The computations of the cuts in the backward pass of SDDP are detailed both in the non-risk averse and in the risk averse setting.

Our developments can be easily extended to other sampling-based decomposition methods such as AND and DOASA.

The Abridged Nested Decomposition (AND) algorithm proposed by Birge and Donohue [BD01] is a variant of SDDP that also involves sampling in the forward pass. This algorithm determines in a different manner the sequence of states and scenarios in the forward pass. The numerical simulations in Birge and Donohue [BD01] report lower computational time on average for the AND algorithm in comparison with SDDP.

When the number of immediate descendant nodes is large (possibly infinite) and when the problem has many stages, we can also (or even have to) sample in the backward pass. In this case, for a given node on a forward path k, not all the optimal simplex multipliers associated to the descendant subproblems are computed. Only the descendant subproblems associated with some realizations are solved. As explained in the Cut Calculation Algorithm (CCA) in Philpott and Guan [PG08], it is however possible in this situation to replace the "missing" multipliers by some coefficients so that the cuts built still lie below the corresponding recourse functions. This gives rise to Dynamic Outer Approximation Sampling Algorithms (DOASA) described in Philpott and Guan [PG08].

The paper is organized as follows. In the second section, we introduce the class of multiperiod extended polyhedral risk measures and study their properties: dual representations are derived and used to provide criteria for convexity and coherence and, in the one-period case, for convexity and consistency with second order stochastic dominance. In Section 3, we derive dynamic programming equations for a risk averse problem defined in terms of extended polyhedral risk measures. We also provide conditions that guarantee the convergence of SDDP in this risk averse setting. In Section 4, we recall the SDDP algorithm for a class of stochastic linear programs (SLP). Finally, in Section 5, we propose to use SDDP to approximate the risk averse recourse functions from Section 3 for the SLP considered in Section 4. In particular, formulas for the cuts in the backward pass are given. We show that under some assumptions, some of the cut coefficients have explicit formulas that are independent of the sampled scenarios.

We mention that after writing our paper we became aware of two recent and closely related papers: Collado et. al [CPR] based on scenario decomposition and Shapiro [Sha10] which suggests to use SDDP to approximate risk-averse recourse functions defined from a nested risk-averse formulation of a multistage stochastic program.

We start by setting down some notation:

- For $x \in \mathbb{R}^n$, the vectors x^+ and x^- are defined by $x^+(i) = \max(x(i), 0)$ and $x^-(i) = \max(-x(i), 0)$ for $i = 1, \ldots, n$;
- For a nonempty set $X \subseteq \mathbb{R}^n$, the polar cone X^* is defined by $X^* = \{x^* : \langle x, x^* \rangle \leq 0 \ \forall x \in X\}$ where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^n$;
- e is a column vector of all ones;
- If A is an $m_1 \times n$ matrix and B an $m_2 \times n$ matrix, $(A; B)$ denotes the $(m_1 + m_2) \times n$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$;
- For vectors $x_1, \ldots, x_T \in \mathbb{R}^n$ and $1 \leq t_1 \leq t_2 \leq T$, we denote $(x_{t_1}, \ldots, x_{t_2}) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ by $x_{t_1:t_2}$;
- For $x, y \in \mathbb{R}^n$, the vector $x \circ y \in \mathbb{R}^n$ is defined by $(x \circ y)(i) = x(i)y(i), \ i = 1, \ldots, n$;
- I_n is the $n \times n$ identity matrix and $0_{m,n}$ is an $m \times n$ matrix of zeros;
- δ_{ij} is the Kronecker delta defined for i, j integers by $\delta_{ij} = 1$ if $i = j$ and 0 otherwise;
- $\mathcal{Q}_{t+1}$ denotes a (generic) recourse function used at time step $t = 1, \ldots, T$, i.e., $\mathcal{Q}_{T+1} \equiv 0$ and if $t < T$ then $\mathcal{Q}_{t+1}(x_t, \xi_{[t]})$ represents a cost over the period $t + 1, \ldots, T$. Various recourse functions at t will be defined using the same notation $\mathcal{Q}_{t+1}$. Which $\mathcal{Q}_{t+1}$ is relevant will be clear from the context.

As is usually done in the SP literature and to alleviate notation, we use the same notation for a random variable and for a particular realization of this random variable, the context allowing us to know which concept is being referred to.

2. Extended polyhedral risk measures

We consider multiperiod risk functionals ρ whose arguments are sequences of random variables. We confine ourselves to discrete-time processes with a finite time horizon as in Ruszczyński and Shapiro [RS06a]. Such risk functionals have to assess the riskiness of a finite sequence $z_1, \ldots, z_T$ of random variables for some integer $T \geq 2$. To reflect the evolution of information as time goes by, we assume that z_t is measurable with respect to some σ-field $\mathcal{F}_t$, where $\mathcal{F}_1, \ldots, \mathcal{F}_T$ form a filtration, i.e., $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_T = \mathcal{F}$, with $\mathcal{F}_1 = \{\emptyset, \Omega\}$. In this setting, z_1 is deterministic and a multiperiod risk functional ρ will be seen as a mapping $\rho : \times_{t=1}^T L_p(\Omega, \mathcal{F}_t, \mathbb{P}) \to \bar{\mathbb{R}}$ for some $p \in [1, +\infty)$.

Remark 2.1. *Throughout the paper, the arguments* $(z_1, \ldots, z_T)$ *of the risk functionals will be interpreted as accumulated revenues (for which higher values are preferred). More precisely, if* $\tilde{z}_t$ *is the revenue for time step* t*, we consider the accumulated revenues* $z_t = \sum_{\tau=1}^{t} \tilde{z}_\tau,\ t = 1, \ldots, T$.

For future use, we recall the definition of multiperiod convex risk measures (from Artzner et al. [ADE$^+$], [ADE$^+$07], Föllmer and Schied [FS04]) which are multiperiod risk functionals of special interest when the random variables z_t represent revenues (accumulated or not).

Definition 2.2. *A functional* ρ *on* $\times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ *is called a* multiperiod convex risk measure *if the following conditions (i)–(iii) hold:*

(i) Monotonicity*: if* $z_t \leq \tilde{z}_t$ *a.s,* $t = 1, \ldots, T$*, then* $\rho(z_1, \ldots, z_T) \geq \rho(\tilde{z}_1, \ldots, \tilde{z}_T)$*;*
(ii) Translation invariance*: for each* $r \in \mathbb{R}$ *we have* $\rho(z_1 + r, \ldots, z_T + r) = \rho(z_1, \ldots, z_T) - r$*;*
(iii) Convexity*: for each* $\lambda \in [0, 1]$ *and* $z, \tilde{z} \in \times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ *we have* $\rho(\lambda z + (1 - \lambda)\tilde{z}) \leq \lambda\rho(z) + (1 - \lambda)\rho(\tilde{z})$.

It is called a multiperiod coherent risk measure *if in addition condition (iv) holds:*

(iv) Positive homogeneity*: for each* $\lambda \geq 0$ *we have* $\rho(\lambda z_1, \ldots, \lambda z_T) = \lambda\rho(z_1, \ldots, z_T)$.

In the literature, there appear different requirements instead of the translation invariance (ii) above, e.g. Fritelli and Scandalo [FS05] and Pflug and Römisch [PR07].
Convex duality can be used to obtain dual representations of multiperiod convex risk measures. Next, we cite such a representation that uses the set $\mathcal{D}_T$ of generalized density functions where

$$\mathcal{D}_T := \{\lambda \in \times_{t=1}^{T} L_1(\Omega, \mathcal{F}_t, \mathbb{P}) \ : \ \lambda_t \geq 0 \text{ a.s.}, t = 1, \ldots, T, \ \sum_{t=1}^{T} \mathbb{E}[\lambda_t] = 1\}.$$

Theorem 2.3. *Let* $\rho : \times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P}) \to \bar{\mathbb{R}}$ *and assume that* ρ *is proper (i.e.* ρ *is finite on the nonempty set* $\operatorname{dom} \rho = \{z : \rho(z) < \infty\}$*) and lower semi-continuous. Then* ρ *is a multiperiod convex risk measure if and only it admits the representation*

$$\rho(z) = \sup \ \Big\{\mathbb{E}\Big(- \sum_{t=1}^{T} \lambda_t z_t\Big) - \rho^*(\lambda) \ : \ \lambda \in \mathcal{P}_\rho\Big\}, \tag{5}$$

for some convex closed subset $\mathcal{P}_\rho \subseteq \mathcal{D}_T$ *of the space* $\times_{t=1}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P})$ *(*$\frac{1}{p} + \frac{1}{q} = 1$*) on which the conjugate* ρ^* *of* ρ *is given, too. The functional* ρ *is coherent if and only if the conjugate* ρ^* *in (5) is the indicator function of* $\mathcal{P}_\rho$.

A proof of the above theorem can be found in e.g. Ruszczyński and Shapiro [RS06b]. We are now in a position to define the class of multiperiod extended polyhedral risk measures.

Definition 2.4. *A risk measure* ρ *on* $\times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ *is called* multiperiod extended polyhedral *if there exist matrices* $A_t, B_{t,\tau}$*, vectors* a_t, c_t*, and functions* $h_t(z) = (h_{t,1}(z), \ldots, h_{t,n_{t,2}}(z))^\top$ *for given functions* $h_{t,1}, \ldots, h_{t,n_{t,2}} : L_p(\Omega, \mathcal{F}_t, \mathbb{P}) \to L_{p'}(\Omega, \mathcal{F}_t, \mathbb{P})$ *with* $1 \leq p' \leq p$ *such that*

$$\rho(z_1, \ldots, z_T) = \left\{ \begin{array}{l} \inf \ \mathbb{E}[\sum_{t=1}^{T} c_t^\top y_t] \\ y_t \in L_p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{k_t}), \ t = 1, \ldots, T, \\ A_t y_t \leq a_t \ a.s., t = 1, \ldots, T, \ \sum_{\tau=0}^{t-1} B_{t,\tau} y_{t-\tau} = h_t(z_t) \ a.s., \ t = 2, \ldots, T. \end{array} \right. \tag{6}$$

Another less general extension of polyhedral risk measures is due to Eichhorn [Eic07]. Like a multiperiod polyhedral risk measure Eichhorn and Römisch [ER05], a multiperiod extended polyhedral risk measure is given as the optimal value of a T-stage linear stochastic program where the arguments of the risk measure appear on the right hand side of the dynamic constraints. Multiperiod polyhedral risk measures constitute a particular case with $a_t = 0, \ t = 2, \ldots, T, \ B_{t,\tau}$ row vectors, and $h_t(z_t) = h_{t,1}(z_t) = z_t$ (i.e., $n_{t,2} = 1$).

The need to consider the extended versions from Definition 2.4 is twofold:

(i) `Modelling`: some (popular) risk measures are extended polyhedral but not polyhedral in the sense of Eichhorn and Römisch [ER05] (see examples in the end of this section).

(ii) `Algorithmic issues`: as announced in the introduction, dynamic programming equations can be written for risk averse versions of (1) defined in terms of extended polyhedral risk measures. Moreover, the convergence of a class of decomposition algorithms applied to the corresponding nested formulation of the risk averse problem will be proved in Section 3 for a subclass of extended polyhedral risk measures that contain some non-polyhedral risk measures. For this subclass, we have $h_t(z_t) = z_t b_t + \tilde{b}_t$ for some vectors $b_t, \tilde{b}_t$.

In view of (ii) above, extended polyhedral risk measures with $h_t(z_t) = z_t b_t + \tilde{b}_t$ play a particular role when algorithmic issues come into play. In the rest of this section, we study properties of such risk functionals. In this context, the matrices $A_t, B_{t,\tau}$ and the vectors $a_t, b_t, \tilde{b}_t$, and c_t are fixed and deterministic. They have to be chosen such that ρ exhibits desirable risk measure properties. In particular, conditions on these parameters for the corresponding extended polyhedral risk measure to be coherent are given in the Corollary 2.6 of Theorem 2.5 which follows. This theorem gives dual representations for stochastic program (6) when $h_t(z_t) = z_t b_t + \tilde{b}_t$ for some vectors $b_t, \tilde{b}_t$. In the sequel, the dimensions of a_t and b_t are respectively denoted by $n_{t,1}$ and $n_{t,2}$.

Theorem 2.5. *Let ρ be a functional of the form (6) on $\times_{t=1}^T L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ with $p \in [1, \infty)$ and $h_t(z_t) = z_t b_t + \tilde{b}_t$ for some vectors $b_t, \tilde{b}_t$. Assume*

(i) *complete recourse: $\{y_1 : A_1 y_1 \leq a_1\} \neq \emptyset$ and for every $t = 2, \ldots, T$, it holds that $\{B_{t,0} y_t : A_t y_t \leq a_t\} = \mathbb{R}^{n_{t,2}}$;*

(ii) *dual feasibility: $\{(u, v) : u \in \times_{t=1}^T \mathbb{R}^{n_{t,1}}, v \in \times_{t=2}^T \mathbb{R}^{n_{t,2}}, c_t + A_t^\top u_t + \sum_{\tau=\max(2,t)}^T B_{\tau,\tau-t}^\top v_{\tau-1} = 0,\ t = 1, \ldots, T\} \neq \emptyset$.*

Then ρ is finite, convex, and continuous on $\times_{t=1}^T L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ and with $\frac{1}{p} + \frac{1}{q} = 1$ the following dual representation holds:

(7)
$$\rho(z) = \left\{ \begin{array}{l} \sup\ -\mathbb{E}[\sum_{t=1}^T \lambda_{1,t}^\top a_t + \sum_{t=2}^T \lambda_{2,t-1}^\top (z_t b_t + \tilde{b}_t)] \\ \lambda_1 \in \times_{t=1}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,1}}),\ \lambda_2 \in \times_{t=2}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,2}}),\ \lambda_{1,t} \geq 0\ a.s.,\ t = 1, \ldots, T, \\ c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^T B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1} | \mathcal{F}_t] = 0\ a.s.,\ t = 1, \ldots, T. \end{array} \right.$$

We also have

$$\rho(z) = \sup \left\{ \mathbb{E}\left[\sum_{t=1}^T z_t^* z_t\right] - \rho^*(z^*)\ :\ z^* \in \times_{t=1}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}) \right\} \tag{8}$$

where ρ^* is the conjugate of ρ. Next, for every $z^* \in \mathrm{dom}(\rho^*)$, $\rho^*(z^*)$ is given by

$$\rho^*(z^*) = \left\{ \begin{array}{l} \inf\ \mathbb{E}[\sum_{t=1}^T \lambda_{1,t}^\top a_t + \sum_{t=2}^T \lambda_{2,t-1}^\top \tilde{b}_t] \\ \lambda_1 \in \times_{t=1}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,1}}),\ \lambda_2 \in \times_{t=2}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,2}}), \\ z_t^* = -\lambda_{2,t-1}^\top b_t\ a.s.,\ t = 2, \ldots, T,\ \lambda_{1,t} \geq 0\ a.s.,\ t = 1, \ldots, T, \\ c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^T B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1} | \mathcal{F}_t] = 0\ a.s.,\ t = 1, \ldots, T \end{array} \right. \tag{9}$$

where

(10)
$$\mathrm{dom}(\rho^*) = \left\{ \begin{array}{l} z^* \in \times_{t=1}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}) \text{ such that} \\ \exists\ \lambda_1 \in \times_{t=1}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,1}}),\ \lambda_2 \in \times_{t=2}^T L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,2}}) \text{ satisfying} \\ \lambda_{1,t} \geq 0\ a.s.,\ t = 1, \ldots, T, \\ c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^T B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1} | \mathcal{F}_t] = 0\ a.s.,\ t = 1, \ldots, T, \quad \text{and} \\ z_1^* = 0,\ z_t^* = -\lambda_{2,t-1}^\top b_t\ a.s.\ t = 2, \ldots, T \end{array} \right\}.$$

Proof. We use results on Lagrangian and conjugate duality. Consider the following Banach spaces and their duals

$$\begin{array}{ll} E := \times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{k_t}) & E^* = \times_{t=1}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{k_t}) \\ Z := \times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P}) & Z^* = \times_{t=1}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}) \end{array}$$

with bilinear forms

$$\langle e, e^* \rangle_{E/E^*} = \sum_{t=1}^{T} \mathbb{E}[e_t^\top e_t^*] \quad \text{and} \quad \langle z, z^* \rangle_{Z/Z^*} = \sum_{t=1}^{T} \mathbb{E}[z_t z_t^*].$$

Let us introduce the Lagrange multipliers $\lambda_1 \in \times_{t=1}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,1}})$ (with $\lambda_1 \geq 0$ *a.s.*) and $\lambda_2 \in \times_{t=2}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,2}})$ associated to the constraints of (6) and the Lagrangian

$$\begin{array}{lll} L(y, \lambda_1, \lambda_2) & := & \mathbb{E}\left[\sum_{t=1}^{T} c_t^\top y_t + \lambda_{1,t}^\top (A_t y_t - a_t) + \sum_{t=2}^{T} \lambda_{2,t-1}^\top (\sum_{\tau=0}^{t-1} B_{t,\tau} y_{t-\tau} - z_t b_t - \tilde{b}_t)\right] \\ & = & \mathbb{E}\left[\sum_{t=1}^{T} (c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^{T} B_{\tau,\tau-t}^\top \lambda_{2,\tau-1})^\top y_t\right] \\ & & +\mathbb{E}\left[-\sum_{t=1}^{T} \lambda_{1,t}^\top a_t - \sum_{t=2}^{T} \lambda_{2,t-1}^\top (z_t b_t + \tilde{b}_t)\right]. \end{array}$$

The dual functional is defined by

$$\theta(\lambda_1, \lambda_2) := \inf_{y \in E} L(y, \lambda_1, \lambda_2) \tag{11}$$

and the Lagrangian dual of (6) is the problem

$$\sup_{\lambda_1, \lambda_2} \left\{\theta(\lambda_1, \lambda_2) \ : \ \lambda_1 \in \times_{t=1}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,1}}), \ \lambda_2 \in \times_{t=2}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,2}}), \lambda_1 \geq 0 \, a.s.\right\}. \tag{12}$$

Due to [Ruszczyński and Shapiro, [RS03], Proposition 5, Chapter 1] the conditional expectation operator $\mathbb{E}[\cdot|\mathcal{F}_t]$ and the operation of minimization can be interchanged in (11) which gives for $\theta(\lambda_1, \lambda_2)$ the expression

$$-\mathbb{E}\left[\sum_{t=1}^{T} \lambda_{1,t}^\top a_t + \sum_{t=2}^{T} \lambda_{2,t-1}^\top (z_t b_t + \tilde{b}_t)\right] + \mathbb{E}\left[\sum_{t=1}^{T} \inf_{y_t \in \mathbb{R}^{k_t}} (c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^{T} B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1}|\mathcal{F}_t])^\top y_t\right].$$

Next, $\inf_{y_t \in \mathbb{R}^{k_t}} (c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^{T} B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1}|\mathcal{F}_t])^\top y_t$ is 0 if

$$c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^{T} B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1}|\mathcal{F}_t] = 0$$

and $-\infty$ otherwise. The Lagrangian dual (12) can thus be expressed as

$$\begin{array}{l} \sup \ -\mathbb{E}[\sum_{t=1}^{T} \lambda_{1,t}^\top a_t + \sum_{t=2}^{T} \lambda_{2,t-1}^\top (z_t b_t + \tilde{b}_t)] \\ \lambda_1 \in \times_{t=1}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,1}}), \ \lambda_2 \in \times_{t=2}^{T} L_q(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^{n_{t,2}}), \lambda_1 \geq 0 \, a.s. \\ c_t + A_t^\top \lambda_{1,t} + \sum_{\tau=\max(2,t)}^{T} B_{\tau,\tau-t}^\top \mathbb{E}[\lambda_{2,\tau-1}|\mathcal{F}_t] = 0 \ a.s., \ t = 1, \ldots, T. \end{array} \tag{13}$$

From weak duality and dual feasibility, we obtain $\rho(z) > -\infty$ and due to the complete recourse assumption $\rho(z) < +\infty$. It follows that $\rho(z)$ is finite. More precisely, dual feasibility and complete recourse imply that there is no duality gap: the optimal value of (6), i.e., $\rho(z)$, is the optimal value of (13). This shows (7).

Next, we use conjugate duality. Let us introduce the vectors $c=(c_1,\ldots,c_T)^\top$, $a=(a_1,\ldots,a_T)^\top$, $\tilde{b}=(\tilde{b}_2,\ldots,\tilde{b}_T)^\top$ and the matrices $A=\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_T\end{pmatrix}$, $\mathcal{B}=\begin{pmatrix} 0 & b_2 & & \\ \vdots & & \ddots & \\ 0 & & & b_T\end{pmatrix}$, and

$$B=\begin{pmatrix} B_{2,1} & B_{2,0} & 0 & \ldots & 0\\ B_{3,2} & B_{3,1} & B_{3,0} & \ddots & \vdots\\ \vdots & \vdots & \vdots & \ddots & 0\\ B_{T,T-1} & B_{T,T-2} & B_{T,T-3} & \ldots & B_{T,0}\end{pmatrix}.$$

Let also $Y=\{y\in E\ :\ Ay(\omega)\le a,\ \text{for a.e. } \omega\in\Omega\}$ and

$$\begin{array}{rccl}\varphi: & E\times Z & \rightarrow & \bar{\mathbb{R}}\\ & (y,z) & \rightarrow & \varphi(y,z)=\langle y,c\rangle_{E/E^*}+\delta_Y(y)+\delta_{\{0\}}(By-\mathcal{B}z-\tilde{b}),\end{array}$$

where δ denotes the indicator function taking values 0 and $+\infty$ only. Since Y is closed and convex, φ is lower semi-continuous and convex. With this notation, we can express $\rho(z)$ as $\rho(z)=\inf_{y\in E}\ \varphi(y,z)$ and due to [Bonnans and Shapiro [BS00], Proposition 2.143] ρ is convex. Since ρ is finite valued, [Bonnans and Shapiro [BS00], Proposition 2.152] guarantees the continuity of ρ. Since ρ is proper, convex and lower semi-continuous, by the Fenchel-Moreau Theorem we have that $\rho^{**}=\rho$ where ρ^{**} is the biconjugate of ρ, i.e.,

$$\rho(z)=\rho^{**}(z)=\sup\left\{\langle z,z^*\rangle_{Z/Z^*}-\rho^*(z^*)\ :\ z^*\in Z^*\right\} \tag{14}$$

which is (8). Next, $\rho^*(z^*)=\varphi^*(0,z^*)$ where the conjugate φ^* of φ is given by

$$\begin{array}{rcl}\varphi^*(y^*,z^*) & = & \sup\left\{\langle y,y^*\rangle_{E/E^*}+\langle z,z^*\rangle_{Z/Z^*}-\varphi(y,z)\ :\ y\in E,\ z\in Z\right\}\\ & = & \sup\left\{\langle y,y^*-c\rangle_{E/E^*}+\langle z,z^*\rangle_{Z/Z^*}\ :\ Ay\le a\ a.s.,\ By=\mathcal{B}z+\tilde{b}\ a.s.\right\}.\end{array}$$

It follows that

$$\rho^*(z^*)=\left\{\begin{array}{l}\sup\ \mathbb{E}[\sum_{t=1}^T(z_t z_t^*-c_t^\top y_t)]\\ y_t\in L_p(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{k_t}),\ z_t\in L_q(\Omega,\mathcal{F}_t,\mathbb{P}),\ t=1,\ldots,T,\\ A_t y_t\le a_t\ a.s., t=1,\ldots,T,\ \sum_{\tau=0}^{t-1}B_{t,\tau}y_{t-\tau}=z_t b_t+\tilde{b}_t\ a.s.,\ t=2,\ldots,T.\end{array}\right. \tag{15}$$

Due to (i) and (ii), complete recourse and dual feasibility also hold for (15) for every $z^*\in\ \mathrm{dom}(\rho^*)$ where $\mathrm{dom}(\rho^*)$ is given by (10). Using once again Lagrangian duality for problem (15), we obtain for $\rho^*(z^*)$ dual representation (9). □

Theorems 2.3 and 2.5 allow us to provide a criterion for an extended polyhedral risk measure to be multiperiod coherent.

Corollary 2.6. *Let ρ be a functional on $\times_{t=1}^T L_p(\Omega,\mathcal{F}_t,\mathbb{P})$ of the form (6) with all a_t null and $h_t(z_t)=z_t b_t$ for some vector b_t. Let the conditions of Theorem 2.5 be satisfied (complete recourse and dual feasibility) and let*

$$\mathcal{M}_\rho=\left\{\begin{array}{l}\lambda\in\times_{t=1}^T L_q(\Omega,\mathcal{F}_t,\mathbb{P})\ \text{ such that there exist}\\ \mu_1\in\times_{t=1}^T L_q(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{n_{t,1}}),\ \mu_2\in\times_{t=2}^T L_q(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^{n_{t,2}})\ \text{satisfying}\\ \mu_{1,t}\ge 0\,a.s.\ t=1,\ldots,T,\\ c_t+A_t^\top\mu_{1,t}+\sum_{\tau=\max(2,t)}^T B_{\tau,\tau-t}^\top\mathbb{E}[\mu_{2,\tau-1}|\mathcal{F}_t]=0\ a.s.,\ t=1,\ldots,T,\ \ \textit{and}\\ \lambda_1=0,\ \lambda_t=\mu_{2,t-1}^\top b_t\ a.s.,\ t=2,\ldots,T\end{array}\right\}$$

be the (convex) set of dual multipliers. If $\mathcal{M}_\rho\subseteq\mathcal{D}_T$ then ρ is a multiperiod coherent risk measure.

Proof. Using representation (7) of Theorem 2.5, we can write $\rho(z) = \sup_{\lambda \in \mathcal{M}_\rho} -\sum_{t=1}^{T} \mathbb{E}[\lambda_t z_t]$. We conclude using Theorem 2.3 with $\mathcal{P}_\rho = \mathcal{M}_\rho$. □

Using representation (8) of Theorem 2.5, the properties of ρ can also be characterized by properties of $\mathrm{dom}(\rho^*)$ where $\mathrm{dom}(\rho^*)$ is given by (10).

Corollary 2.7. *Let ρ be a functional on $\times_{t=1}^{T} L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ of the form (6) with $h_t(z_t) = z_t b_t + \tilde{b}_t$ for some vectors $b_t, \tilde{b}_t$ and let the conditions of Theorem 2.5 be satisfied (complete recourse and dual feasibility). The following holds:*

(i) *ρ is* monotone $\Longleftrightarrow$ *for all $z^* \in \mathrm{dom}(\rho^*)$ we have $z_t^* \leq 0$ a.s. for $t = 1, \ldots, T$;*
(ii) *ρ is* translation invariant $\Longleftrightarrow$ *for all $z^* \in \mathrm{dom}(\rho^*)$ we have $\sum_{t=1}^{T} \mathbb{E}[z_t^*] = -1$;*
(iii) *ρ is* positively homogeneous $\Longleftrightarrow$ *for all $z^* \in \mathrm{dom}(\rho^*)$ we have $\rho^*(z^*) = 0$.*

When $T = 2$, since z_1 is deterministic, Definition 2.4 corresponds to one-period extended polyhedral risk measures that assess the riskiness of one random variable z only. For later reference we recall the definition of such risk measures which extend the class of one-period polyhedral risk measures.

Definition 2.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $h(z) = (h_1(z), \ldots, h_{n_{2,2}}(z))^\top$ for given functions $h_1, \ldots, h_{n_{2,2}} : L_p(\Omega, \mathcal{F}, \mathbb{P}) \to L_{p'}(\Omega, \mathcal{F}, \mathbb{P})$ with $1 \leq p' \leq p$. A risk measure ρ on $L_p(\Omega, \mathcal{F}, \mathbb{P})$ with $p \in [1, \infty)$ is called* extended polyhedral *if there exist matrices $A_1, A_2, B_{2,0}, B_{2,1}$, and vectors a_1, a_2, c_1, c_2 such that for every random variable $z \in L_p(\Omega, \mathcal{F}, \mathbb{P})$*

$$\rho(z) = \left\{ \begin{array}{l} \inf\ c_1^\top y_1 + \mathbb{E}[c_2^\top y_2] \\ y_1 \in \mathbb{R}^{k_1},\ y_2 \in L_p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{k_2}), \\ A_1 y_1 \leq a_1,\ A_2 y_2 \leq a_2 \ a.s. \\ B_{2,1} y_1 + B_{2,0} y_2 = h(z) \ a.s. \end{array} \right. \tag{16}$$

For one-period risk measures, dual representations from Theorem 2.5 specialize as follows:

Corollary 2.9. *Let ρ be a functional of the form (16) on $L_p(\Omega, \mathcal{F}, \mathbb{P})$ with some $p \in [1, \infty)$ and $h(z) = z b_2 + \tilde{b}_2$ for some vectors $b_2, \tilde{b}_2$. Assume*

(i) *complete recourse: $\{y_1 : A_1 y_1 \leq a_1\} \neq \emptyset$ and $\{B_{2,0} y_2 : A_2 y_2 \leq a_2\} = \mathbb{R}^{n_{2,2}}$;*
(ii) *dual feasibility: $\{(u, v) : u \in \mathbb{R}^{n_{1,1}} \times \mathbb{R}^{n_{2,1}}, v \in \mathbb{R}^{n_{2,2}}, c_t + A_t^\top u_t + B_{2,2-t}^\top v = 0,\ t = 1, 2\} \neq \emptyset$.*

Then ρ is finite, convex, continuous, and with $\frac{1}{p} + \frac{1}{q} = 1$, ρ admits the dual representation

$$\rho(z) = \left\{ \begin{array}{l} \sup\ -\lambda_1^\top a_1 - \mathbb{E}[\lambda_2^\top a_2 + \lambda_3^\top (z b_2 + \tilde{b}_2)] \\ \lambda_1 \in \mathbb{R}^{n_{1,1}},\ \lambda_2 \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,1}}),\ \lambda_3 \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,2}}), \\ c_1 + A_1^\top \lambda_1 + B_{2,1}^\top \mathbb{E}[\lambda_3] = 0 \\ c_2 + A_2^\top \lambda_2 + B_{2,0}^\top \lambda_3 = 0 \ a.s. \\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ a.s. \end{array} \right.$$

We also have

$$\rho(z) = \sup \{\mathbb{E}[z^* z] - \rho^*(z^*)\ :\ z^* \in\ L_q(\Omega, \mathcal{F}, \mathbb{P})\} \tag{17}$$

where ρ^* is the conjugate of ρ. Next, for every $z^* \in \mathrm{dom}(\rho^*)$, $\rho^*(z^*)$ is given by

$$\rho^*(z^*) = \left\{ \begin{array}{l} \inf\ \mathbb{E}[\lambda_1^\top a_1 + \lambda_2^\top a_2 + \lambda_3^\top \tilde{b}_2] \\ \lambda_1 \in \mathbb{R}^{n_{1,1}},\ \lambda_2 \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,1}}),\ \lambda_3 \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,2}}), \\ z^* = -\lambda_3^\top b_2 \ a.s.,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0 \ a.s. \\ c_1 + A_1^\top \lambda_1 + B_{2,1}^\top \mathbb{E}[\lambda_3] = 0 \\ c_2 + A_2^\top \lambda_2 + B_{2,0}^\top \lambda_3 = 0 \ a.s. \end{array} \right. \tag{18}$$

where

$$\text{dom}(\rho^*) = \left\{ \begin{array}{l} z^* \in L_q(\Omega, \mathcal{F}, \mathbb{P}) \text{ such that there exist} \\ \lambda_1 \in \mathbb{R}^{n_{1,1}},\ \lambda_2 \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,1}}),\ \lambda_3 \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,2}}) \text{ satisfying} \\ c_1 + A_1^\top \lambda_1 + B_{2,1}^\top \mathbb{E}[\lambda_3] = 0,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0 \ a.s. \\ c_2 + A_2^\top \lambda_2 + B_{2,0}^\top \lambda_3 = 0 \ a.s., \quad \text{and} \quad z^* = -\lambda_3^\top b_2 \ a.s. \end{array} \right\}. \tag{19}$$

Proof. It suffices to use Theorem 2.5 with $T = 2$. □

Definition 2.2 specializes as follows to the one-period case.

Definition 2.10. *A functional $\rho : L_p(\Omega, \mathcal{F}, \mathbb{P}) \to \bar{\mathbb{R}}$ is called a* convex risk measure *if it satisfies the following three conditions for all $z, \tilde{z} \in L_p(\Omega, \mathcal{F}, \mathbb{P})$:*

(i) Monotonicity*: if $z \leq \tilde{z}$ a.s., then $\rho(z) \geq \rho(\tilde{z})$;*
(ii) Translation invariance*: for each $r \in \mathbb{R}$ we have $\rho(z + r) = \rho(z) - r$;*
(iii) Convexity*: for all $\mu \in [0, 1]$ we have $\rho(\mu z + (1 - \mu)\tilde{z}) \leq \mu\rho(z) + (1 - \mu)\rho(\tilde{z})$.*

Such a functional ρ is said to be coherent *if it is positively homogeneous, i.e., $\rho(\mu z) = \mu\rho(z)$ for all $\mu \geq 0$ and $z \in L_p(\Omega, \mathcal{F}, \mathbb{P})$.*

Using Theorems 2.3 and Corollary 2.9, a sufficient criterion can be provided for a one-period extended polyhedral risk measure to be coherent:

Corollary 2.11. *Let ρ be a functional on $L_p(\Omega, \mathcal{F}, \mathbb{P})$ of the form (16) with a_1, a_2 null, $p \in [1, \infty)$ and $h(z) = zb_2$ for some vector b_2. Let the conditions of Corollary 2.9 be satisfied (complete recourse and dual feasibility) and let $\mathcal{M}_\rho$ be the following (convex) set of dual multipliers:*

$$\mathcal{M}_\rho = \left\{ \begin{array}{l} \lambda \in L_q(\Omega, \mathcal{F}, \mathbb{P}) \text{ such that there exist} \\ (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^{n_{1,1}} \times L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,1}}) \times L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^{n_{2,2}}) \text{ satisfying} \\ c_1 + A_1^\top \mu_1 + B_{2,1}^\top \mathbb{E}[\mu_3] = 0 \\ c_2 + A_2^\top \mu_2 + B_{2,0}^\top \mu_3 = 0 \ a.s.,\ \mu_1 \geq 0,\ \mu_2 \geq 0 \ a.s. \text{ with } \ \lambda = \mu_3^\top b_2 \end{array} \right\}. \tag{20}$$

If $\mathcal{M}_\rho \subseteq \mathcal{D}_1$ then ρ is a (one-period) coherent risk measure.

Proof. From Corollary 2.9, we obtain $\rho(z) = \sup_{\lambda \in \mathcal{M}_\rho} -\mathbb{E}[\lambda z]$ and the result follows taking $\mathcal{P}_\rho = \mathcal{M}_\rho$ in Theorem 2.3. □

A dual representation of the second-stage problem for (16) will prove useful for obtaining further properties of one-period risk measures of the form (16):

Proposition 2.12. *Let ρ be a functional of the form (16) on $L_p(\Omega, \mathcal{F}, \mathbb{P})$ with some $p \in [1, \infty)$ and $h(z) = zb_2 + \tilde{b}_2$ for some vectors $b_2, \tilde{b}_2$. Let the conditions of Corollary 2.9 be satisfied (complete recourse and dual feasibility). Assume the feasible set $\mathcal{D}$ of the dual of the second-stage problem is nonempty where*

$$\mathcal{D} = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^{n_{2,2}} \times \mathbb{R}^{n_{2,1}} \ : \ \lambda_2 \leq 0,\ \ B_{2,0}^\top \lambda_1 + A_2^\top \lambda_2 = c_2\}. \tag{21}$$

Then ρ is finite, convex, continuous, and is given by

$$\rho(z) = \inf_{A_1 y_1 \leq a_1} \left\{ c_1^\top y_1 + \mathbb{E}[\sup_{\lambda \in \mathcal{D}} \ \lambda_1^\top (zb_2 + \tilde{b}_2 - B_{2,1} y_1) + \lambda_2 a_2] \right\}.$$

Proof. Finiteness, convexity and continuity follow from Corollary 2.9. Next, we write $\rho(z)$ as

$$\rho(z) = \inf_{y_1} \{c_1^\top y_1 + \mathbb{E}[\mathcal{Q}_2(y_1, z)] \ : \ A_1 y_1 \leq a_1\} \tag{22}$$

where for each y_1 such that $A_1 y_1 \leq a_1$ and for each $z \in \mathbb{R}$ we have defined

$$\mathcal{Q}_2(y_1, z) = \inf_{y_2} \{c_2^\top y_2 \ : \ B_{2,0} y_2 = zb_2 + \tilde{b}_2 - B_{2,1} y_1,\ A_2 y_2 \leq a_2\}.$$

Finally, since $\mathcal{D} \neq \emptyset$, by duality, we can express $\mathcal{Q}_2(y_1, z)$ as

$$\mathcal{Q}_2(y_1,z) = \sup_{(\lambda_1,\lambda_2)} \{\lambda_1^\top (zb_2 + \tilde{b}_2 - B_{2,1}y_1) + \lambda_2^\top a_2 \ : \lambda_2 \leq 0,\ B_{2,0}^\top \lambda_1 + A_2^\top \lambda_2 = c_2\}. \tag{23}$$

□

The following proposition provides a sufficient criterion for some extended polyhedral risk measures to be convex risk measures when

$$Y_1 = \{y_1 : A_1 y_1 \leq a_1\} \tag{24}$$

is not necessarily a cone (a_1 need not be 0).

Proposition 2.13. *Let ρ be a functional on $L_p(\Omega, \mathcal{F}, \mathbb{P})$ of the form (16) with $p \in [1, \infty)$ and $h(z) = zb_2 + \tilde{b}_2$ for some vectors $b_2, \tilde{b}_2$. Let the conditions of Corollary 2.9 be satisfied (complete recourse and dual feasibility) and let $\mathcal{D}$ be defined as in Proposition 2.12. Assume*

- (i) *$\mathcal{D} \neq \emptyset$ with $\mathcal{D} \subseteq \{b_2\}^* \times \mathbb{R}^{n_{2,1}}$,*
- (ii) *$c_1 \neq 0$ and b_2 is of the form $b_2 = -B_{2,1}^i / c_1(i)$ for at least one $i \in I = \{j \ : \ c_1(j) \neq 0\}$ with $y_1(i)$ unconstrained and where $B_{2,1}^i$ denotes the ith column of $B_{2,1}$.*

Then ρ is a finite-valued convex risk measure.

Proof. Let Y_1 be defined by (24). Finiteness and convexity of ρ follow from Corollary 2.9. The monotonicity of ρ follows from (i). Indeed, if $z, \tilde{z} \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ satisfy $z \leq \tilde{z}$ a.s.; then for every $y_1 \in Y_1$ and every $(\lambda_1, \lambda_2) \in \mathcal{D}$ we have

$$\lambda_1^\top (zb_2 + \tilde{b}_2 - B_{2,1}y_1) + \lambda_2^\top a_2 \geq \lambda_1^\top (\tilde{z}b_2 + \tilde{b}_2 - B_{2,1}y_1) + \lambda_2^\top a_2.$$

With the notation of Proposition 2.12 and with $\varphi(y_1, z) = c_1^\top y_1 + \mathbb{E}[\mathcal{Q}_2(y_1, z)]$, it follows that for every $y_1 \in Y_1$, we have $\mathbb{E}[\mathcal{Q}_2(y_1, z)] \geq \mathbb{E}[\mathcal{Q}_2(y_1, \tilde{z})]$, $\varphi(y_1, z) \geq \varphi(y_1, \tilde{z})$, and $\rho(z) = \inf_{y_1 \in Y_1} \varphi(y_1, z) \geq \inf_{y_1 \in Y_1} \varphi(y_1, \tilde{z}) = \rho(\tilde{z})$. The translation invariance condition follows from (ii). Indeed, eventually after reordering the components of y_1, c_1, and the columns of $B_{2,1}$, we can always assume that the index i satisfying (ii) is the last k_1th index, i.e., that $c_1, B_{2,1}$, and Y_1 are of the form $c_1 = (\hat{c}_1, \bar{c}_1)^\top$ with $\bar{c}_1 \in \mathbb{R}^*$, $B_{2,1} = [\hat{B}_{2,1}, -\bar{c}_1 b_2]$, and $Y_1 = \{y_1 = (\hat{y}_1, \bar{y}_1) \ : \ \hat{A}_1 \hat{y}_1 \leq a_1, \bar{y}_1 \in \mathbb{R}\}$. We then have for each $r \in \mathbb{R}$, for each $z \in L_p(\Omega, \mathcal{F}, \mathbb{P})$ and setting $\tilde{y}_1 = \bar{y}_1 + \frac{r}{\bar{c}_1} \in \mathbb{R}$

$$\begin{aligned}
\rho(z+r) &= \inf_{\hat{A}_1\hat{y}_1 \leq a_1,\ \bar{y}_1 \in \mathbb{R}} \{\hat{c}_1^\top \hat{y}_1 + \bar{c}_1 \bar{y}_1 + \mathbb{E}[\sup_{(\lambda_1,\lambda_2)\in\mathcal{D}} \lambda_1^\top ((z+r)b_2 + \tilde{b}_2 - \hat{B}_{2,1}\hat{y}_1 + \bar{y}_1 \bar{c}_1 b_2) + \lambda_2^\top a_2]\} \\
&= \inf_{\hat{A}_1\hat{y}_1 \leq a_1,\ \tilde{y}_1 \in \mathbb{R}} \{\hat{c}_1^\top \hat{y}_1 + \bar{c}_1 \tilde{y}_1 + \mathbb{E}[\sup_{(\lambda_1,\lambda_2)\in\mathcal{D}} \lambda_1^\top (zb_2 + \tilde{b}_2 - \hat{B}_{2,1}\hat{y}_1 + \tilde{y}_1 \bar{c}_1 b_2) + \lambda_2^\top a_2]\} - r \\
&= \rho(z) - r.
\end{aligned}$$

□

Proposition 2.13 extends the corresponding result in Eichhorn and Römisch [ER05]. Proposition 2.14 below shows that condition (i) in Proposition 2.13 ensures in fact a stronger type of monotonicity than (i) in Definition 2.10. Such monotonicity is based on *stochastic dominance rules* Müller and Stoyan [MS02]. For real-valued random variables $z, \tilde{z} \in L_1(\Omega, \mathcal{F}, \mathbb{P})$, stochastic dominance rules are defined by classes of measurable real-valued functions on $\mathbb{R}$. The stochastic dominance rule with respect to class $\mathcal{F}$ is defined by

$$z \preceq_{\mathcal{F}} \tilde{z} \quad :\Longleftrightarrow \quad \forall\, f \in \mathcal{F} \ : \ [\text{ if } \mathbb{E}[f(z)] \text{ and } \mathbb{E}[f(\tilde{z})] \text{ exist then } \mathbb{E}[f(z)] \leq \mathbb{E}[f(\tilde{z})]]$$

for each $z, \tilde{z} \in L_1(\Omega, \mathcal{F}, \mathbb{P})$. Important special cases are the classes $\mathcal{F}_{nd}$ of nondecreasing functions and $\mathcal{F}_{ndc}$ of nondecreasing concave functions which respectively characterize first and second order stochastic dominance rules:

$$\begin{array}{lll} z \preceq_{FSD} \tilde{z} & :\Longleftrightarrow z \preceq_{\mathcal{F}_{nd}} \tilde{z} & \Longleftrightarrow \quad \mathbb{P}(z \le t) \ge \mathbb{P}(\tilde{z} \le t) \quad \forall\, t \in \mathbb{R}, \\ z \preceq_{SSD} \tilde{z} & :\Longleftrightarrow z \preceq_{\mathcal{F}_{ndc}} \tilde{z} & \Longleftrightarrow \quad \mathbb{E}[\min(z,t)] \le \mathbb{E}[\min(\tilde{z},t)] \quad \forall\, t \in \mathbb{R}. \end{array}$$

In particular, it is said that a risk measure ρ is *consistent with second order stochastic dominance* Ogryczak and Ruszczyński [OR02] if $z \preceq_{SSD} \tilde{z}$ implies $\rho(z) \ge \rho(\tilde{z})$.

Proposition 2.14. *Let ρ be a functional on $L_p(\Omega, \mathcal{F}, \mathcal{P})$ of the form* (16) *with $p \in [1, \infty)$ and $h(z) = zb_2 + \tilde{b}_2$ for some vectors $b_2, \tilde{b}_2$. Let the conditions of Corollary 2.9 be satisfied (complete recourse and dual feasibility) and let $\mathcal{D}$ be defined as in Proposition 2.12. Assume $\mathcal{D} \neq \emptyset$ with $\mathcal{D} \subseteq \{b_2\}^* \times \mathbb{R}^{n_{2,1}}$. Then ρ is consistent with second order stochastic dominance.*

Proof. With Y_1 defined as in (24), let g be the function defined for every $y_1 \in Y_1$ and $z \in \mathbb{R}$ by

$$g(y_1, z) = c_1^\top y_1 + \sup_{(\lambda_1,\lambda_2)\in\mathcal{D}} \{\lambda_1^\top(zb_2 + \tilde{b}_2 - B_{2,1}y_1) + \lambda_2^\top a_2\}. \tag{25}$$

For every $y_1 \in Y_1$, $g(y_1, \cdot)$ is convex and since $\mathcal{D} \subseteq \{b_2\}^* \times \mathbb{R}^{n_{2,1}}$ it is also nonincreasing. Let $z \preceq_{SSD} \tilde{z}$. For every $y_1 \in Y_1$, since $-g(y_1, \cdot)$ is concave and nondecreasing $\mathbb{E}[-g(y_1, z)] \le \mathbb{E}[-g(y_1, \tilde{z})]$ and $\rho(z) = \inf_{y_1\in Y_1} \mathbb{E}[g(y_1, z)] \ge \inf_{y_1\in Y_1} \mathbb{E}[g(y_1, \tilde{z})] = \rho(\tilde{z})$. □

For a one-period risk measure of the form (16) with $h(z) = zb_2 + \tilde{b}_2$ for some vectors $b_2, \tilde{b}_2$, the first stage solution set $S(\rho(z)) \subseteq Y_1$ is given by

$$S(\rho(z)) = \{y_1 \in Y_1 \ : \ \rho(z) = c_1^\top y_1 + \sup_{(\lambda_1,\lambda_2)\in\mathcal{D}} \{\lambda_1^\top(zb_2 + \tilde{b}_2 - B_{2,1}y_1) + \lambda_2^\top a_2\}\}. \tag{26}$$

For algorithmic issues considered in Sections 3 and 5, it can be useful to have at hand conditions that guarantee the boundedness of $S(\rho(z))$. This question is addressed in the following proposition:

Proposition 2.15. *Let ρ be a functional on $L_p(\Omega, \mathcal{F}, \mathbb{P})$ of the form* (16) *with $p \in [1, \infty)$, a_2 null and $h(z) = zb_2$ for some vector b_2. Let the conditions of Corollary 2.9 be satisfied (complete recourse and dual feasibility) and assume that $S(\rho(0))$ is nonempty and bounded. Then $S(\rho(z))$ is nonempty, convex, and compact for any $z \in L_p(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. The proof follows the proof of Proposition 2.9 in Eichhorn and Römisch [ER05], with, in our case, g given by (25). □

We provide examples of extended polyhedral risk measures. The above criteria for coherence and consistency with second order stochastic dominance are applied.

Example 2.16 (Spectral risk measures and CVaR)**.** *Let $F_z(x) = \mathbb{P}(z \le x)$ be the distribution function of random variable z and let $F_z^{\leftarrow}(p) = \inf\{x \, : \, F_z(x) \ge p\}$ be the usual generalized inverse of F_z. Given a risk spectrum $\phi \in L_1([0,1])$ the* spectral risk measure *ρ_ϕ generated by ϕ is given by Acerbi* [Ace02]

$$\rho_\phi(z) = -\int_0^1 F_z^{\leftarrow}(p)\phi(p)dp.$$

Spectral risk measures have been used in a number of applications (portfolio selection Acerbi and Simonetti [AS]*, insurance Cotter and Kevin* [CD06]*). The Conditional Value-at-Risk (CVaR) of level $0 < \varepsilon < 1$, also called Average Value-at-Risk (AVaR) in Föllmer and Schied* [FS04]*, is a particular spectral risk measure with a piecewise risk function ϕ having a jump at ε: $\phi(u) = \frac{1}{\varepsilon} 1_{0 \le u \le \varepsilon}$ (Acerbi* [Ace02]*). Let us consider more generally a piecewise risk function $\phi(\cdot)$ with J jumps at*

$0 < p_1 < p_2 < \ldots < p_J < 1$. *Setting* $\Delta\phi_k = \phi(p_k^+) - \phi(p_k^-) = \phi(p_k) - \phi(p_{k-1})$, *for* $k = 1, \ldots, J$, *with* $p_0 = 0$, *we assume*

$$(i)\ \phi(\cdot) \text{ is positive, } (ii)\ \Delta\phi_k < 0,\ k = 1, \ldots, J,\ (iii) \int_0^1 \phi(u)du = 1.$$

With this choice of ϕ, *we can express* $\rho_\phi(z)$ *as the optimal value of a linear programming problem, Acerbi and Simonetti* [AS]*:*

$$\rho_\phi(z) = \inf_{x \in \mathbb{R}^J} \sum_{k=1}^{J} \Delta\phi_k [p_k x_k - \mathbb{E}\,[x_k - z]^+] - \phi(1)\mathbb{E}[z]. \tag{27}$$

When $J = 1, \Delta\phi_1 = -1/\varepsilon, p_1 = \varepsilon$, *and* $\phi(1) = 0$, *the above formula reduces to the formula for the CVaR given by Rockafellar and Uryasev* [RU02]*:*

$$CVaR^\varepsilon[z] = \inf_{x \in \mathbb{R}} \left[x + \frac{1}{\varepsilon}\,\mathbb{E}[z + x]^- \right]. \tag{28}$$

A spectral risk measure with a piecewise risk function satisfying (i), (ii), and (iii) above is a coherent extended polyhedral risk measure. Indeed, with respect to (16), *we have* $c_1 = \Delta\phi \circ p$ *with* $\Delta\phi = (\Delta\phi_1, \ldots, \Delta\phi_J)^\top$, $c_2 = (-\Delta\phi; 0_{J,1}; -\phi(1))$, $B_{2,1} = (I_J; 0_{1,J}), B_{2,0} = (-I_J, I_J, 0_{J,1}; 0_{1,2J}, 1)$, $A_2 = (-I_{2J}, 0_{2J,1})$, *and* $h(z) = ze$. *The matrix* A_1 *and the vectors* a_1 *and* a_2 *are null,* b_2 *is a* $(J+1)$-*vector of ones and* $\tilde{b}_2 = 0$. *Notice that when* $J > 1$ *it is not polyhedral in the sense of Eichhorn and Römisch* [ER05]. *The complete recourse and dual feasibility assumptions from Corollary 2.9 are easily checked. This theorem provides for* ρ_ϕ *the dual representation*

$$\rho_\phi(z) = \left\{ \begin{array}{l} \sup\ -\mathbb{E}[\lambda z] \\ \lambda = \mu^\top e + \phi(1),\ \mu \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^J), \\ \mathbb{E}[\mu] = -\Delta\phi \circ p,\ 0 \le \mu \le -\Delta\phi\ a.s. \end{array} \right. \tag{29}$$

Let $\mathcal{M}_{\rho_\phi}$ *be the set of dual multipliers from Corollary 2.11 for* ρ_ϕ. *For every* $\lambda \in \mathcal{M}_{\rho_\phi}$, *we have* $\lambda \ge 0$ *a.s. and*

$$\begin{array}{rcl} \mathbb{E}[\lambda] & = & \mathbb{E}[\phi(1) + \mu^\top e] = \phi(1) - \sum_{i=1}^{J} \Delta\phi_i p_i = \phi(1) - \sum_{i=1}^{J} (\phi(p_i) - \phi(p_{i-1}))p_i \\ & = & \phi(0)p_1 + \sum_{i=1}^{J-1} \phi(p_i)(p_{i+1} - p_i) + (1 - p_J)\phi(1) = \int_0^1 \phi(u)du = 1. \end{array}$$

It follows that $\mathcal{M}_{\rho_\phi} \subseteq \mathcal{D}_1$ *and using Corollary 2.11,* ρ_ϕ *is a coherent one-period risk measure. Next, the set* $\mathcal{D}$ *in Proposition 2.14 is given by* $\mathcal{D} = \{(\lambda_1, \lambda_2) \in \mathbb{R}^{J+1} \times \mathbb{R}^{2J}\ :\ \lambda_2 \le 0,\ \lambda_{1,J+1} = -\phi(1),\ \lambda_{1,1:J} = \lambda_{2,J+1:2J},\ \lambda_{1,1:J} = -\lambda_{2,1:J} + \Delta\phi\}$. *For every* $(\lambda_1, \lambda_2) \in \mathcal{D}$, *we have* $\lambda_1^\top b_2 = \lambda_1^\top e \le 0$. *It follows that* $\mathcal{D} \subseteq \{b_2\}^* \times \mathbb{R}^{n_{2,1}}$ *and due to Corollary 2.14,* ρ_ϕ *is consistent with second order stochastic dominance. When* $J = 1, \Delta\phi_1 = -1/\varepsilon, p_1 = \varepsilon$, *and* $\phi(1) = 0$, $\rho_\phi = CVaR^\varepsilon$ *and we recover results given in Eichhorn and Römisch* [ER05]*: the CVaR is consistent with second order stochastic dominance and is an extended polyhedral risk measure of the form* (16) *with* $c_1 = 1$, $c_2 = (\frac{1}{\varepsilon}; 0)$, $B_{2,1} = -1$, $B_{2,0} = (-1, 1), A_2 = -I_2$, $h(z) = z$, *and* A_1, a_1, a_2 *null. The dual representation* (29) *becomes*

$$CVaR^\varepsilon(z) = \sup\{-\mathbb{E}[\lambda z] : \lambda \in L_q(\Omega, \mathcal{F}, \mathbb{P}), 0 \le \lambda \le \frac{1}{\varepsilon}\ a.s.,\ \mathbb{E}[\lambda] = 1\}.$$

Example 2.17 (Optimized certainty equivalent (OCE) and expected utility.)**.** *Given a concave non-decreasing utility function* u, *the optimized certainty equivalent* $S_u(z)$ *of the random variable* z *is defined in Ben-Tal and Teboulle* [BTT07] *by* $S_u(z) = \sup_{y_1 \in \mathbb{R}}\ y_1 + \mathbb{E}[u(z - y_1)]$. *Considering* *for u a piecewise affine concave function, we can express the convex function $-u$ as Rockafellar and Wets,* [RW98]*[Example 3.54]*

$$-u(x) = \inf\{c^\top y \;:\; y \in \mathbb{R}^k, y \geq 0, e^\top y = 1,\; b^\top y = x\} \tag{30}$$

for some vectors $b, c \in \mathbb{R}^k$. It follows that if u is a piecewise affine concave function, $\rho(z) = -S_u(z)$ is given by

$$\rho(z) = \left\{ \begin{array}{l} \inf \; -y_1 + \mathbb{E}[c^\top y_2] \\ y_1 \in \mathbb{R},\; y_2 \in \mathbb{R}^k,\; y_2 \geq 0,\; e^\top y_2 = 1,\; b^\top y_2 = z - y_1. \end{array} \right. \tag{31}$$

In this case, the opposite of the OCE is an extended one-period polyhedral risk measure with h affine: $c_1 = -1$, $c_2 = c$, $A_2 = [-I_k; e^\top; -e^\top]$, $a_2 = [0_{k,1}; 1; -1]$, $B_{2,1} = 1$, $B_{2,0} = b^\top$, $b_2 = 1$, and A_1, a_1, and $\tilde{b}_2$ null. Notice that it is not polyhedral in the sense of Eichhorn and Römisch [ER05] *and that complete recourse does not hold. However, properties of the OCE, given in Ben-Tal and Teboulle* [BTT07]*, are easily checked:* monotonicity *follows from the definition of $-S_u$ and the fact that u is non-decreasing;* translation invariance *follows from the change of variable $\bar{y}_1 = y_1 - r$ in* (31) *(for $\rho(z + r)$) or in the definition of $-S_u(z + r)$;* convexity *can be checked directly from the definition of S_u (or using representation* (31) *and [Bonnans and Shapiro* [BS00]*, Proposition 2.143], as in the proof of Theorem 2.5).*

Let us consider as a special case a piecewise linear utility function of the form

$$u(x) = \gamma_1(x)^+ - \gamma_2(-x)^+ \text{ where } 0 \leq \gamma_1 < 1 < \gamma_2; \tag{32}$$

(note that $u(x) < x$ for $x \neq 0$). The corresponding risk measure $\rho(z) = -S_u(z)$ is an extended polyhedral risk measure with $c_1 = -1$, $c_2 = (-\gamma_1; \gamma_2)$, $B_{2,1} = 1, B_{2,0} = [1 \; -1]$, $A_2 = -I_2$, $h(z) = z$, and A_1, a_2, a_2 null. Since complete (and even simple) recourse and dual feasibility hold, Corollary 2.9 provides the following dual representation:

$$\rho(z) = -S_u(z) = \sup\{-\mathbb{E}[\lambda z] : \lambda \in L_q(\Omega, \mathcal{F}, \mathbb{P}),\; \mathbb{E}[\lambda] = 1,\; \gamma_1 \leq \lambda \leq \gamma_2 \; a.s.\}.$$

Using Corollary 2.11, we deduce that when u is of the form (32)*, $\rho(z) = -S_u(z)$ is a coherent risk measure. More generally, it is shown in Ben-Tal and Teboulle* [BTT07] *that if u is a strongly risk averse function (see Ben-Tal and Teboulle* [BTT07]*), $\rho(z) = -S_u(z)$ is coherent if and only if u is of the form* (32)*. For $0 < \varepsilon < 1$, $CVaR^\varepsilon$ constitutes a particular case with $\gamma_1 = 0$ and $\gamma_2 = \frac{1}{\varepsilon}$. The set $\mathcal{D}$ in Proposition 2.14 is given by $\mathcal{D} = \{(\lambda_1, \lambda_2) : -\gamma_2 \leq \lambda_1 \leq -\gamma_1,\; \lambda_2 \leq 0\}$. Since for every $(\lambda_1, \lambda_2) \in \mathcal{D}$ we have $\lambda_1^\top b_2 = \lambda_1^\top e \leq 0$, using Proposition 2.14 we conclude that $-S_u(z)$ is consistent with second order stochastic dominance.*

For any concave utility function u, the risk measure $\rho(z) = -\mathbb{E}(u(z))$ is an extended polyhedral risk measure with $h = u$, $B_{2,0} = c_2 = 1$, while the other parameters are null. In the particular case when u is a piecewise affine concave function, representation (30) *shows that $-\mathbb{E}(u(z))$ can be written as an extended polyhedral risk measure with $h(z) = z$ and that complete recourse does not hold. However, a dual representation of ρ can be derived from the dual representation*

$$-u(x) = \sup\{-\lambda_1 x - \lambda_2 : \lambda \in \mathbb{R}^2,\; \lambda_1 b + \lambda_2 e \leq -c\} \tag{33}$$

of $-u$. Applying the expectation operator to both sides of the above equation and using Rockafellar and Wets, [RW98]*[Theorem 14.60] (for switching the inf and expectation operators), we obtain for ρ the dual representation*

$$\rho(z) = \sup\{-\mathbb{E}[\lambda_1 z + \lambda_2] : \lambda \in L_q(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^2),\; \lambda_1 b + \lambda_2 e \leq -c \; a.s.\}.$$

Since $-u$ is non-increasing, for every (λ_1, λ_2) in the feasible set of (33) *we have $\lambda_1 \geq 0$ (otherwise, there would be positive subgradients of $-u$ at large enough points). It follows that in the above representation of ρ, $\lambda_1 \geq 0$ a.s., which implies that ρ is monotone, convex, and consistent with second order stochastic dominance. The* expected regret *or* expected loss *$\rho(z) = \mathbb{E}(z - \beta)^-$ for some* *target β is a special case (already considered in Eichhorn and Römisch* [ER05]*) with utility function $u(z) = -(z-\beta)^-$. Finally, notice that $\rho(z) = \mathbb{E}[(z-\mathbb{E}[z])^k]$ for some $1 \le k \le p-1$, is an extended polyhedral risk measure with $h(z) = (z-\mathbb{E}[z])^k$.*

Example 2.18 (Multiperiod extended polyhedral risk measures).
We consider functionals ρ on $\times_{t=1}^T L_p(\Omega, \mathcal{F}_t, \mathbb{P})$ ($p \in [1,\infty)$) of the form $\rho(z) = CVaR^\varepsilon(\Phi(z))$ where the function Φ is defined on $\mathbb{R}^T$ and maps to the extended real numbers.
Then ρ is a finite-valued coherent multiperiod risk measure if the function Φ is (i) concave, (ii) monotone with respect to the (canonical) partial ordering in $\mathbb{R}^T$, (iii) positively homogeneous, (iv) satisfies the property $\Phi(\zeta_1 + r, \dots, \zeta_T + r) = \Phi(\zeta_1, \dots, \zeta_T) + r$ for all $r \in \mathbb{R}$ and $\zeta \in \mathbb{R}^T$ and (v) has linear growth, i.e., for some constant $L > 0$ it holds $|\Phi(\zeta)| \le L \sum_{t=1}^T |\zeta_t|$ for every $\zeta \in \mathbb{R}^T$.
There are two important special cases for the function Φ:

(a) *$\Phi(\zeta) = \sum_{t=1}^T \gamma_t \zeta_t$ with nonnegative γ_t, $t = 1, \dots, T$, such that $\sum_{t=1}^T \gamma_t = 1$. According to the dual representation of $CVaR^\varepsilon$ we obtain*

$$\begin{aligned} \rho(z) &= CVaR^\varepsilon\Big(\sum_{t=1}^T \gamma_t z_t\Big) = \sup\Big\{-\sum_{t=1}^T \gamma_t \mathbb{E}(\lambda z_t) : \lambda \in L_q(\Omega, \mathcal{F}, \mathbb{P}),\, \mathbb{E}(\lambda) = 1,\, 0 \le \lambda \le \frac{1}{\varepsilon} \text{ a.s.}\Big\} \\ &= \sup\Big\{-\sum_{t=1}^T \mathbb{E}(\lambda_t z_t) : \lambda_t \in L_q(\Omega, \mathcal{F}_t, \mathbb{P}),\, \mathbb{E}(\lambda_t) = \gamma_t,\, 0 \le \lambda_t \le \frac{\gamma_t}{\varepsilon},\, t = 1, \dots, T, \\ &\qquad \gamma_t \mathbb{E}(\lambda_{t+1}|\mathcal{F}_t) = \gamma_{t+1}\lambda_t \text{ a.s.},\, t = 1, \dots, T-1\Big\}, \end{aligned}$$

where $\lambda_t = \gamma_t \mathbb{E}(\lambda|\mathcal{F}_t)$, $t = 1, \dots, T$, and $\frac{1}{p} + \frac{1}{q} = 1$. Hence, ρ is a multiperiod extended polyhedral coherent risk measure according to Theorems 2.3 and 2.5.

(b) *$\Phi(\zeta) = \min_{t=1,\dots,T} \zeta_t$ for $\zeta \in \mathbb{R}^T$. Here, we use the representation (28) and obtain*

$$\begin{aligned} \rho(z) &= CVaR^\varepsilon\Big(\min_{t=1,\dots,T} z_t\Big) = \inf\Big\{x + \frac{1}{\varepsilon}\mathbb{E}\Big(\Big[\min_{t=1,\dots,T} z_t + x\Big]^-\Big) : x \in \mathbb{R}\Big\} \\ &= \inf\Big\{x + \frac{1}{\varepsilon}\mathbb{E}\Big(\max_{t=1\dots,T}\Big\{0, -x - z_t\Big\}\Big) : x \in \mathbb{R}\Big\} \\ &= \inf\Big\{x + \frac{1}{\varepsilon}\mathbb{E}(v_T) : v_t \in L_p(\Omega, \mathcal{F}_t, \mathbb{P}),\, -x - z_t \le v_t,\, v_{t-1} \le v_t,\, t = 1, \dots, T,\, v_0 = 0,\, x \in \mathbb{R}\Big\}. \end{aligned}$$

The latter linear stochastic program may be rewritten in the form (6) and ρ is a multiperiod extended polyhedral coherent risk measure. It has been first studied by Eichhorn in [Eic07].

3. Risk averse dynamic programming

3.1. **General setting.** When using a multiperiod extended polyhedral risk measure to deal with uncertainty in the multistage stochastic programming framework (4), we consider accumulated revenues $z_t = -\sum_{\tau=1}^t f_\tau(x_\tau, \xi_\tau)$ and the sigma-algebras $\mathcal{F}_t = \sigma(\xi_j, j \le t)$ for $t = 1, \dots, T$. Recall that x_0 and $\chi_1(x_0, \xi_1)$ are deterministic and that for any time step $t = 1, \dots, T$, we denote by $\xi_{[t]}$ the available realizations of the process up to this time step, i.e., $\xi_{[t]} = (\xi_j, j \le t)$.

We also denote by $\mathcal{Z}_t$ the space of $\mathcal{F}_t$-measurable functions (these sets are embedded: $\mathcal{Z}_1 \subset \ldots \subset \mathcal{Z}_T$). Next, for $t = 1, \dots, T$, we assume the following:

(H1) the functions $f_t : \mathbb{R}^{N_{t,x}} \times \mathbb{R}^{M_t} \to \mathbb{R}$ are continuous and $\chi_t : \mathbb{R}^{N_{t-1,x}} \times \mathbb{R}^{M_t} \rightrightarrows \mathbb{R}^{N_{t,x}}$ are measurable, bounded and closed valued multifunctions.

We are now in a position to define a risk averse problem for (1) via a multiperiod risk measure. Let $\rho : \mathcal{Z}_1 \times \ldots \mathcal{Z}_T \to \mathbb{R}$ be a multiperiod risk measure and let us introduce the risk averse problem

$$\begin{array}{l} \inf \ \rho \left(-f_1(x_1, \xi_1), -\sum_{\tau=1}^{2} f_\tau(x_\tau(\xi_{[\tau]}), \xi_\tau), \ldots, -\sum_{\tau=1}^{T} f_\tau(x_\tau(\xi_{[\tau]}), \xi_\tau) \right) \\ x_t(\xi_{[t]}) \in \chi_t(x_{t-1}(\xi_{[t-1]}), \xi_t), \ t = 1, \ldots, T. \end{array} \tag{34}$$

In the above problem, the optimization is performed over $\mathcal{F}_t$-measurable functions $x_t, t = 1, \ldots, T$ satisfying the constraints and such that $f_t(x_t(\cdot), \cdot) \in \mathcal{Z}_t$. The sequence of measurable mappings $x_t(\cdot)$, $t = 1, \ldots, T$, is called a policy. The $\mathcal{F}_t$-measurability of $x_t(\cdot)$ implies the nonanticipativity of the policy, i.e., implies that x_t is a function of $\xi_{[t]}$. The policy obtained from (34) will be said to be risk averse. A policy is said to be *feasible* if the constraints $x_t(\xi_{[t]}) \in \chi_t(x_{t-1}(\xi_{[t-1]}), \xi_t)$, $t = 1, \ldots, T$, are satisfied with probability one.

In this section, our objective is to provide a class of form (1) problems and a class of multiperiod risk measures ρ having the following two properties:

(P1) Dynamic programming (DP) equations can be written for (34).
(P2) The SDDP algorithm applied to problem (34) decomposed by stages converges to an optimal solution of (34).

We intend to enforce (P2) obtaining DP equations that satisfy conditions given in Philpott and Guan [PG08]. These conditions imply the following:

(P3) The recourse functions are given as the optimal value of a non-risk averse stochastic program (the objective function is an expectation) where the randomness appears on the right hand side of the constraints only.

Property ($P3$) leads us naturally to use the class of extended polyhedral risk measures introduced in the previous section.

3.2. **Extended polyhedral risk measures.** Taking for ρ a multiperiod extended polyhedral risk measure of the form (6), problem (34) can be written

$$\begin{array}{l} \inf \ \mathbb{E}[\sum_{t=1}^{T} \ c_t^\top y_t] \\ A_t y_t \le a_t \ a.s., \ t = 1, \ldots, T, \\ \sum_{\tau=0}^{t-1} \ B_{t,\tau} y_{t-\tau} = h_t(-\sum_{\tau=1}^{t} f_\tau(x_\tau, \xi_\tau)) \ a.s. \ t = 2, \ldots, T, \\ x_t \in \chi_t(x_{t-1}, \xi_t) \ a.s. \ t = 1, \ldots, T. \end{array} \tag{35}$$

Remark 3.1. *In* (35)*, the dependence of x_t and y_t with respect to $\xi_{[t]}$ was suppressed to alleviate notation. This will in general be done in the sequel.*

We first check that (P1) and (P3) hold for problem (35) above. Since we want to write dynamic programming equations, and for further use, we start with the following simple remark:

Remark 3.2. *Let us consider the following T-stage optimization problem*

$$P \left\{ \begin{array}{l} \inf \ f(x_1, \ldots, x_T) \\ x_t \in X(x_0, \ldots, x_{t-1}), \ t = 1, \ldots, T. \end{array} \right.$$

We decompose f as $f(x) = \sum_{k=1}^{T} \ f_k(x_{1:k})$, where f_k is the sum of all the functions in the sum of functions defining f which depend on x_k but not on $x_{k+1:T}$ (for a given k, f_k is 0 if no such functions exist). Dynamic programming equations for P can be written as follows:

$$\mathcal{Q}_t(x_{0:t-1}) = \left\{ \begin{array}{l} \inf\limits_{x_t} f_t(x_{1:t}) + \mathcal{Q}_{t+1}(x_{0:t}) \\ x_t \in X(x_{0:t-1}) \end{array} \right.$$

for $t = 1, \ldots, T$, with $\mathcal{Q}_{T+1} \equiv 0$.

The application of Remark 3.2 to (35) yields the following DP equations: for $t = 1, \ldots, T$, $\mathcal{Q}_t(x_{0:t-1}, \xi_{[t-1]}, y_{1:t-1})$ is given by

(36)
$$\mathcal{Q}_t(x_{0:t-1}, \xi_{[t-1]}, y_{1:t-1}) = \mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf_{x_t, y_t} \; c_t^\top y_t + \mathcal{Q}_{t+1}(x_{0:t}, \xi_{[t]}, y_{1:t}) \\ A_t y_t \leq a_t \\ (1-\delta_{t1}) \left(\sum_{\tau=0}^{t-1} \; B_{t,\tau} y_{t-\tau} - h_t(-\sum_{\tau=1}^{t} f_\tau(x_\tau, \xi_\tau)) \right) = 0 \\ x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right)$$

where here, and in what follows, $\mathcal{Q}_{T+1} \equiv 0$. Since these DP equations correspond to the stagewise decomposition of risk averse problem (35), the recourse functions $\mathcal{Q}_t$ in (36) are said to be *risk averse.*

Let us now take as a special case for ρ the multiperiod risk measure defined by

$$\rho(z_1, \ldots, z_T) = -\theta_1 \mathbb{E}[z_T] + \sum_{t=2}^{T} \theta_t \rho^t(z_t) \tag{37}$$

for some non-negative weights $\theta_t, t = 1, \ldots, T$, summing to one ($\sum_{t=1}^{T} \theta_t = 1$) and for some one-period coherent extended polyhedral risk measures $\rho^t : \mathcal{Z}_t \to \mathbb{R},\; t = 2, \ldots, T$.

Remark 3.3. *We easily check that* ρ *in* (37) *is a multiperiod (coherent) extended polyhedral risk measure.*

Observe that since ρ^t is coherent and z_1 deterministic, we have $\rho^t(z_t - z_1) = \rho^t(z_t) + z_1$ and $\rho(z_1, \ldots, z_T)$ in (37) can be expressed as $\rho(z_1, \ldots, z_T) = -z_1 - \theta_1 \mathbb{E}[z_T - z_1] + \sum_{t=2}^{T} \theta_t \rho^t(z_t - z_1)$. This expression reveals that the corresponding objective function in (34) is the sum of the first stage (deterministic) cost and of a convex combination of the mean future cost and of risk measures of future partial costs. With this choice of ρ, problem (34) becomes

$$\begin{array}{l} \inf f_1(x_1, \xi_1) + \theta_1 \mathbb{E}[\sum_{t=2}^{T} f_t(x_t, \xi_t)] + \sum_{t=2}^{T} \theta_t \rho^t(-\sum_{k=2}^{t} f_k(x_k, \xi_k)) \\ x_t \in \chi_t(x_{t-1}, \xi_t), \; t = 1, \ldots, T. \end{array} \tag{38}$$

Plugging the expression (16) of the risk measure ρ^t (taking the same for all time steps) into (38), the latter can be written

$$\begin{array}{l} \inf\limits_{x_t, w_t, y_t} f_1(x_1, \xi_1) + \sum_{t=2}^{T} \theta_t c_1^\top w_t + \mathbb{E}[\theta_1 \sum_{t=2}^{T} f_t(x_t, \xi_t) + \sum_{t=2}^{T} \theta_t c_2^\top y_t] \\ B_{2,1} w_t + B_{2,0} y_t = h(-\sum_{k=2}^{t} f_k(x_k, \xi_k)), \; t = 2, \ldots, T, \\ A_1 w_t \leq a_1, \;\; A_2 y_t \leq a_2, \; t = 2, \ldots, T, \\ x_t \in \chi_t(x_{t-1}, \xi_t), \; t = 1, \ldots, T. \end{array}$$

In turn, the above optimization problem can be expressed as

$$\begin{array}{l} \inf\limits_{x_1, w_{2:T}} f_1(x_1, \xi_1) + \sum_{t=2}^{T} \theta_t c_1^\top w_t + \mathcal{Q}_2(x_1, \xi_{[1]}, w_2, \ldots, w_T) \\ A_1 w_t \leq a_1, \; t = 2, \ldots, T, \; x_1 \in \chi_1(x_0, \xi_1), \end{array} \tag{39}$$

where

$$\mathcal{Q}_2(x_1, \xi_{[1]}, w_{2:T}) = \left\{ \begin{array}{l} \inf\limits_{x_t, y_t} \mathbb{E}[\theta_1 \sum_{t=2}^{T} f_t(x_t, \xi_t) + \sum_{t=2}^{T} \theta_t c_2^\top y_t] \\ B_{2,1} w_t + B_{2,0} y_t = h(-\sum_{k=2}^{t} f_k(x_k, \xi_k)), \; A_2 y_t \leq a_2, \, t = 2, \ldots, T, \\ x_t \in \chi_t(x_{t-1}, \xi_t), \; t = 2, \ldots, T. \end{array} \right. \tag{40}$$

The application of Remark 3.2 to optimization problem (40) yields the following DP equations: for $t = 2, \ldots, T$, $\mathcal{Q}_t(x_{1:t-1}, \xi_{[t-1]}, w_{t:T})$ is given by

$$\mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf\limits_{x_t, y_t} \; \theta_1 f_t(x_t, \xi_t) + \theta_t c_2^\top y_t + \mathcal{Q}_{t+1}(x_{1:t}, \xi_{[t]}, w_{t+1:T}) \\ B_{2,1} w_t + B_{2,0} y_t = h(-\sum_{k=2}^t f_k(x_k, \xi_k)), \; A_2 y_t \le a_2, \; x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right). \tag{41}$$

In DP equations (36) and (41) obtained for respectively risk averse problems (35) and (38), the state variables memorize the relevant history of the process and of the decisions. For (36) (resp. (41)), we can reduce the size of the state vector replacing the history of the decisions $x_{1:t-1}$ by x_{t-1} and z_{t-1} (resp. x_{t-1} and $\tilde{z}_{t-1}$ with $\tilde{z}_{t-1} = z_{t-1} - z_1$). Variable $\tilde{z}_{t-1}$ represents the total revenue (opposite of the cost) from time step 2 until time step $t-1$ (i.e., the total income until time step $t-1$ for the time steps where the data are random). Variables $\tilde{z}_t$ satisfy $\tilde{z}_t = \tilde{z}_{t-1} - f_t(x_t, \xi_t)$ for $t = 2, \ldots, T$, with $\tilde{z}_1$ set equal to 0. With this notation, DP equations (36) for problem (35) become

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, z_{t-1}, y_{1:t-1}) = \mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf_{x_t, y_t, z_t} \; c_t^\top y_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}, z_t, y_{1:t}) \\ (1 - \delta_{t1}) \left(\sum_{\tau=0}^{t-1} B_{t,\tau} y_{t-\tau} - h_t(z_t) \right) = 0, \; A_t y_t \le a_t \\ z_t = z_{t-1} - f_t(x_t, \xi_t), \; x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right) \tag{42}$$

for $t = 1, \ldots, T$, with $z_0 = 0$. As for the dynamic programming equations (39) and (41), they simplify as follows: in (39), $\mathcal{Q}_2(x_1, \xi_{[1]}, w_2, \ldots, w_T)$ needs to be replaced by $\mathcal{Q}_2(x_1, \xi_{[1]}, \tilde{z}_1, w_2, \ldots, w_T)$ and for $t = 2, \ldots, T$, we have

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, \tilde{z}_{t-1}, w_{t:T}) = \mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf\limits_{x_t, \tilde{z}_t, y_t} \; -\delta_{tT} \theta_1 \tilde{z}_t + \theta_t c_2^\top y_t + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}, \tilde{z}_t, w_{t+1:T}) \\ B_{2,1} w_t + B_{2,0} y_t = h(\tilde{z}_t), \; A_2 y_t \le a_2 \\ \tilde{z}_t = \tilde{z}_{t-1} - f_t(x_t, \xi_t), \; x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right). \tag{43}$$

Remark 3.4. *Comparing the non-risk averse dynamic programming equations* (3) *with the risk averse ones* (42) *or* (39) *and* (43)*, we see that additional decision and state variables are introduced in the latter cases. More precisely, at the first time step, in the non-risk averse case the decision* x_1 *is taken while in risk averse case* (42) *(resp.* (39) *and* (43)*), additional decision variables* y_1 *and* z_1 *(resp.* $(w_2, \ldots, w_T)$*) are needed. This first stage problem is deterministic for all models.*

For time step $t = 2, \ldots, T$*, in risk averse case* (42) *(resp.* (39) *and* (43)*), the state vector is augmented with partial cost* z_{t-1} *and with the variables* $(y_1, \ldots, y_{t-1})$ *(resp. partial cost* $\tilde{z}_{t-1}$ *and the variables* $(w_t, \ldots, w_T)$*). For both risk averse models, additional decisions* z_t *(or* $\tilde{z}_t$*) and* y_t *are needed for stages* $t = 2, \ldots, T$*. This is summarized in the table below:*

		First stage	***Stages*** $\mathbf{t = 2, \ldots, T}$
Decision variables	NRA	x_1	x_t
	RA_1	(x_1, z_1, y_1)	(x_t, z_t, y_t)
	RA_2	$(x_1, w_2, \ldots, w_T)$	$(x_t, \tilde{z}_t, y_t)$
State variables	NRA	$(x_0, \xi_{[0]})$	$(x_{t-1}, \xi_{[t-1]})$
	RA_1	$(x_0, \xi_{[0]})$	$(x_{t-1}, \xi_{[t-1]}, z_{t-1}, y_1, \ldots, y_{t-1})$
	RA_2	$(x_0, \xi_{[0]})$	$(x_{t-1}, \xi_{[t-1]}, \tilde{z}_{t-1}, w_t, \ldots, w_T)$

TABLE 1. Decision and state variables for the non-risk averse (NRA) DP equations (3) as well as for the risk averse ones (42) (RA_1), and (39) and (43) (RA_2).

Remark 3.5. *Other special cases for the multiperiod risk measure* ρ *in* (34) *for which DP equations can be written are the risk measures from Example 2.18.*

Properties (P1) and (P3) thus hold for (35) and hold for (38) when using extended one-period polyhedral risk measures for ρ^t. We now concentrate on (P2). So far, all the developments of this section were valid for a problem of the form (1). To ensure that (P2) holds, we consider the special case when (1) is a stochastic linear program (SLP). Indeed, the convergence of SDDP algorithm and of related sampling-based algorithms is proved in Philpott and Guan [PG08] for SLP. We observe that if (1) is a SLP, risk averse problem (35) (resp. (38)) is a SLP if and only if

$$h_t(z) = zb_t + \tilde{b}_t, \text{ for some } b_t, \tilde{b}_t \in \mathbb{R}^{n_{t,2}} \text{ (resp. } h(z) = zb_2 + \tilde{b}_2, \text{ for some } b_2, \tilde{b}_2 \in \mathbb{R}^{n_{2,2}}). \tag{44}$$

Of interest for applications, we now specialize the above DP equations (43) taking extended polyhedral risk measures with $h(\cdot)$ of the kind (44) above. As seen in the previous section, spectral risk measures with piecewise constant spectra are of this kind. We provide the DP equations obtained in this case using directly (27).

3.3. **Spectral risk measures.** Let ϕ be a piecewise risk spectrum satisfying (i), (ii), and (iii) given in Example 2.16 and let $\Delta\phi_k = \phi(p_k) - \phi(p_{k-1}), k = 1, \ldots, J$. If we take for ρ^t a spectral risk measure ρ_ϕ (the same for all time steps), using (27) we can decompose (38) by stages and express it under the form

$$\begin{array}{l} \inf \ f_1(x_1, \xi_1) + \displaystyle\sum_{t=2}^{T} \theta_t c_1^\top w_t + \mathcal{Q}_2(x_1, \xi_{[1]}, \tilde{z}_1, w_2, \ldots, w_T), \\ x_1 \in \chi_1(x_0, \xi_1), \ w_t \in \mathbb{R}^J, \ t = 2, \ldots, T, \end{array} \tag{45}$$

with $\tilde{z}_1 = 0, c_1 = \Delta\phi \circ p$, and where for $t = 2, \ldots, T$,

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, \tilde{z}_{t-1}, w_{t:T}) = \mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf\limits_{x_t, \tilde{z}_t} \ \tilde{f}_t(\tilde{z}_t, w_t) + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}, \tilde{z}_t, w_{t+1:T}) \\ \tilde{z}_t = \tilde{z}_{t-1} - f_t(x_t, \xi_t), \ x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right) \tag{46}$$

with

$$\tilde{f}_t(\tilde{z}_t, w_t) = -(\delta_{tT}\theta_1 + \phi(1)\theta_t)\tilde{z}_t - \theta_t \, \Delta\phi^\top (w_t - \tilde{z}_t e)^+.$$

When the risk spectrum ϕ has one jump, we obtain the CVaR.

3.4. **Conditional Value-at-Risk.** When taking $\rho^t = \text{CVaR}^{\varepsilon_t}$ and using (28), we can express (38) under the form

$$\begin{array}{l} \inf\limits_{x_1, w_{2:T}} f_1(x_1, \xi_1) - \displaystyle\sum_{t=2}^{T} \theta_t w_t + \mathcal{Q}_2(x_1, \xi_{[1]}, \tilde{z}_1, w_2, \ldots, w_T) \\ x_1 \in \chi_1(x_0, \xi_1), \ w_t \in \mathbb{R}, \ t = 2, \ldots, T, \end{array} \tag{47}$$

with $\tilde{z}_1 = 0$ and where for $t = 2, \ldots, T$,

$$\mathcal{Q}_t(x_{t-1}, \xi_{[t-1]}, \tilde{z}_{t-1}, w_{t:T}) = \mathbb{E}_{\xi_t|\xi_{[t-1]}} \left(\begin{array}{l} \inf\limits_{x_t, \tilde{z}_t} \ -\delta_{tT}\theta_1 \tilde{z}_t + \dfrac{\theta_t}{\varepsilon_t}(w_t - \tilde{z}_t)^+ + \mathcal{Q}_{t+1}(x_t, \xi_{[t]}, \tilde{z}_t, w_{t+1:T}) \\ \tilde{z}_t = \tilde{z}_{t-1} - f_t(x_t, \xi_t), \ x_t \in \chi_t(x_{t-1}, \xi_t) \end{array} \right). \tag{48}$$

3.5. **Convergence of SDDP in a risk averse setting.** The convergence of SDDP algorithm and of related sampling-based algorithms is proved in Philpott and Guan [PG08] for SLP with the following properties:

- (A1) Random data only appear in the right-hand side of the constraints.
- (A2) The supports of the distributions of the underlying random vectors are discrete and finite.
- (A3) Random vectors are interstage independent or satisfy a certain type of interstage dependence (see Philpott and Guan [PG08]).
- (A4) The feasible set of the linear program is nonempty and bounded in each stage.

In what follows, we consider multistage stochastic linear programs of the form (1) where

$$f_t(x_t, \xi_t) = d_t^\top x_t \text{ and } \chi_t(x_{t-1}, \xi_t) = \{x_t : x_t \geq 0,\ C_t x_t = \xi_t - D_t x_{t-1}\}. \quad (49)$$

For these programs, Assumption (A1) holds and it can be noted that if (A1) holds for (1) then (A1) holds for risk averse problems (35) and (38). In the remainder of the paper, we assume (A2) and (A3). We also assume that (A4) holds for (1), which, in our context, can be expressed as follows:

(A4) For $t = 1, \ldots, T$, for any feasible state x_{t-1} and for any realization ξ_t^i of ξ_t, the set

$$\chi_t(x_{t-1}, \xi_t^i) = \{x_t \mid x_t \geq 0,\ C_t x_t = \xi_t^i - D_t x_{t-1}\}$$

is bounded and nonempty.

To apply the convergence results from Philpott and Guan [PG08] in our risk averse setting, (A4) should also hold for risk averse problems (35) or (38). For (35), (A4) takes the following form:

(A5) $\{y_1 : A_1 y_1 \leq a_1\}$ is bounded and for all $t = 2, \ldots, T$, for any feasible states $x_1, y_1, \ldots, x_{t-1}, y_{t-1}$, for any sequence of realizations $\xi_1^i, \ldots, \xi_t^i$ of $\xi_1, \ldots, \xi_t$, the set $\{y_t : A_t y_t \leq a_t, B_{t,0} y_t = h_t(-\sum_{\tau=1}^{t} f_\tau(x_\tau, \xi_\tau^i)) - \sum_{\tau=1}^{t-1} B_{t,\tau} y_{t-\tau}$ for some $x_t \in \chi_t(x_{t-1}, \xi_t^i)\}$ is bounded and nonempty.

For (38), remembering Proposition 2.15, a condition implying (A4) is the following:

(A6) For $t = 2, \ldots, T$, the sets $S(\rho^t(0))$ are nonempty and bounded where $S(\rho^t(0))$ is defined in (26). $\{y_1 : A_1 y_1 \leq a_1\}$ is bounded and for all $t = 2, \ldots, T$, for any feasible $x_1, y_1, \ldots, x_{t-1}, y_{t-1}, w_{2:T}$, for any sequence of realizations $\xi_1^i, \ldots, \xi_t^i$ of $\xi_1, \ldots, \xi_t$, the set $\{y_t : A_t y_t \leq a_t, \exists\ x_t \in \chi_t(x_{t-1}, \xi_t^i), B_{2,0} y_t = h(-\sum_{\tau=2}^{t} f_\tau(x_\tau, \xi_\tau^i)) - B_{2,1} w_t\}$ is bounded and nonempty.

Indeed, with respect to the non-risk averse setting, recall that the additional decision variables for (38) are $\tilde{z}_t$ (bounded, due to (A4)), y_t, and w_t. Variables $w_t, t = 2, \ldots, T$ are first stage decision variables and due to Proposition 2.15, if $S(\rho^t(0))$ is nonempty and bounded then optimal w_t are bounded. Next, condition (A6) guarantees the boundedness of optimal y_t.

However, even if the feasible set at each stage for (35) or (38) is not bounded, we may be able to show, in some cases, that these feasible sets can be replaced by bounded feasible sets without changing the problems, i.e., that the solutions are bounded. Such is the case for problems (45) and (47). Indeed, for these problems, the only additional variables with respect to the non-risk averse case are $\tilde{z}_t$ (bounded, due to (A4)) and first stage variables $w_2, \ldots, w_T$. For the spectral risk measure $\rho^t = \rho_\phi, t = 2, \ldots, T$, considered in (45), the sets $S(\rho^t(0)) = S(\rho_\phi(0)) = \{0\}, t = 2, \ldots, T$, are nonempty and bounded. Using Proposition 2.15, optimal values of w_t in (45) are bounded. This result can also be easily proved directly:

Lemma 3.6. *Let assumption (A4) hold and let ϕ be a piecewise risk spectrum satisfying (i), (ii), and (iii) given in Example 2.16. Let $w_2^*, \ldots, w_T^*$ be optimal values of $w_2, \ldots, w_T$ for* (45). *Then $w_t^*(k)$ is finite for every $t = 2, \ldots, T$, and $k = 1, \ldots, J$.*

Proof. Since $\chi_t, t = 1, \ldots, T$, are bounded and $\Delta\phi < 0$, we can bound from below the objective function of (45) by $L_1(w) = K_1 + \sum_{t=2}^{T} \theta_t(\Delta\phi \circ p)^\top w_t$ and $L_2(w) = K_2 + \sum_{t=2}^{T} \theta_t(\Delta\phi \circ (p-e))^\top w_t$ for some constants K_1 and K_2. Since $\Delta\phi \circ p < 0$, if one component $w_t(k) = -\infty$ then $L_1(w) = +\infty$, the objective function is therefore $+\infty$, and such $w_t(k)$ cannot be an optimal value of $w_t(k)$. Similarly, since $\Delta\phi \circ (p - e) > 0$, if one $w_t(k) = +\infty$ then $L_2(w) = +\infty$, the objective function is $+\infty$ and such $w_t(k)$ cannot be an optimal value of $w_t(k)$. □

The following corollary is an immediate consequence of this lemma:

Corollary 3.7. *Let assumption (A4) hold. Let $w_2^*, \ldots, w_T^*$ be optimal values of $w_2, \ldots, w_T$ for* (47). *Then w_t^* is finite for every $t = 2, \ldots, T$.*

It follows that we can add (sufficiently large) box constraints on w_t in (45) and (47) without changing the optimal solutions of (45) and (47). Gathering our observations, we come to the following proposition:

Proposition 3.8. *[Convergence of SDDP in a risk averse setting] Consider multistage stochastic linear programs of the form* (1) *with* f_t *and* χ_t *given by* (49). *Assume that for such multistage programs, Assumptions (A1), (A2), (A3), and (A4) hold. Consider the risk averse formulations* (45), (46) *and* (47), (48). *Then an SDDP algorithm applied on these DP equations will converge if the sampling procedures satisfy the FPSP and BPSP assumptions (see Philpott and Guan* [PG08]*).*

The same convergence result holds for the following two risk averse formulations:

(1) assuming (A5), for risk averse program (35) *decomposed by stages as* (42) *with* $h_t(\cdot)$ *given by* (44)*;*
(2) assuming (A6), for risk averse program (38) *decomposed by stages as* (39), (43) *with* $h(\cdot)$ *given by* (44).

In the next two sections, we focus on solution methods for interstage independent SLP. The developments can however be easily adapted to the case when the process affinely depends on previous values. Our notation follows closely Birge and Donohue [BD01] and Philpott and Guan [PG08].

4. Decomposition algorithms for a class of non-risk averse stochastic programs

Since the supports of the distributions of the random vectors $\xi_2, \ldots, \xi_T$ are discrete and finite, optimization problem (34) is finite dimensional and the evolution of the uncertain parameters over the optimization period can be represented by a scenario tree having a finite number of scenarios that can happen in the future for $\xi_2, \ldots, \xi_T$. The root node of the scenario tree corresponds to the first time step with ξ_1, x_0 and $\chi_1(x_0, \xi_1)$ deterministic.

For a given stage t, each node of the scenario tree corresponds to a possible realization of ξ_t and the set of nodes is denoted by Ω_t. The children nodes of a node at stage $t \geq 1$ are the nodes that can happen at stage $t+1$ if we are at this node at t. A sampled scenario $(\xi_1, \ldots, \xi_T)$ corresponds to a particular succession of nodes such that ξ_t is a possible value for the process at t and ξ_{t+1} is a child of ξ_t. A given node in the tree at stage t is identified with a scenario $(\xi_1, \ldots, \xi_t)$ going from the root node to this node. In this situation, the sigma algebra $\mathcal{F}_T$ is the set of all subsets of Ω_T. More generally, $\mathcal{F}_t$ is the set of all subsets of the set whose ith atom is the set of scenarios that pass through a given node $i \in \Omega_t$.

We consider multistage stochastic linear programs with f_t and χ_t of the form (49) with an interstage independent process ξ_t. In this situation, for $t = 1, \ldots, T$, the recourse function $\mathcal{Q}_t(x_{t-1})$ is given by $\mathcal{Q}_t(x_{t-1}) = \mathbb{E}_{\xi_t}[Q_t(x_{t-1}, \xi_t)]$ with

$$(50) \qquad [LP_t] \quad Q_t(x_{t-1}, \xi_t) = \left\{ \begin{array}{l} \inf\limits_{x_t} d_t^\top x_t + \mathcal{Q}_{t+1}(x_t) \\ C_t x_t = \xi_t - D_t x_{t-1} \ , \ x_t \geq 0, \end{array} \right.$$

and $\mathcal{Q}_{T+1} \equiv 0$. Following the notation in Philpott and Guan [PG08], though problem $[LP_t]$ depends on the choice of x_{t-1} and ξ_t, we write $[LP_t]$ instead of $[LP_t(x_{t-1}, \xi_t)]$ to alleviate notation.

We assume *relatively complete recourse* for (50), which means that for any feasible sequence of decisions $(x_1, \ldots, x_t)$ to any t-stage scenario $(\xi_1, \xi_2, \ldots, \xi_t)$, there exists a sequence of feasible decisions $(x_{t+1}, \ldots, x_T)$ with probability one.

We now detail the computations of optimality cuts as well as the computation of a confidence interval on an upper bound on the expected value of a given first stage solution.

4.1. **SDDP: backward pass.** The cuts are computed for time step $T+1$ down to time step 2. For time step $T+1$, since $\mathcal{Q}_{T+1}^i = \mathcal{Q}_{T+1} = 0$, we have for $\mathcal{Q}_{T+1}$ the cuts $E_T^k = e_T^k = 0$ for $k = (i-1)H, \ldots, iH$. At time step $t = 2, \ldots, T$, the cuts for $\mathcal{Q}_t$ are computed having at hand the approximation $\mathcal{Q}_{t+1}^i$ of $\mathcal{Q}_{t+1}$ which satisfies

$$\mathcal{Q}_{t+1}(x_t) \geq \mathcal{Q}_{t+1}^i(x_t). \tag{51}$$

Plugging (51) into (50), we obtain $\mathcal{Q}_t(x_{t-1}) \geq \mathbb{E}_{\xi_t}\left[Q_t^i(x_{t-1}, \xi_t)\right]$ with

$$Q_t^i(x_{t-1}, \xi_t) = \left\{ \begin{array}{l} \inf\limits_{x_t} d_t^\top x_t + \theta_t \\ C_t x_t = \xi_t - D_t x_{t-1} \\ \overrightarrow{E}_t^i x_t + e\theta_t \geq \overrightarrow{e}_t^i, \; x_t \geq 0. \end{array} \right. \tag{52}$$

On scenario k and time step t, the above problem is solved for $(x_{t-1}, \xi_t) = (x_{t-1}^k, \xi_t^j), \; j = 1, \ldots, q_t$, where $\xi_t^j, \; j = 1, \ldots, q_t$ are all possible realizations of ξ_t at time step t. Since relatively complete recourse and (A4) hold, these linear programs have nonempty feasible sets; their optimal values are finite and both the primal and the dual have the same optimal value. We denote by $\pi_t^{k,j}, \rho_t^{k,j}$ the (row vectors) optimal Lagrange multipliers associated to respectively the equality and cut constraints for problem $\mathcal{Q}_t^i(x_{t-1}^k, \xi_t^j)$. In what follows we also set $p(t,j) = \mathbb{P}(\xi_t = \xi_t^j)$. The following theorem provides the cuts computed for $\mathcal{Q}_t$ at iteration i:

Theorem 4.1. *In the backward pass of iteration i, H valid cuts for $\mathcal{Q}_t, t = 2, \ldots, T$, are given by*

$$\begin{array}{lcl} E_{t-1}^k & = & \sum_{j=1}^{q_t} p(t,j) \pi_t^{k,j} D_t \\ e_{t-1}^k & = & \sum_{j=1}^{q_t} p(t,j)(\pi_t^{k,j} \xi_t^j + \rho_t^{k,j} \overrightarrow{e}_t^i), \end{array}$$

and $E_T^k = e_T^k = 0$ for $k = (i-1)H + 1, \ldots, iH$.

Proof. By duality, $Q_t^i(x_{t-1}, \xi_t^j)$ may be expressed as the optimal value of the following linear program:

$$\begin{array}{l} \sup\limits_{\pi,\rho} \pi \left(\xi_t^j - D_t x_{t-1} \right) + \rho \overrightarrow{e}_t^i \\ \pi C_t + \rho \overrightarrow{E}_t^i \leq d_t^\top \;, \; \rho e = 1 \;, \; \rho \geq 0. \end{array}$$

For the above problem, the optimal solutions are extremal points of the feasible set. Next, the feasible set does not depend on x_{t-1}. This means that for any x_{t-1}, the row vectors $\pi_t^{k,j}$ and $\rho_t^{k,j}$ are extremal points of the feasible set of the problem $Q_t^i(x_{t-1}, \xi_t^j)$. It follows that

$$Q_t^i(x_{t-1}, \xi_t^j) \geq \pi_t^{k,j}(\xi_t^j - D_t x_{t-1}) + \rho_t^{k,j} \overrightarrow{e}_t^i,$$

for $j = 1, \ldots, q_t$. Using these inequalities and since $\mathcal{Q}_t(x_{t-1})$ is bounded from below by the term $\mathbb{E}_{\xi_t}[Q_t^i(x_{t-1}, \xi_t)] = \sum_{j=1}^{q_t} p(t,j) Q_t^i(x_{t-1}, \xi_t^j)$, we obtain a cut of the form $\theta_{t-1}^k + E_{t-1}^k x_{t-1} \geq e_{t-1}^k$ and the result follows. □

Remark 4.2. *Using the convexity of Q_t, we can also express e_{t-1}^k as*

$$e_{t-1}^k = \sum_{j=1}^{q_t} p(t,j) \left[Q_t^i(x_{t-1}^k, \xi_t^j) + \pi_t^{k,j} D_t x_{t-1}^k \right]. \tag{53}$$

4.2. **SDDP: stopping rule and algorithm.** In the backward pass, for the first time step, the first stage problem is solved using the recourse function $\mathcal{Q}_2^i \leq \mathcal{Q}_2$. Since (A4) holds, the optimal value of this problem is finite and provides a lower bound z_{inf} for the optimal mean cost.

In the end of the forward pass, we can compute the total cost $-z_k$ on scenario k for the policy $(x_1, x_2^k, \ldots, x_T^k)$:

$$-z_k = d_1^\top x_1 + \sum_{t=2}^{T} d_t^\top x_t^k,$$

which is an upper bound on the optimal cost on this scenario. If the H scenarios were representing all possible evolutions of $(\xi_1, \ldots, \xi_T)$, then

$$\bar{z} = -\frac{1}{H} \sum_{k=(i-1)H+1}^{iH} z_k$$

would be an upper bound on the optimal mean cost. Since we only have a sample of all the possible scenarios, $\bar{z}$ is an estimation of an upper bound on the optimal mean cost. The standard deviation of the estimator associated to this estimation is measured by

$$\sigma_{\bar{z}} = \frac{1}{H} \sqrt{\sum_{k=(i-1)H+1}^{iH} (\bar{z} - z_k)^2}.$$

A $100(1-\alpha)\%$ confidence interval (with $0 < \alpha < 1$) for an upper bound on the optimal mean cost is then given by

$$[\bar{z} - t_{H-1}(\alpha)\sigma_{\bar{z}}, \bar{z} + t_{H-1}(\alpha)\sigma_{\bar{z}}] \tag{54}$$

where $t_{H-1}(\alpha)$ is the $(1-\frac{\alpha}{2})$-quantile of the Student density with $H-1$ degrees of freedom. The algorithm stops when the lower bound z_{inf} belongs to the confidence interval (54) (stopping rule from Pereira and Pinto [PP91]). Using the previous developments, SDDP algorithm for solving (50) can be formulated as in Figure 1 which follows. We now have all the ingredients to detail the application of SDDP for approximating the risk averse recourse functions from Section 3 for SLP.

5. SDDP for some risk averse stochastic programs

We consider the risk averse recourse functions from Section 3 in the case when f_t and χ_t are given by (49) and $h_t(\cdot)$ and $h(\cdot)$ in respectively (42) and (43) are given by (44). Recall that risk averse DP equations (42), or (39),(43) that define these recourse functions satisfy (P3) (like the non-risk averse DP equations (3) but with additional state and control variables). We assume that relatively complete recourse holds for (1) and that the assumptions of Proposition 3.8 hold. In this context, relatively complete recourse also holds for risk averse problems (42) or (39),(43). The SDDP algorithm presented in the previous section can thus be easily adapted to the risk averse DP equations to obtain approximations of the corresponding risk averse recourse functions. These adaptations and some specific comments are given below. In particular, we show that when ρ^t in (38) is a spectral risk measure, we obtain closed-form expressions for some cut coefficients under some assumptions.

5.1. **Extended polyhedral risk measures.** With respect to the previous section, at iteration i of SDDP, the forward pass additionally provides decisions $y_1^k, \ldots, y_T^k$ as well as the partial costs $z_1^k, \ldots, z_T^k$ on scenario $k = (i-1)H+1, \ldots, iH$. In the backward pass of iteration i, H cuts are

Step 0: **INITIALISATION.** Set $i=1$ (iteration number), $\bar{z}=\infty$, $z_{\inf}=-\infty$, $E_t^0=0$ and e_t^0 is a lower bound on $\mathcal{Q}_t$ for $t=2,\ldots,T+1$. Go to Step 1.

Step 1: **FORWARD PASS.**
Sample H scenarios $(\xi_1^k,\ldots,\xi_T^k), k=(i-1)H+1,\ldots,iH$.
Ct=0, Ct_Sq=0.
For $k=(i-1)H+1,\ldots,iH$,
  For $t=1,\ldots,T$,
    Solve problem $[AP_t^{i,k}]$ and store an optimal solution x_t^k of this problem.
  End For
  Ct=Ct+ $\sum_{t=1}^T d_t^\top x_t^k$, Ct_Sq=Ct_Sq+ $\sum_{t=1}^T (d_t^\top x_t^k)^2$.
End For
$\bar{z}=\frac{\text{Ct}}{H}$, $\sigma_{\bar{z}}=\frac{1}{H}\sqrt{\text{Ct\_Sq}-H\bar{z}^2}$. Go to Step 2.

Step 2: **BACKWARD PASS.**
For $t=T+1$ down to 1,
  For $k=(i-1)H+1,\ldots,iH$,
    If $(t=T+1)$ then set E_t^0 and e_t^0 to 0.
    Else if $(t\geq 2)$
      For $j=1,\ldots,q_t$,
        Compute $\mathcal{Q}_t^i(x_{t-1}^k,\xi_t^j)$ given by (52) and let $\pi_t^{k,j},\rho_t^{k,j}$ be optimal dual multipliers.
      End For
      Build a cut for $\mathcal{Q}_t$ of the form $\theta_{t-1}^k+E_{t-1}^k x_{t-1}\geq e_{t-1}^k$ with E_{t-1}^k and e_{t-1}^k given in Theorem 4.1.
    Else
      Set $z_{\inf}$ to the optimal value of the first stage problem.
    End If
  End For
End For
Go to Step 3.

Step 3: **STOPPING RULE.**
If $|z_{\inf}-\bar{z}|\leq t_{H-1}(\alpha)\sigma_{\bar{z}}$ then stop.
Else $i\leftarrow i+1$ and go to Step 1. **End If**

Figure 1. SDDP algorithm with relatively complete recourse for an interstage independent SLP.

computed for $\mathcal{Q}_t$ at $(x_{t-1}^k,z_{t-1}^k,y_1^k,\ldots,y_{t-1}^k), k=(i-1)H+1,\ldots,iH$. Using the notation previously introduced, the lower bounding approximations $\mathcal{Q}_t^i$ of $\mathcal{Q}_t$ have the form

$$\mathcal{Q}_t^i(x_{t-1},z_{t-1},y_{1:t-1})=\max_{j=0,1,\ldots,iH}[-E_{t-1}^j x_{t-1}-Z_{t-1}^j z_{t-1}-\sum_{\tau=1}^{t-1}Y_{t-1}^{j,\tau}y_\tau+e_{t-1}^j]$$

with $Z_{t-1}^j\in\mathbb{R}$ and for some row vectors $Y_{t-1}^{j,\tau}$ of appropriate dimensions. Following the developments of the previous section, for $t=2,\ldots,T$, we can bound from below $\mathcal{Q}_t(x_{t-1},z_{t-1},y_{1:t-1})$ by $\mathbb{E}_{\xi_t}[Q_t^i(x_{t-1},z_{t-1},y_{1:t-1},\xi_t)]$ with $Q_t^i(x_{t-1},z_{t-1},y_{1:t-1},\xi_t)$ given as the optimal value of the following linear program:

$$
(55)\quad \begin{array}{ll}
\inf_{x_t, y_t, z_t, \tilde{\theta}_t} & c_t^\top y_t + \tilde{\theta}_t \\
A_t y_t \leq a_t, \ x_t \geq 0 & \\
\sum_{\tau=0}^{t-1} B_{t,\tau} y_{t-\tau} - z_t b_t = \tilde{b}_t & (a) \\
z_t + d_t^\top x_t = z_{t-1} & (b) \\
C_t x_t = \xi_t - D_t x_{t-1} & (c) \\
\overrightarrow{E}_t^i x_t + \overrightarrow{Z}_t^i z_t + e\tilde{\theta}_t \geq -\sum_{\tau=1}^{t} \overrightarrow{Y}_t^{i,\tau} y_\tau + \overrightarrow{e}_t^i & (d)
\end{array}
$$

where $\overrightarrow{Z}_t^i = (Z_t^0, Z_t^1, \ldots, Z_t^{iH})^\top$ and $\overrightarrow{Y}_t^{i,\tau}$ is the matrix whose $(j+1)$th line is $Y_t^{j,\tau}$ for $j = 0, \ldots, iH$. We denote by $\sigma_t^{k,j}, \mu_t^{k,j}, \pi_t^{k,j}$, and $\rho_t^{k,j}$ the (row vectors) optimal Lagrange multipliers associated to constraints (55)-(a), (55)-(b),(55)-(c), and (55)-(d) for the problem defining $Q_t^i(x_{t-1}^k, z_{t-1}^k, y_{1:t-1}^k, \xi_t^j)$. With this notation, the following theorem provides the cuts computed for $\mathcal{Q}_t$ at iteration i:

Theorem 5.1. *Let $\mathcal{Q}_t, t = 2, \ldots, T+1$, be the risk averse recourse functions given by* (42) *with $h_t(\cdot)$ given by* (44)*. In the backward pass of iteration i of the SDDP algorithm, the following cuts are computed for these recourse functions. For $t = T+1$, we set $E_{t-1}^k, Z_{t-1}^k, Y_{t-1}^{k,\tau}$ and e_{t-1}^k to 0 for $k = (i-1)H+1, \ldots, iH$ and $\tau = 1, \ldots, T$. For $t = 2, \ldots, T$, and $k = (i-1)H+1, \ldots, iH$, E_{t-1}^k is given by Theorem 4.1 and*

$$
Z_{t-1}^k = -\sum_{j=1}^{q_t} p(t,j)\mu_t^{k,j}, \quad Y_{t-1}^{k,\tau} = \sum_{j=1}^{q_t} p(t,j)(\sigma_t^{k,j} B_{t,t-\tau} + \rho_t^{k,j} \overrightarrow{Y}_t^{i,\tau}), \ \tau = 1, \ldots, t-1.
$$

Next, e_{t-1}^k is given by (53) *in Remark 4.2 where $Q_t^i(x_{t-1}^k, \xi_t^j)$ needs to be replaced by*

$$
Q_t^i(x_{t-1}^k, z_{t-1}^k, y_{1:t-1}^k, \xi_t^j) - \mu_t^{k,j} z_{t-1}^k + \sum_{\tau=1}^{t-1} (\sigma_t^{k,j} B_{t,t-\tau} + \rho_t^{k,j} \overrightarrow{Y}_t^{i,\tau}) y_\tau^k.
$$

Proof. The proof is similar to the proof of Theorem 4.1. □

When applying SDDP on DP equations (39), (43), with respect to the previous section, at iteration i of the SDDP algorithm, the forward pass additionally computes the first stage decisions $w_2^i, \ldots, w_T^i$ as well as the partial costs $\tilde{z}_1^k, \ldots, \tilde{z}_T^k$ on scenario $k = (i-1)H+1, \ldots, iH$. In the backward pass of iteration i, H cuts are computed for $\mathcal{Q}_t$ at $(x_{t-1}^k, \tilde{z}_{t-1}^k, w_t^i, \ldots, w_T^i), k = (i-1)H+1, \ldots, iH$. The lower bounding approximations $\mathcal{Q}_t^i$ of $\mathcal{Q}_t$ now have the form

$$
\mathcal{Q}_t^i(x_{t-1}, \tilde{z}_{t-1}, w_{t:T}) = \max_{j=0,1,\ldots,iH} [-E_{t-1}^j x_{t-1} - Z_{t-1}^j \tilde{z}_{t-1} + \sum_{\tau=1}^{T-t+1} W_{t-1}^{j,\tau} w_{t+\tau-1} + e_{t-1}^j]
$$

with $Z_{t-1}^j \in \mathbb{R}$ and where $W_{t-1}^{j,\tau}$ is a row vector of appropriate dimension. We can bound from below $\mathcal{Q}_t(x_{t-1}, \tilde{z}_{t-1}, w_{t:T})$ by $\mathbb{E}_{\xi_t}[Q_t^i(x_{t-1}, \tilde{z}_{t-1}, w_{t:T}, \xi_t)]$ with $Q_t^i(x_{t-1}, \tilde{z}_{t-1}, w_{t:T}, \xi_t)$ given as the optimal value of the following linear program:

$$
(56)\quad \begin{array}{ll}
\inf_{x_t, y_t, \tilde{z}_t, \tilde{\theta}_t} & -\delta_{tT}\theta_1 \tilde{z}_t + \theta_t c_2^\top y_t + \tilde{\theta}_t \\
A_2 y_t \leq a_2, \ x_t \geq 0 & \\
B_{2,1} w_t + B_{2,0} y_t = \tilde{z}_t b_2 + \tilde{b}_2 & (a) \\
\tilde{z}_t + d_t^\top x_t = \tilde{z}_{t-1} & (b) \\
C_t x_t = \xi_t - D_t x_{t-1} & (c) \\
\overrightarrow{E}_t^i x_t + \overrightarrow{Z}_t^i \tilde{z}_t + e\tilde{\theta}_t \geq \sum_{\tau=1}^{T-t} \overrightarrow{W}_t^{i,\tau} w_{t+\tau} + \overrightarrow{e}_t^i & (d)
\end{array}
$$

where $\overrightarrow{W}_t^{i,\tau}$ is the matrix whose $(j+1)$th line is $W_t^{j,\tau}$ for $j = 0, \ldots, iH$.

We denote by $\sigma_t^{k,j}, \mu_t^{k,j}, \pi_t^{k,j}$, and $\rho_t^{k,j}$ the (row vectors) optimal Lagrange multipliers associated to constraints (56)-(a), (56)-(b),(56)-(c), and (56)-(d) for the problem defining $Q_t^i(x_{t-1}^k, \tilde{z}_{t-1}^k, w_{t:T}^i, \xi_t^j)$. With this notation, the following theorem provides the cuts computed for $\mathcal{Q}_t$ at iteration i:

Theorem 5.2. *Let $\mathcal{Q}_t, t = 2, \ldots, T+1$, be the risk averse recourse functions given by* (43) *with $h(\cdot)$ given by* (44). *In the backward pass of iteration i of the SDDP algorithm, the following cuts are computed for these recourse functions. For $t = T+1$, we set E_{t-1}^k, Z_{t-1}^k, and e_{t-1}^k to 0 for $k = (i-1)H+1, \ldots, iH$. For $t = 2, \ldots, T$ and $k = (i-1)H+1, \ldots, iH$, E_{t-1}^k is given in Theorem 4.1 and*

$$(57) \qquad Z_{t-1}^k = -\sum_{j=1}^{q_t} p(t,j)\mu_t^{k,j}, \quad W_{t-1}^{k,1} = -\sum_{j=1}^{q_t} p(t,j)\sigma_t^{k,j} B_{2,1},$$

$$(58) \qquad W_{t-1}^{k,\tau} = \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\overrightarrow{W}_t^{i,\tau-1}, \ \tau = 2, \ldots, T-t+1.$$

Further, e_{t-1}^k is given by (53) *in Remark 4.2 where $Q_t^i(x_{t-1}^k, \xi_t^j)$ needs to be replaced by*

$$Q_t^i(x_{t-1}^k, \tilde{z}_{t-1}^k, w_{t:T}^i, \xi_t^j) - \mu_t^{k,j}\tilde{z}_{t-1}^k + \sigma_t^{k,j} B_{2,1} w_t^i - \sum_{\tau=1}^{T-t} \rho_t^{k,j}\overrightarrow{W}_t^{i,\tau} w_{t+\tau}^i.$$

Proof. The proof is similar to the proof of Theorem 4.1. □

Since (P3) holds, the stopping criterion discussed in Section 4 can still be used.

As a special case, we can consider for ρ the risk measures from Example 2.18.

5.2. **Spectral risk measures.** We consider the application of SDDP to the DP equations (45), (46) (when a spectral risk measure is used, a particular case of the previous section). In this particular case, we intend to give for some particular choices of the first stage variable w^1, the exact expressions (independent of the sampled scenarios) of Z_{t-1}^k and $W_{t-1}^{k,\tau}$ for every $t = 2, \ldots, T$, $k = 1, \ldots, H$, and $\tau = 1, \ldots, T-t+1$. Recall that though the first stage feasible set for (45) is not bounded, the optimal values of $w_{2:T}$ are bounded. For numerical reasons, in the course of SDDP, well-chosen box constraints on $w_t, t = 2, \ldots, T$ are added (at the first stage, and that do not modify the optimal value of (45)) without changing the cut calculations (since these latter are performed for stages $t = 2, \ldots, T$ where w_t are state variables).

Let us define for $t = 1, \ldots, T$, $x^t = (x_1, \ldots, x_t)$, $\xi^t = (\xi_1, \ldots, \xi_t)$, and let us introduce the set χ^t of admissible decisions up to time step t:

$$\chi^t = \{x^t : \exists \tilde{\xi}^t \text{ realization of } \xi^t \ : \ x_\tau \geq 0 \text{ and } C_\tau x_\tau = \tilde{\xi}_\tau - D_\tau x_{\tau-1}, \tau = 1, \ldots, t\}.$$

Since (A4) holds, the sets χ^t are compact and since $f^t(x^t) = \sum_{\tau=2}^t d_\tau^\top x_\tau$ is continuous, we can introduce the pairs $(C_t^u, C_t^\ell) \in \mathbb{R}^2$ defined by

$$C_t^u = \left\{ \begin{array}{l} \max \ f^t(x^t) \\ x^t \in \chi^t, \end{array} \right. \qquad C_t^\ell = \left\{ \begin{array}{l} \min \ f^t(x^t) \\ x^t \in \chi^t. \end{array} \right.$$

Following the developments of Section 4 and introducing slack variables, $\mathcal{Q}_t(x_{t-1}, \tilde{z}_{t-1}, w_{t:T})$ is bounded from below by $\mathbb{E}_{\xi_t}[Q_t^i(x_{t-1}, \tilde{z}_{t-1}, w_{t:T}, \xi_t)]$ with $Q_t^i(x_{t-1}, \tilde{z}_{t-1}, w_{t:T}, \xi_t)$ given as the optimal value of the following linear program:

$$(59)\quad \begin{array}{lr} \inf\limits_{x_t,\tilde z_t,v_t,\tilde\theta_t} \; -(\delta_{tT}\theta_1+\phi(1)\theta_t)\tilde z_t-\theta_t\Delta\phi^\top v_t+\tilde\theta_t & \\ v_t\ge 0,\; v_t\ge w_t-\tilde z_t e,\; x_t\ge 0, & \\ \tilde z_t+d_t^\top x_t=\tilde z_{t-1} & (a)\\ C_t x_t=\xi_t-D_t x_{t-1} & (b)\\ \overrightarrow{E}_t^{i} x_t+\overrightarrow{Z}_t^{i}\tilde z_t+e\tilde\theta_t\ge \sum_{\tau=1}^{T-t}\overrightarrow{W}_t^{i,\tau}w_{t+\tau}+\overrightarrow{e}_t^{\,i} & (c)\end{array}$$

with $W_t^{j,\tau}\in\mathbb{R}^{1\times J}$. Let $\sigma_t^{k,j},\tilde\sigma_t^{k,j},\mu_t^{k,j},\pi_t^{k,j}$, and $\rho_t^{k,j}$, be the (row vectors) optimal Lagrange multipliers respectively for the constraints $v_t\ge w_t^1-z_te$, $v_t\ge 0$, (59)-(a), (59)-(b), and (59)-(c) for the problem defining $Q_t^1(x_{t-1}^k,\tilde z_{t-1}^k,w_{t:T}^1,\xi_t^j)$ (of the form (59)).

The objective of the forward pass is to build states where cuts are computed in the backward pass. At the first iteration, instead of building these states using the approximate recourse functions $\mathcal{Q}_t^0$, we can choose arbitrary feasible states $x_t,\tilde z_t,w_t$ (which is a simple task since relatively complete recourse holds). With this variant of the first iteration $\mathcal{Q}_t^i(x_{t-1})=\max_{j=1,\dots,iH}[-E_{t-1}^j x_{t-1}+e_{t-1}^j]$. If we choose first stage variables $w_{2:T}^1$ such that (i) $w_t^1>-C_t^\ell e$ for $t=2,\dots,T$ (resp. such that (ii) $w_t^1<-C_t^u e$ for $t=2,\dots,T$) then $Z_{t-1}^{k,\tau}$ and $W_{t-1}^{k,\tau}$ for $k=1,\dots,H$, can be computed using Lemma 5.3-(i) (resp. Lemma 5.3-(ii)) which follows. For instance, if the costs are positive then item (i) is fulfilled with $w_t^1=0$ and item (ii) taking for each component of w_t^1 the opposite of a strict upper bound on the worst cost.

Lemma 5.3. *[Cut calculation at the first iteration] Let us consider the risk averse recourse functions $\mathcal{Q}_t$ given by* (46)*. Valid cuts for $\mathcal{Q}_t$ are given by Theorem 5.2. Moreover, in the following two cases, we obtain closed-form expressions for Z_{t-1}^k and $W_{t-1}^{k,\tau}$ (independent of the sampled scenarios):*

(i) If for $t=2,\dots,T$, $w_t^1>-C_t^\ell e$, then for $t=2,\dots,T$, $\mathcal{P}(t)$ holds where

$$\mathcal{P}(t):\left\{\begin{array}{l}\forall\, k=1,\dots,H, Z_{t-1}^k=\theta_1+\phi(0)\sum_{\ell=t}^T\theta_\ell,\\ \forall\, k=1,\dots,H, W_{t-1}^{k,\tau}=-\theta_{t+\tau-1}\Delta\phi^\top,\ \tau=1,\dots,T-t+1.\end{array}\right.$$

(ii) If for $t=2,\dots,T$, $w_t^1<-C_t^u e$, then for $t=2,\dots,T$, $\tilde{\mathcal{P}}(t)$ holds where

$$\tilde{\mathcal{P}}(t):\left\{\begin{array}{l}\forall\, k=1,\dots,H, Z_{t-1}^k=\theta_1+\phi(1)\sum_{\ell=t}^T\theta_\ell,\\ \forall\, k=1,\dots,H, W_{t-1}^{k,\tau}=0,\ \tau=1,\dots,T-t+1.\end{array}\right.$$

Proof. Let us fix $t\in\{2,\dots,T\}$, $k\in\{1,\dots,H\}$, and $j\in\{1,\dots,q_t\}$. We denote by $x_t,\tilde z_t,v_t,\tilde\theta_t$ an optimal solution to the problem defining $Q_t^1(x_{t-1}^k,\tilde z_{t-1}^k,w_{t:T}^1,\xi_t^j)$ above (the dependence of the solution with respect to k,j is suppressed to alleviate notation).

The KKT conditions for problem $Q_t^1(x_{t-1}^k,\tilde z_{t-1}^k,w_{t:T}^1,\xi_t^j)$ imply

$$(60)\qquad -\delta_{tT}\theta_1-\phi(1)\theta_t-\mu_t^{k,j}-\sigma_t^{k,j}e-\rho_t^{k,j}\overrightarrow{Z}_t^{1}=0,$$

$$(61)\qquad -\theta_t\Delta\phi^\top-\tilde\sigma_t^{k,j}-\sigma_t^{k,j}=0,$$

$$(62)\qquad \sigma_t^{k,j}\circ(-\tilde z_te+w_t^1-v_t)^\top=0,$$

$$(63)\qquad \tilde\sigma_t^{k,j}\circ v_t^\top=0,$$

where for $t=T$ we have set $\rho_t^{k,j}=0$. Next, since $\tilde z_t$ can be written as $\tilde z_t=-f^t(x^t)$ for some $x^t\in\chi^t$, in case (i), we have $\tilde z_te\le -C_t^\ell e<w_t^1$. Further $v_t=\max(0,w_t^1-\tilde z_te)=w_t^1-\tilde z_te>0$. Using (61) and (63) we then get

$$(64)\qquad \tilde\sigma_t^{k,j}=0 \text{ and } \sigma_t^{k,j}=-\theta_t\Delta\phi^\top.$$

Let us now first show (i) by backward induction on t. Plugging the value of $\sigma_T^{k,j}$ given in (64) into (60) we obtain

$$\mu_T^{k,j} = -\theta_1 - \phi(1)\theta_T + \theta_T e^\top \Delta\phi = -\theta_1 + \theta_T(-\phi(1) + \sum_{\ell=1}^{J}[\phi(p_\ell) - \phi(p_{\ell-1})]) = -\theta_1 - \theta_T\phi(0).$$

Using the above relation and (57) yields $Z_{T-1}^k = -\sum_{j=1}^{q_T} p(T,j)\mu_T^{k,j} = \theta_T\phi(0) + \theta_1$. Further, using (57) with $-B_{2,1}$ the identity matrix, we obtain

$$W_{T-1}^{k,1} = \sum_{j=1}^{q_T} p(T,j)\sigma_T^{k,j} = -\sum_{j=1}^{q_T} p(T,j)\theta_T\Delta\phi^\top = -\theta_T\Delta\phi^\top. \tag{65}$$

This shows $\mathcal{P}(T)$. Let us now assume that $\mathcal{P}(t+1)$ holds for some $t \in \{2,\ldots,T-1\}$ and let us show that $\mathcal{P}(t)$ holds. First notice that (65) still holds with T substituted with t, i.e., $W_{t-1}^{k,1} = -\theta_t\Delta\phi^\top$. Further, for $\tau = 2,\ldots,T-t+1$,

$$\begin{array}{rcll} W_{t-1}^{k,\tau} & = & \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\overrightarrow{W}_t^{1,\tau-1}, & \text{from (58)},\\ & = & -\sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\theta_{t+\tau-1}e\Delta\phi^\top, & \text{using } \mathcal{P}(t+1),\\ & = & -\sum_{j=1}^{q_t} p(t,j)\theta_{t+\tau-1}\Delta\phi^\top = -\theta_{t+\tau-1}\Delta\phi^\top, & \text{since } \rho_t^{k,j}e = 1. \end{array}$$

Also

$$\begin{array}{rcll} Z_{t-1}^{k} & = & -\sum_{j=1}^{q_t} p(t,j)\mu_t^{k,j}, & \text{from (57)},\\ & = & -\sum_{j=1}^{q_t} p(t,j)(-\phi(1)\theta_t + \theta_t\Delta\phi^\top e - \rho_t^{k,j}\overrightarrow{Z}_t^1), & \text{using (60) and (64)},\\ & = & -\sum_{j=1}^{q_t} p(t,j)(-\phi(0)\theta_t - \rho_t^{k,j}\overrightarrow{Z}_t^1), & \text{using the definition of } \Delta\phi,\\ & = & \phi(0)\theta_t + \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}(\theta_1 + \phi(0)\sum_{\ell=t+1}^{T}\theta_\ell)e, & \text{using } \mathcal{P}(t+1),\\ & = & \theta_1 + \phi(0)\sum_{\ell=t}^{T}\theta_\ell & \text{since } \rho_t^{k,j}e = 1. \end{array}$$

We have thus shown $\mathcal{P}(t)$ which achieves the proof of (i).

Let us now assume that $w_t^1 < -C_t^u e$ for $t = 2,\ldots,T$ and let us show (ii). Let us fix $t \in \{2,\ldots,T\}$, $k \in \{1,\ldots,H\}$, and $j \in \{1,\ldots,q_t\}$. As before, we denote by $x_t, \tilde{z}_t, v_t, \tilde{\theta}_t$ an optimal solution to the problem defining $Q_t^1(x_{t-1}^k, \tilde{z}_{t-1}^k, w_{t:T}^1, \xi_t^j)$. In this case, $\tilde{z}_t e \geq -C_t^u e > w_t^1$ and $v_t = \max(0, w_t^1 - \tilde{z}_t e) = 0$. Using (61) and (62), we see that

$$\tilde{\sigma}_t^{k,j} = -\theta_t\Delta\phi^\top \text{ and } \sigma_t^{k,j} = 0. \tag{66}$$

Using (57) with $B_{2,1} = -I_J$, we get $W_{t-1}^{k,1} = 0$. We show (ii) by backward induction. For $t = T$, plugging the value of $\sigma_T^{k,j}$ (66) into (60) gives $\mu_T^{k,j} = -\theta_1 - \phi(1)\theta_T$, which, together with (57), gives $Z_{T-1}^k = \theta_1 + \phi(1)\theta_T$. We have already proved that $W_{T-1}^{k,1} = 0$ and thus $\tilde{\mathcal{P}}(T)$ holds. Let us now assume that $\mathcal{P}(t+1)$ holds for some $t \in \{2,\ldots,T-1\}$ and let us show that $\mathcal{P}(t)$ holds. Since $\overrightarrow{W}_t^{1,\tau-1} = 0$, we obtain $W_{t-1}^{k,\tau} = \sum_{j=1}^{q_t} p(t,j)\rho_t^{k,j}\overrightarrow{W}_t^{1,\tau-1} = 0$ for $\tau = 2,\ldots,T-t+1$. Plugging $\sigma_t^{k,j} = 0$ into (60) and using (57) gives

$$\begin{array}{rcll} Z_{t-1}^{k} & = & \sum_{j=1}^{q_t} p(t,j)(\phi(1)\theta_t + \rho_t^{k,j}\overrightarrow{Z}_t^1), & \\ & = & \sum_{j=1}^{q_t} p(t,j)(\theta_1 + \phi(1)\sum_{\ell=t}^{T}\theta_\ell), & \text{using } \tilde{\mathcal{P}}(t+1) \text{ and } \rho_t^{k,j}e = 1,\\ & = & \theta_1 + \phi(1)\sum_{\ell=t}^{T}\theta_\ell. & \end{array}$$

This shows $\tilde{\mathcal{P}}(t)$ and achieves the proof of (ii). □

Remark 5.4. *Lemma 5.3 can be used as a debugging tool to check the implementation of the SDDP for risk averse problem* (38) *when ρ^t is a spectral risk measure (which can be a heavy implementation when many constraints and interstage dependent SLP are considered). More precisely, we can check* *that in cases (i) and (ii) the formulas for* $Z_{t-1}^{k,\tau}$ *and* $W_{t-1}^{k,\tau}$ *given in Lemma 5.3 match the formulas from Theorem 5.2.*

5.3. **Conditional Value-at-Risk.** When ρ^t in (38) is $CVaR^{\varepsilon_t}$, we obtain a result analogous to Lemma 5.3:

Lemma 5.5. *Let us consider the risk averse recourse functions* $\mathcal{Q}_t$ *given by* (48). *Valid cuts for* $\mathcal{Q}_t$ *are given by Theorem 5.2. Moreover, in the following two cases, we obtain closed-form expressions for* Z_{t-1}^{k} *and* $W_{t-1}^{k,\tau}$ *(independent of the sampled scenarios):*

(i) If for $t = 2, \ldots, T$, $w_t^1 > -C_t^\ell$, *then for* $t = 2, \ldots, T$, $\mathcal{P}(t)$ *holds where*

$$\mathcal{P}(t) : \begin{cases} \forall\, k = 1, \ldots, H, Z_{t-1}^{k} = \theta_1 + \sum_{\ell=t}^{T} \frac{\theta_\ell}{\varepsilon_\ell}, \\ \forall\, k = 1, \ldots, H, W_{t-1}^{k,\tau} = \frac{\theta_{t+\tau-1}}{\varepsilon_{t+\tau-1}},\ \tau = 1, \ldots, T-t+1. \end{cases}$$

(ii) If for $t = 2, \ldots, T$, $w_t^1 < -C_t^u$, *then for* $t = 2, \ldots, T$, $\tilde{\mathcal{P}}(t)$ *holds where*

$$\tilde{\mathcal{P}}(t) :\ \forall\, k = 1, \ldots, H, Z_{t-1}^{k} = \theta_1, \text{ and } W_{t-1}^{k,\tau} = 0,\ \tau = 1, \ldots, T-t+1.$$

Proof. The proof is similar to the proof of Lemma 5.3. □

Remark 5.6. *In the particular case when the CVaR levels* $\varepsilon_t = \varepsilon \in]0, 1[$ *are the same at each time step, Lemma 5.5 is a particular case of Lemma 5.3 with* $\phi(1) = 0, \phi(0) = \frac{1}{\varepsilon}$, *and* $\Delta\phi = -1/\varepsilon \in \mathbb{R}$.

6. Conclusion

The class of extended polyhedral risk measures was introduced in this paper. Dual representations of these risk measures were obtained and used to provide conditions for coherence, convexity and consistency with second order stochastic dominance.

This class allowed us to write risk averse dynamic programming equations for some risk averse problems with risk measures taken from this class. We then detailed a stochastic dual dynamic programming algorithm for approximating the corresponding risk averse recourse functions for some stochastic linear programs. In particular, conditions were given to guarantee convergence. The developments of Sections 4 and 5 can be easily adapted if the recourse functions are approximated using other sampling-based decomposition algorithms such as AND (Birge and Donohue [BD01]) and DOASA (Philpott and Guan [PG08]).

A forthcoming work will assess the proposed approach on a mid-term multistage production management problem [GS].

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Vincent Guigues: PUC-Rio, Departamento de Engenharia Industrial, Rua Marquês de São Vicente, 225, Gávea, Rio de Janeiro, Brasil, vguigues@puc-rio.br, IMPA, Instituto de Matemática Pura e Aplicada, 110 Estrada Dona Castorina, Jardim Botanico, Rio de Janeiro, Brasil, vguigues@impa.br and Werner Römisch: Humboldt-University Berlin, Institute of Mathematics, 10099 Berlin, Germany, romisch@math.hu-berlin.de