MEASURES OF INFORMATION IN MULTI-STAGE STOCHASTIC PROGRAMMING

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Abstract. Multistage stochastic programs, which involve sequences of decisions over time, are usually hard to solve in realistically sized problems. In the two-stage case, several approaches based on different levels of available information has been adopted in literature such as the Expected Value Problem, EV, the Sum of Pairs Expected Values, SPEV, the Expectation of Pairs Expected Value, EPEV, solving series of sub-problems more computationally tractable than the initial one, or the Expected Skeleton Solution Value, ESSV and the Expected Input Value, EIV which evaluate the quality of the deterministic solution in term of its structure and upgradeability.

In this paper we generalize the definition of the above quantities to the multistage stochastic formulation when the right hand side of constraints are stochastic: we introduce the Multistage Expected Value of the Reference Scenario, MEVRS, the Multistage Sum of Pairs Expected Values, MSPEV and the Multistage Expectation of Pairs Expected Value, MEPEV by means of the new concept of auxiliary scenario and redefinition of pairs subproblems probability. We show that theorems proved in [2] and [3] for two stage case are valid also in the multi-stage case. Measures of quality of the average solution such as the Multistage Loss Using Skeleton Solution, MLUSS and the Multistage Loss of Upgrading the Deterministic Solution, MLUDS are introduced and related to the standard Value of Stochastic Solution, VSS at stage t.

A set of theorems providing chains of inequalities among the new quantities are proved. These bounds may help in evaluating whether it is worth the additional computations for the stochastic program versus the simplified approaches proposed. Numerical results on a case study related to a simple transportation problem are shown.

Key words. Multistage stochastic programming, Expected value problem, Value of stochastic solution, Skeleton solution

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1. Introduction. Stochastic programs, especially multistage programs, which involve sequences of decisions over time, are usually hard to solve in realistically sized problems. In the simpler two-stage case, several approaches and measures of levels of available information on a future realization has been adopted in the literature (see for instance [1], [2], [4], [7], [8], [9], [10], [11], [14], [15], [17], [18] or [19]). The standard measure is given by the Value of the Stochastic Solution – VSS – [2] which indicates the expected gain from solving a stochastic model rather than its deterministic counterpart, in which random parameters are replaced by their expected values. A large VSS means that uncertainty is important for the optimal solution, and the deterministic solution is “bad”. Bounds on VSS were introduced in [2] by means of the Sum of Pairs Expected Values Solutions SPEV and Expectation of Pairs Expected Value EPEV by solving pairs subproblems much less complex than the general recourse problem; these bounds may be valuable in determining whether the additional computations for the stochastic program are warranted.
Even when VSS is high, and hence stochastic program is appropriate, in real case problems can happen that all we may have access is the deterministic solution. A qualitative understanding of the deterministic solution is then important because it could actually carries out a lot of information: in [12] the structure and upgradeability of the deterministic solution has been analyzed for the two-stage case by means of the Loss Using the Skeleton Solution LUSS and the Loss of Upgrading the Deterministic Solution LUDS in relation to the standard VSS. LUSS and LUDS give deeper information than VSS on the structure of the problem and could be useful to take a fast “good” decision instead of using expensive direct techniques.

The aim of this paper is to extend to the multistage case the measures of information already valid for the two-stage case in [3] and [12], inspired by [4], [5] and also by [16].

Because of the computational intractability of most multistage problems, we believe it is very useful for the multistage case to consider different approximations of the recourse problem and evaluations, at different levels of information, of how the deterministic solution performs in the stochastic framework. This applies to algorithmic developments as well as practical use of models in management for industry and government.

An extension to multistage case of the classical VSS defined for the two-stage one, has been already introduced in [5] through a chain of values $VSS^t$ which takes into account the information until stage $t$ of the associated deterministic model. These results are valid if, in the formulation of the multistage case, only the right hand side constraints are stochastic.

We discuss approximations of the optimal stochastic solution such as the Multistage Expected Value of the Reference Scenario, MEVRS, the Multistage Sum of Pairs Expected Values, MSPEV, and the Multistage Expectation of Pairs Expected Value, MEPEV. They are introduced by means of the new concept of auxiliary scenario and redefinition of pairs subproblem probability. The proposed approaches allow to bound the optimal stochastic objective function by solving pairs subproblems that are less complex than the original one. Moreover, those approaches help to quantify if it is worth the additional computation of the former problem.

Beside the standard Value of Stochastic Solution, $VSS^t$ at stage $t$, measures of quality of the average solution such as the Multistage Loss Using Skeleton Solution, $MLUSS^t$ and the Multistage Loss of Upgrading the Deterministic Solution, $MLUDS^t$ are discussed.

The above measures could be useful because they help to qualitatively understand the behavior of the deterministic solution relative to the stochastic one, they reveal some general properties of the underlying problem as well as how the stochastic model performs when the problem is not even solvable.

As pointed out in [5] the generalization of such measures entails several issues: first of all the decision of the variables to be fixed from the deterministic solution. The trivial case would be to fix just the first stage variables and leaving the other ones free to adapt to the particular scenario. This procedure, nevertheless, can become a paradox in some cases since it could be that the first stage deterministic solution performs better than the stochastic one since the nonanticipativity constraints are relaxed in later stages.

In order to update the estimation at each stage and add more information, the above classes of measures are also defined with a rolling horizon approach already considered in [4] and [16]: the Rolling Horizon Value of Stochastic Solution, RHVSS, the Rolling
Rolling Horizon Loss Using Skeleton Solution, RHLUSS and the Rolling Horizon Loss of Upgrading the Deterministic Solution, RHLUDS are presented.

Chains of inequalities among the new quantities are proved to evaluate if it is worth the additional computations for the stochastic program versus the simplified approaches. An example of such chains of inequalities are shown in Figure 4.3 for a practical logistic problem.

We finally remark that all this class of measures is often used to describe problem classes, even though, they are instance dependent. As in the two-stage case [12], we assume that if a particular performance measure is high (or low) for a given selection of instances, then it will also be high (or low) for other instances that have similar characteristics, such as larger instances of the same problem.

The paper is organized as follow: basic definitions and notations are introduced in Section 2. Section 3 contains the generalization to the multistage case of the characteristics, such as larger instances of the same problem. Section 5 concludes the paper.

2. Notations and basic definitions. We introduce the notation that we are going to use.

The following mathematical model represents the nested formulation of a multistage linear stochastic program in which a decision maker has to take a sequence of decisions \( x^1, x^2, \ldots, x^H \), in order to minimize (expected) costs:

\[
RP = \min_{x} E_{\xi} \mathcal{H}_{-1} z(x, \xi^H) \\
= \min_{x^1} c^1 x^1 + E_{\xi^1} \left[ \min_{x^2} c^2 x^2 (\xi^1) + E_{\xi^2} \left[ \cdots + \min_{x^H} c^H x^H (\xi^{H-1}) \right] \right] \\
\text{s.t. } Ax^1 = h^1, \\
T^1(\xi^1) x^1 + W^2(\xi^1) x^2 (\xi^1) = h^2 (\xi^1), \\
\vdots \\
T^{H-1}(\xi^{H-1}) x^{H-1} (\xi^{H-1}) + W^H (\xi^{H-1}) x^H (\xi^{H-1}) = h^H (\xi^{H-1}), \\
x^1 \geq 0; \quad x^t(\xi^{t-1}) \geq 0, \quad t = 2, \ldots, H;
\]

with \( c^t \in \mathbb{R}^{m_1}, h^1 \in \mathbb{R}^{m_1}, A \in \mathbb{R}^{m_1 \times m_1}, t = 2, \ldots, H \). \( E_{\xi^t} \) denotes the expectation with respect to a random vector \( \xi^t \), defined on a probability space \((\Xi^t, \mathcal{A}^t, p)\) with support \( \Xi^t \subseteq \mathbb{R}^{m_t} \) and given probability distribution \( p \) on the \( \sigma \)-algebra \( \mathcal{A}^t \) (with \( \mathcal{A}^t \subset \mathcal{A}^{t+1} \)). We use the same notation \( \xi^t \) to denote a random vector and its particular realization. Which of these two meanings will be used in a particular situation will be clear from the context.

We denote:

- \( h^t \in \mathbb{R}^{m_t}, c^t \in \mathbb{R}^{m_t}, T^{t-1} \in \mathbb{R}^{m_{t-1} \times m_t} \), \( W^t \in \mathbb{R}^{m_t \times m_t} \), \( t = 2, \ldots, H \);
- \( \xi^t = (\xi^1, \ldots, \xi^t), t = 1, \ldots, H - 1 \);
- \( x = (x^1, x^2, \ldots, x^H) \) with \( x^t \in \mathbb{R}^{m_t}, t = 1, \ldots, H \) and \( x^t_j \) the \( j \)-th component of \( x^t \).

In general \( c^t = c^t(\xi^{t-1}) \) for \( t = 2, \ldots, H \). The decision \( x^t \) at stage \( t = 1, \ldots, H \) depends from the history up to time \( t \), more precisely the decision process has the form:

\[
\text{decision}(x^1) \rightarrow \text{observation}(\xi^1) \rightarrow \text{decision}(x^2) \rightarrow \text{observation}(\xi^2) \rightarrow \ldots \\
\ldots \rightarrow \text{decision}(x^{t-1}) \rightarrow \text{observation}(\xi^{t-1}) \rightarrow \text{decision}(x^t).
\]
The solution obtained by solving problem (2.1) is denoted with $x^*$ and called here and now solution.

Another possible way is to write the corresponding dynamic programming equations (see [19]). At the last stage the values of all problem data $\xi_{H-1}$ are already known and the values of the earlier decision $x^1, x^2, \ldots, x^{H-1}$ have been chosen. The problem becomes:

$$\begin{align*}
\min_{x^H} & \quad c^H x^H(\xi_{H-1}) \\
\text{s.t.} & \quad T^{H-1}(\xi_{H-1}) x^{H-1}(\xi_{H-1}) + W^{H}(\xi_{H-1}) x^H(\xi_{H-1}) = h^H(\xi_{H-1}), \\
& \quad x^H(\xi_{H-1}) \geq 0.
\end{align*}$$

(2.2)

The optimal value of problem (2.2), denoted $Q^H(x^{H-1}, \xi_{H-1})$ depends on the decision $x^{H-1}$ at the previous stage and realization of the data process $\xi_{H-1}$. Problem (2.1) is then solved recursively calculating the cost-to-go functions $Q_t(x^{t-1}, \xi_t)$, going backward in time. At stage $t = 2, \ldots, H-1$ the problem is formulated as follows:

$$\begin{align*}
\min_{x^t} & \quad c^t x^t(\xi^{t-1}) + E_{\xi^t} \left[ Q^{t+1}(x^t, \xi_t) \right] \\
\text{s.t.} & \quad T^{t-1}(\xi^{t-1}) x^{t-1}(\xi^{t-1}) + W^t(\xi^{t-1}) x^t(\xi^{t-1}) = h^t(\xi^{t-1}), \\
& \quad x^t(\xi^{t-1}) \geq 0 \quad t = 2, \ldots, H - 1,
\end{align*}$$

(2.3)

where $E_{\xi^t}$ represents the conditional expectation and its optimal value is denoted $Q^t(x^{t-1}, \xi^{t-1})$.

On the top of all these problems we have to find the first decision variable $x^1$:

$$\begin{align*}
\min_{x^1} & \quad c^1 x^1 + E_{\xi^1} \left[ Q^2(x^1, \xi^1) \right] \\
\text{s.t.} & \quad Ax^1 = h^1, \\
& \quad x^1 \geq 0.
\end{align*}$$

(2.4)

We introduce, for later use, the feasible region of problem (2.4):

$$K^1 := \left\{ x^1 \left| \begin{array}{l}
Ax^1 = h^1 \\
E_{\xi^1} \left[ Q^2(x^1, \xi^1) \right] < +\infty
\end{array} \right\} ,$$

and the feasible region at stage $t = 2, \ldots, H - 1$ of problem (2.3):

$$K^t := \left\{ x^t(\xi^{t-1}) \left| \begin{array}{l}
T^{t-1}(\xi^{t-1}) x^{t-1}(\xi^{t-1}) + W^t(\xi^{t-1}) x^t(\xi^{t-1}) = h^t(\xi^{t-1}) \\
E_{\xi^t} \left[ Q^{t+1}(x^t, \xi_t) \right] < +\infty
\end{array} \right\} .$$

The multistage wait-and-see problem, where the decision maker knows at the first
stage the realizations of all the random variables, takes the following form:

\[ WS = E_{\xi} \min_{x} z(x, \xi^{H-1}) \]

\[ = E_{\xi} \min_{x, (\xi^{H-1}), \ldots, x^{n}(\xi^{H-1})} c^{1}x^{1}(\xi^{H-1}) + \ldots + c^{H}x^{H}(\xi^{H-1}) \]

s.t. \[ Ax = h^{1} , \]
\[ T^{1}(\xi^{1})x^{1}(\xi^{H-1}) + W^{2}(\xi^{1})x^{2}(\xi^{H-1}) = h^{2}(\xi^{1}) , \]
\[ \vdots \]
\[ T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-1}) + W^{H}(\xi^{H-1})x^{H}(\xi^{H-1}) = h^{H}(\xi^{H-1}) , \]
\[ x^{1} \geq 0 ; \quad x^{t}(\xi^{H-1}) \geq 0 , \quad t = 2, \ldots, H ; \]

notice that this decision process is anticipative, since all the decisions \( x^{1}, x^{2}, \ldots, x^{H} \)
depend on all the realization of \( \xi^{H-1} \).

The Expected Value problem \( EV \) is obtained by replacing all random variables by their
expected values and solving a deterministic program, with \( \xi = E(\xi^{1}, \xi^{2}, \ldots, \xi^{H-1}) = (E\xi^{1}, E\xi^{2}, \ldots, E\xi^{H-1}) \):

\[ EV = \min_{x, \xi} z(x, \xi) \]

\[ = \min_{x, \xi} c^{1}x^{1} + \ldots + c^{H}x^{H} \]

s.t. \[ Ax = h^{1} , \]
\[ T^{1}(\xi^{1})x^{1} + W^{2}(\xi^{1})x^{2} = h^{2}(\xi^{1}) , \]
\[ \vdots \]
\[ T^{H-1}(\xi^{H-1})x^{H-1} + W^{H}(\xi^{H-1})x^{H} = h^{H}(\xi^{H-1}) , \]
\[ x^{1} \geq 0 , \quad t = 1, \ldots, H ; \]

Theorem 2.1. (see Madansky [11]) For the \( H \) - stage problem (2.1) the following
inequalities hold true:

\[ WS \leq RP \leq EEV , \]  

(2.7)

where \( EEV \) denotes the optimal value of the \( RP \) model when all the decision variables
until stage \( H \) are fixed at the optimal values obtained by using the average scenario.

Notice that the inequality \( EV \leq WS \) also holds true when the only random elements
are \( h^{2}(\xi^{1}), \ldots, h^{H}(\xi^{H-1}) \).

We introduce the Expected result at stage \( t \) of using the Expected Value solution
\( EEV^{t} \), \( t = 1, \ldots, H - 1 \) given by the optimal value of the \( RP \) model where the
decision variables until stage \( t \), \( x^{(t,t)} = (x^{1}, x^{2}, \ldots, x^{t}) \), \( t = 1, \ldots, H - 1 \) are fixed
at the optimal values obtained by the average scenario \( x^{(1,t)} = (\bar{x}^{1}, \bar{x}^{2}, \ldots, \bar{x}^{t}) \)
\( t = 1, \ldots, H - 1 \). This is an alternative definition with respect to the one introduced in
[5]. The Value of the Stochastic Solution at stage \( t \), \( VSS^{t} \) is then defined as:

\[ VSS^{t} = EEV^{t} - RP , \quad t = 1, \ldots, H - 1 . \]

(2.8)

Theorem 2.2. (See Escudero et al. [5]) For multistage stochastic linear programs
with deterministic constraint matrices and deterministic objective coefficients,
the following inequalities are satisfied:

\[ VSS^{t} \leq EV - EEV^{t} , \quad t = 1, \ldots, H - 1 . \]

(2.9)
\textbf{Proof}

See [5].

Again, we notice that \((2.9)\) holds true here when the only random elements are \(h^2(\xi^1), \ldots, h^H(\xi^{H-1})\) and that the problems \(EEV^t, \ t = 1, \ldots, H - 1\) could be infeasible because too many variables are fixed from the deterministic problem.

In order to proceed with numerical computation, it is useful to have a discretization of the underlying random process. In the case \(\xi^{H-1} = (\xi^1, \ldots, \xi^{H-1})\) is a random variable evolving as a discrete-time stochastic process with finite support, then the information structure can be described in the form of a \textit{scenario tree} \(\mathcal{T}\) where at each stage \(t\) there is a discrete number of atoms (nodes) \(j_t\) where a specific realization of the uncertain parameter (represented by the random variable) takes place. There are \(H\) levels (stages) in the tree, that correspond to specific time periods. The final nodes \(j_H\) are called leaf nodes. Each node, except the root, is connected to a unique node at stage \(t - 1\) called the ancestor node and to nodes at stage \(t + 1\) called the successors nodes. Each non-leaf node \(j\) is the root of a subtree \(\mathcal{T}(j)\). For each node \(j\) at stage \(t\), we denote its ancestor with \(a(j)\) and with \(\pi(a(j), j)\) the conditional probability of the random process being in node \(j\) given its history up to the ancestor node \(a(j)\).

Scenario is a path through nodes from the root node to a leaf node. We indicate with \(\pi\), the probability of scenario \(s\) passing through nodes \(j_1, j_2, \ldots, j_H\) (where \(j_t, \ t = 1, \ldots, H\) is the generic node at stage \(t\)) defined by \(\pi_s = \pi_{j_1, j_2} \cdot \pi_{j_2, j_3} \cdots \pi_{j_{H-1}, j_H}\).

We also indicate with \(p_{j,t}\) the probability of node \(j\) at stage \(t\): if node \(j\) at stage \(t\) is reachable through nodes \(j_1\) at stage 1, node \(j_2\) at stage 2, \ldots, node \(j_{t-1}\) at stage \(t - 1\), that is given by \(p_{j,t}^j = \pi_{j_1, j_2} \cdot \pi_{j_2, j_3} \cdots \pi_{j_{t-1}, j_t}\). Moreover, \(\sum_{j} p_{j,t} = 1\) where \(n_t\) is the number of nodes at stage \(t\). We indicate \(x_{j,t}^i\) the decision in the node \(j\) at stage \(t\).

Let \(\xi_1, \ldots, \xi_S\) be the possible realizations (or scenarios) of \(\xi^{H-1}\), \(\Xi\) the support of possible scenarios and \(\xi_k^{(1,j)} = (\xi^1_k, \xi^2_k, \ldots, \xi^H_k), \ i = 1, \ldots, S, \ j = 1, \ldots, H\) with \(\xi_k^i\) being the \(k\)-stage of the \(i\)-realization, \(k = 1, \ldots, j\).

Using the scenario notation described above, the \textit{multistage wait-and-see} problem \((2.5)\) takes the following form:

\[
WS = \mathbb{E}_{\xi^t} \min_{x^1(\xi^t), \ldots, x^H(\xi^t)} c^1 x^1(\xi^t) + \ldots + c^H x^H(\xi^t) \\
\text{s.t. } Ax^1(\xi^t) = h^1, \\
T^1(\xi^{(1,1)}_1) x^1(\xi^t) + W^2(\xi^{(1,1)}_2) x^2(\xi^t) = h^2(\xi^{(1,1)}_1), \\
\vdots \\
T^{H-1}(\xi^{(1,H-1)}_t) x^{H-1}(\xi^t) + W^H(\xi^{(1,H-1)}_t) x^H(\xi^t) = h^H(\xi^{(1,H-1)}_t), \\
x^1(\xi^t) \geq 0, \ x^t(\xi^t) \geq 0, \ t = 2, \ldots, H, \ i = 1, \ldots, S.
\]

Another reduced formulation of problem \((2.1)\) is given by the so-called \textit{two-stage relaxation} where the nonanticipativity constraints in the second and other stages are relaxed.

We define a new scenario tree where all random elements of stages \(2, \ldots, H - 1\) are estimated by their expected values and solve the obtained model. We denote this new scenario tree as \(\xi^{t,+} = (\xi^1, \xi^2, \ldots, \xi^t), \ t = 2, \ldots, H - 1\). We also define another scenario tree where all random elements of stages \(t, \ldots, H - 1, \ t = 2, \ldots, H - 1\) are estimated by their expected values. We denote for later use this second scenario tree as \(\xi^{t,+} = (\xi^1, \xi^2, \ldots, \xi^t, \ldots, \xi^{H-1}), \ t = 2, \ldots, H - 1\).
The two stage relaxation problem is given by the two-stage model with $H$ time periods evaluated on scenario tree just defined $\xi_t$:  

$$
TP = \min_{x^1} c_1 x^1 + E_{\xi_{H-1}} \left[ \min_{x^2,...,x^H} c_2 x^2(\xi^1) + c_3 x^3(\xi^2) + \ldots + c_H x^H(\xi^{H-1}) \right]
$$

s.t. $Ax^1 = h^1$

$$
T^1(\xi^1)x^1 + W^2(\xi^1)x^2(\xi^1) = h^2(\xi^1),
$$

$$
\vdots
$$

$$
T^{H-1}(\xi^{H-1})x^{H-1}(\xi^{H-1}) + W^H(\xi^{H-1})x^H(\xi^{H-1}) = h^H(\xi^{H-1}),
$$

$$
x^1 \geq 0; \quad x^i(\xi^{t-1}) \geq 0, \quad t = 2, \ldots, H.
$$

3. Performance measures in multistage problems. In this section we propose performance measures for multistage stochastic linear programs. They are divided in measures of information, where the same problem is solved and compared with and without a piece of available information on the future, measures of the quality of the deterministic solution which can be identified in the class of measures of different approaches with the same level of information (see [4]), and rolling horizon measures which update the estimation and add more information at each stage.

3.1. Measures of information in multistage problems. First, we intend to generalize measures introduced in [6] for the deterministic solution of the modified wait and see approach and in [3] for the stochastic two-stage ($T = 2$) case. We consider a simplified version of the stochastic program, where only the right hand side is stochastic ($h = h(\xi)$).

Instead of using a scenario given by the expected variable values, one may choose a specific realization $\xi_r$ (the scenario $r = 1, \ldots, S$) of the random variable $\xi^{H-1}$, called the reference scenario, and solve problem (2.1) along that one.

Let the PAIRS subproblem of scenarios $\xi_r$ and $\xi_k$ ($k = 1, \ldots, S$) be defined as follows:

$$
\min z^P(\mathbf{x}, \xi_r, \xi_k) = c_1 x^1 + e^2 \pi_r x^2(\xi_r) + (1 - \pi_r) e^2 x^2(\xi_k)
$$

s.t. $Ax^1 = h^1$

$$
T_r x^1 + W_r x^2(\xi_r) = \xi_r,
$$

$$
T_k x^1 + W_k x^2(\xi_k) = \xi_k,
$$

$$
x^1 \geq 0; \quad x^2(\xi_r) \geq 0; \quad x^2(\xi_k) \geq 0.
$$

In [2] and [3] the Sum of Pairs Expected Values, denoted by $SPEV$, is then defined as:

$$
SPEV = \frac{1}{1 - \pi_r} \sum_{k=1, k \neq r}^S \pi_k \min z^P(\mathbf{x}, \xi_r, \xi_k).
$$

Since the pairs subproblems (3.1) are much less complex than the general recourse problem (2.1), it is worthwhile to generalize the definition of $SPEV$ to multistage stochastic programs in order to evaluate the convenience of the additional computations required by the stochastic programming models versus a simplified approach.

We generalize the definition of $SPEV$ for multistage stochastic programs. We fix an auxiliary scenario $a$ with the following characteristics:

1. $\xi_a = \xi_r$ i.e. the values of the random parameters are the same along the nodes of this scenario;
2. If $\hat{H}$ is the first stage where scenarios $r$ and $k$ branch we define:

$$p^t_a = \begin{cases} 
1 & \text{if } t = 1, \ldots, \hat{H} - 1 \\
\pi_r & \text{if } t = \hat{H}, \ldots, H
\end{cases} \quad \pi_{j,t+j+1} = \begin{cases} 
\pi_r & \text{if } t = \hat{H} \\
1 & \text{if } t \neq \hat{H}
\end{cases}$$

3. $\pi_a = \pi_r$.

Figure 3.1 shows an example of probabilities computation on the pair subproblem made by scenarios $a = 1$ and $k = 2$ of the four stage scenario tree of Figure 4.2.

![Pair subtree](image)

**Fig. 3.1.** Pair subtree $(\xi_a, \xi_k) = (\xi_1, \xi_2)$ of scenario tree represented in Figure 4.2.

We then solve the following pair subproblems defined as:

$$\min z^P(x, \xi_a, \xi_k) = c^1 x^1 + \sum_{t=2}^{\hat{H}-1} c^t x^t_a(\xi_a) + \sum_{t=\hat{H}}^H \left[ \pi_a c^t x^t_a(\xi_a) + (1 - \pi_a) c^t x^t_k(\xi_k) \right]$$

s.t. $Ax^1 = h^1,$

$$T^t_a x^t_a + W^t a x^t_a = h^t_a,$$

$$T^t_k x^t_k + W^t k x^t_k = h^t_k,$$

$$x^1 \geq 0; \quad x^t_a \geq 0, \quad x^t_k \geq 0, \quad t = 2, \ldots, H;$$

where $\mathbf{x}^{(2,H)}_k = (x^2_k, x^3_k, \ldots, x^H_k)$. Let $\mathbf{x}^{a,k} = (x^1_k, \mathbf{x}^{(2,H)}_a, \mathbf{x}^{(2,H)}_k)$ denote an optimal solution to the pair subproblem and $z^P(\mathbf{x}^{a,k}, \xi_a, \xi_k)$ the optimal value of this problem. Eventually, the reference scenario may not correspond to any of the given scenarios.

We define the **Multistage Sum of Pairs Expected Values** denoted by **MSPEV** as follows:

$$MSPEV = \frac{1}{1 - \pi_a} \sum_{k=1}^{n_H} \pi_k \min z^P(x, \xi_a, \xi_k).$$

**Proposition 3.1.** If the scenario $\xi_r$ is not in $\Xi$, then $MSPEV = WS$.

**Proof**

If the scenario $\xi_r$ is not in $\Xi$, then $\pi_r = 0$, consequently $\pi_a = 0$ and the pair subprob-
lens $z^p(x,\xi_a,\xi_k)$ reduce to $z(x,\xi_k)$. Hence

$$MSPEV = \sum_{k=1}^{n_H} \pi_k \left[ c^1 x^1_k + \sum_{t=2}^{H} \left[ \pi_a c^t x^t_a(\xi_a) + (1-\pi_a) c^t x^t_k(\xi_k) \right] \right]$$

$$= \sum_{k=1}^{n_H} \pi_k \min z(x,\xi_k) = WS.$$ \hfill (3.5)

In the following $\xi_r \in \Xi$.

**Proposition 3.2.** $WS \leq MSPEV$.

**Proof**

Let $\tilde{x}^{a,k} = (\hat{x}^1_k, x_a(2,H), \hat{x}^2_k(2,H))$ be an optimal solution to the pair subproblem of scenarios $\xi_a$ and $\xi_k$, by definition (3.4) we have:

$$MSPEV = \sum_{k=1}^{n_H} \pi_k \min z^p(x,\xi_a,\xi_k) / (1 - \pi_a)$$

$$\geq \sum_{k=1}^{n_H} \pi_k \min z^a_k \pi_a z^a\ast_k / (1 - \pi_a) + \sum_{k=1}^{n_H} \pi_k \left[ c^1 \hat{x}^1_k + \sum_{t=2}^{H-1} c^t \hat{x}^t_a(\xi_a) + \sum_{t=H}^{H} c^t \hat{x}^t_k(\xi_k) \right]$$

$$\geq WS + \sum_{k=1}^{n_H} \pi_k \left( \sum_{t=2}^{H-1} c^t (\hat{x}^t_a(\xi_a) - \hat{x}^t_k(\xi_k)) \right).$$ \hfill (3.6)

The sum in (3.6):

$$\sum_{k=1}^{n_H} \pi_k \left( \sum_{t=2}^{H-1} c^t (\hat{x}^t_a(\xi_a) - \hat{x}^t_k(\xi_k)) \right) = 0,$$ \hfill (3.7)
if scenarios $k$ and $a$ branch at stage $\hat{H} = 2$, (3.7) is not defined, if $2 < \hat{H} \leq H$, (3.7) reduces to zero because $k$ and $a$ are defined on the same nodes until stage $\hat{H} - 1$, consequently the optimal solutions verify $\hat{x}^*_a(x_a) = \hat{x}^*_k(x_k)$ and the thesis $\text{MSPEV} \geq \text{WS}$ is proved. ■

**Proposition 3.3.** $\text{RP} \geq \text{MSPEV}$. 

**Proof**

Let $(x^{1+}, x^{(2, H)*}_k)$ be an optimal solution to the recourse problem. Then $(x^{1+}, x^{(2, H)*}_a, x^{(2, H)*}_k)$ is feasible for the Pairs subproblem of $\xi_a, \xi_k$, where this implies:

$$c^1 \hat{x}^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}^t_a(\xi_a) + \sum_{t=H}^H \left[ \pi_a c^t \hat{x}^t_a(\xi_a) + (1 - \pi_a) c^t \hat{x}^t_k(\xi_k) \right]$$

$$\leq c^1 x^{1+} + \sum_{t=2}^{\hat{H}-1} c^t x^{t+}_a(\xi_a) + \sum_{t=H}^H \left[ \pi_a c^t x^{t+}_a(\xi_a) + (1 - \pi_a) c^t x^{t+}_k(\xi_k) \right].$$

Now, we obtain

$$\sum_{k=1}^{n_H} \pi_k \left[ c^1 \hat{x}^1 + \sum_{t=2}^{\hat{H}-1} c^t \hat{x}^t_a(\xi_a) + \sum_{t=H}^H \left[ \pi_a c^t \hat{x}^t_a(\xi_a) + (1 - \pi_a) c^t \hat{x}^t_k(\xi_k) \right] \right]$$

$$= (1 - \pi_a) \text{MSPEV}.$$ 

On the other hand:

$$\sum_{k=1}^{n_H} \pi_k \left[ c^1 x^{1+} + \sum_{t=2}^{\hat{H}-1} c^t x^{t+}_a(\xi_a) + \sum_{t=H}^H \left[ \pi_a c^t x^{t+}_a(\xi_a) + (1 - \pi_a) c^t x^{t+}_k(\xi_k) \right] \right]$$

$$= (1 - \pi_a) \left[ c^1 x^{1+} + \sum_{t=2}^{\hat{H}-1} \sum_{k=1}^{n_H} \pi_k c^t x^{t+}_a(\xi_a) + \sum_{t=H}^H \sum_{k=1}^{n_H} \pi_k c^t x^{t+}_a(\xi_a) + \sum_{t=H}^H \sum_{k=1}^{n_H} \pi_k c^t x^{t+}_k(\xi_k) \right]$$

$$= (1 - \pi_a) \text{RP}$$

because

$$\sum_{k=1}^{n_H} \pi_k \sum_{t=2}^{\hat{H}-1} c^t x^{t+}_a(\xi_a) = \sum_{k=1}^{n_H} \pi_k \sum_{t=2}^{\hat{H}-1} c^t x^{t+}_k(\xi_k), \quad (3.8)$$

if scenarios $k$ and $a$ branch at stage $\hat{H} = 2$, (3.8) is not defined, if $2 < \hat{H} \leq H$, (3.8) holds because $k$ and $a$ are defined on the same nodes until stage $\hat{H} - 1$, consequently
the optimal solutions verify $x_a^t(\xi_a) = x_k^t(\xi_k)$. This proves the inequality $RP \geq M_{PEV}$.

Following [3], we introduce some upper bounds on $RP$ for multistage problems, such as the Multistage Expected Value of the Reference Scenario:

$$M EVRS = E_{\xi_{H-1}} \min_{x^H} z(\hat{x}_1^{(1,H-1)}, x^H, \xi_{H-1}) ,$$

(3.9)

where $\hat{x}_1^{(1,H-1)} = (\hat{x}_1^1, \hat{x}_2^2, \ldots, \hat{x}_t^H)$ is the optimal solution until stage $H - 1$ of the deterministic problem $\min_{x} z(x, \xi_r)$ under scenario $r$. The Multistage Value of Stochastic Solution $MVSS$ is defined as:

$$MVSS = M EVRS - RP .$$

(3.10)

Notice that $MVSS \geq 0$ because there are two alternatives:
1. $\hat{x}_r^{(1,H-1)}$ is a feasible solution to the recourse problem;
2. $\hat{x}_r^{(1,H-1)}$ is infeasible and in this case $MEVRS = +\infty$.

The definition of $MEVRS$ can be generalized in a sequence of Multistage Expected Value of the Reference Scenario, $MEVRS^1, MEVRS^2, \ldots, MEVRS^t$ such as

$$MEVRS^t = E_{\xi_{H-1}} \min_{x^{(t+1,H)}} z(\hat{x}_1^{(t,t)}, x^{(t+1,H)}, \xi_{H-1}), t = 1, \ldots, H - 1 ,$$

(3.11)

where $\hat{x}_1^{(t,t)} = (\hat{x}_1^t, \hat{x}_2^t, \ldots, \hat{x}_t^t)$ is the optimal solution until stage $t$ of the deterministic problem $\min_{x} z(x, \xi_r)$ under scenario $r$ (according to this definition $MEVRS = MEVRS^{H-1}$) and

$$MVSS^t = MEVRS^t - RP, \ t = 1, \ldots, H - 1 .$$

(3.12)

The following relation holds true:

**Proposition 3.4.**

$$MEVRS^{t+1} \geq MEVRS^t , \ t = 1, \ldots, H - 2 .$$

(3.13)

**Proof**

Any feasible solution of $MEVRS^{t+1}$ problem is also a solution of $MEVRS^t$ because the feasible region of $MEVRS^{t+1}$ has a set of constraints (at stage $t + 1$), more than $MEVRS^t$ to be satisfied and the relation (3.13) holds true. If $MEVRS^t = +\infty$ the inequality is automatically satisfied. 

As before let $\hat{x}_{a,k} = (\hat{x}_{a,k}^1, \hat{x}_{a,k}^{(2,H)}, \hat{x}_{a,k}^{(2,H)})$ be optimal solutions to the pair subproblems of $\xi_a$ and $\xi_k$, $k = 1, \ldots, n_H, k \neq r$. The Multistage Expectation of Pairs Expected Value is defined as:

$$MEPEV = \min_{k=1, \ldots, n_H \cup \{r}\}} (E_{\xi_{H-1}} \min_{x_{k,H}} z(\hat{x}^1, x^{(2,H)}, \xi_{H-1})) .$$

(3.14)

**Proposition 3.5.**

$$RP \leq MEPEV \leq MEVRS^t ,$$

(3.15)
Proof
We denote by $K = \{ x^t \in K^t \mid t = 1, \ldots, H - 1 \}$ the feasibility set of $RP$, $K \cap \{ \hat{x}^t_k, k = 1, \ldots, n_H \cup \{ r \} \}$ the feasibility set of $MEPEV$ and $K \cap \tilde{x}^t_k$ the one of $MEVRS^t$, which are obviously smaller and smaller and the thesis is proved. 

As a consequence of Proposition (3.4) it follows

$$RP \leq MEPEV \leq MEVRS^1 \leq \cdots \leq MEVRS^{H-1}. \quad (3.16)$$

Putting the previous relations together it holds:

**Theorem 3.6.**

$$0 \leq MEVRS^t - MEPEV \leq MVSS^t \leq MEVRS^t - MSPEV \leq MEVRS^t - WS, \quad t = 1, \ldots, H - 1. \quad (3.17)$$

**3.2. Measures of the quality of deterministic solution in multistage problems.** Measures of the structure and upgradeability of the deterministic solution for the two-stage case, such as the *Loss Using the Skeleton Solution* $LUSS$ and the *Loss of Upgrading the Deterministic Solution* $LUDS$ has been introduced in [12], in relation to the standard $VSS$. The aim of the measures is to find out, even when $VSS$ is large, if the deterministic solution carries useful information for the stochastic case.

We recall the definition of $LUSS$ and $LUDS$ for the two-stage case. Let $J$ be the set of indices for which the components of the expected value solution $\bar{x}(\bar{\xi})$ are at zero or at their lower bound. Then let $\tilde{x}$ be the solution of:

$$\begin{align*}
\min_{x \in X} & \ E_{\xi} \ z(x, \xi) \\
\text{s.t.} & \ x_j = \bar{x}_j(\bar{\xi}), \ j \in J.
\end{align*} \quad (3.18)$$

We then compute the *Expected Skeleton Solution Value*

$$ESV = E_{\xi} (z(\tilde{x}, \xi)), \quad (3.19)$$

and we compare it with $RP$ by means of the *Loss Using the Skeleton Solution*

$$LUSS = ESV - RP. \quad (3.20)$$

Consider the expected value solution $\bar{x}(\bar{\xi})$ as a starting point (input) to the stochastic two-stage model and compare it, in terms of objective functions, without such input. We test if the expected value solution can improve (if not optimal) in the stochastic setting. This is equivalent to adding in the former problem the constraint $x \geq \bar{x}(\bar{\xi})$ and hence solve the following problem with solution $\tilde{x}$:

$$\begin{align*}
\min_{x \in X} & \ E_{\xi} \ z(x, \xi) \\
\text{s.t.} & \ x \geq \bar{x}(\bar{\xi}).
\end{align*} \quad (3.21)$$
We then compute the Expected Input Value

\[ EIV = E_{\xi}(z(\tilde{x}, \xi)) \]  

(3.22)

and we compare it with \( RP \), by means of the Loss of Upgrading the Deterministic Solution:

\[ LUDS = EIV - RP. \]  

(3.23)

We extend the above definitions to the multistage-case by considering the Multistage Loss Using the Skeleton Solution until stage \( t \) \( \text{MLUSS}^t \) and the Multistage Loss of Upgrading the Deterministic Solution until stage \( t \) \( \text{MLUDS}^t \) in relation to \( VSS^t \), \( t = 1, \ldots, H \) defined by (2.8).

The computation of \( \text{MLUSS}^t \) is based on the following procedure: we fix at zero (or at the lower bound) all the variables which are at zero (or at the lower bound) in the expected value solution until stage \( t \), and then solve the stochastic program.

Let \( \mathcal{J}^t, t = 1, \ldots, H - 1 \) be the set of indices for which the components of the expected value solution \( \bar{x}^{(1,t)} \) are at zero or at their lower bound. Then let \( \bar{x}^{(1,t)} \) be the solution of:

\[
\min_{x} E_{\xi} z(x, \xi^{H-1}) \\
\text{s.t. } x_{jt} = \bar{x}_{jt}(\bar{\xi}^t), \ j \in \mathcal{J}^t.
\]  

(3.24)

We then compute the Multistage Expected Skeleton Solution Value at stage \( t \)

\[ \text{MESSV}^t = E_{\xi}^{H-1} \min_{x^{(t+1,H-1)}} z(\bar{x}^{(1,t)}, x^{(t+1,H-1)}, \xi^{H-1}), \ t = 1, \ldots, H - 1, \]  

(3.25)

and we compare it with \( RP \) by means of Multistage Loss Using Skeleton Solution until stage \( t \)

\[ \text{MLUSS}^t = \text{MESSV}^t - RP, \ t = 1, \ldots, H - 1. \]  

(3.26)

Notice that \( \text{MLUSS}^t \geq 0, t = 1, \ldots, H - 1 \) because \( \bar{x}^{(1,t)} \) is a feasible solution to the recourse problem or infeasible such that \( \text{MLUSS}^t = +\infty \). The case \( \text{MLUSS}^t \) close to zero means that the variables chosen by the deterministic solution until stage \( t \) are good but their values may be off. We have:

**Proposition 3.7.**

\[ \text{MLUSS}^{t+1} \geq \text{MLUSS}^t, \ t = 1, \ldots, H - 2. \]  

(3.27)

**Proof**

Any feasible solution of \( \text{MLUSS}^{t+1} \) problem is also a solution of \( \text{MLUSS}^t \) because the feasible region of \( \text{MLUSS}^{t+1} \) has a set of constraints \( x^{(t+1)} = \tilde{x}^{(t+1)}(\xi^{t+1}) \) with \( j \in \mathcal{J}^{t+1} \), larger than \( \text{MLUSS}^t \) to be satisfied and the relation (3.27) holds true. If \( \text{MLUSS}^t = +\infty \) the inequality is automatically satisfied.

We have:

\[ RP \leq \text{MESSV}^t \leq EEV^t, \]  

(3.28)
and consequently,

$$VSS^t \geq MLUSS^t \geq 0.$$  \hfill (3.29)

For multistage stochastic linear programs with deterministic constraint matrices and deterministic objective coefficients, the following inequalities are satisfied (see Escudero et al. [5] (2007)):

$$EEV^t - EV \geq VSS^t.$$  \hfill (3.30)

Notice that the case $MLUSS^t = 0$ (i.e. $MESSV^t = RP$) corresponds to the perfect skeleton solution until stage $t$ in which the condition $x^{t(j)} = \bar{x}^{t(j)}(\xi^t)$, $j \in J^t$ is satisfied by the stochastic solution even without being enforced by the set of constraints.

MLUDS$^t$, $t = 1, \ldots, H-1$ measures if the expected value solution $\bar{x}^t = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^t)$ until stage $t$ can be considered as a starting point (if not optimal) in the stochastic setting. This is equivalent to adding in problem (2.1) the constraint $x^t \geq \bar{x}^t(\xi^t)$ and hence solve the following problem obtaining the solution $\bar{\bar{x}}^t$:

$$\min_{x} E_{\xi^{H-1}} z(x, \xi^{H-1})$$  
$$\text{s.t. } x^t \geq \bar{x}^t(\xi^t).$$  \hfill (3.31)

We then compute the Multistage Expected Input Value until stage $t$

$$MEIV^t = E_{\xi^{H-1}} \min_{x^{(t+1:H-1)}} z(\bar{x}^t, \xi^{H-1})$$  \hfill (3.32)

and we compare it with $RP$, by means of the Multistage Loss of Upgrading the Deterministic Solution until stage $t$:

$$MLUDS^t = MEIV^t - RP.$$  \hfill (3.33)

As in the case of $MLUSS^t$ the following inequalities hold true:

**Proposition 3.8.**

$$MLUDS^{t+1} \geq MLUDS^t, \quad t = 1, \ldots, H - 2,$$  \hfill (3.34)

$$EEV^t \geq MEIV^t \geq RP, \quad t = 1, \ldots, H - 1,$$  \hfill (3.35)

$$EEV^t - EV \geq VSS^t \geq MLUDS^t \geq 0, \quad t = 1, \ldots, H - 1.$$  \hfill (3.36)

**Proof**

See the proof of Proposition 3.7. 

Notice that the case $MLUDS^t = 0$ (i.e. $MEIV^t = RP$) corresponds to the case where the conditions $x^t \geq \bar{x}^t(\xi^t)$ are satisfied by the stochastic solution even without being enforced by these constraints (under the assumption that the stochastic first-stage decision is unique).
3.3. Rolling horizon measures in multistage problems. Multistage problems such as MERV$^t$, MESS$^t$ and MIV$^t$ ($t = 1, \ldots, H - 1$) are often infeasible because they require to fix too many variables from the mean or reference scenario.

An alternative approach is to consider a rolling of time horizon procedure (see [4] and [16]) in order to update the estimations and add more information to the model. We propose the following methodology for the evaluation of the reference scenario; in [13] and in [20] it has been adopted for the deterministic solution.

1. Solve the reference scenario $r$ and store the first stage decision variables $\bar{x}^1$.
2. Define a new scenario tree $\mathcal{T}^{2,ev}$ where all random elements of stages $2, \ldots, H - 1$ are estimated by their expected values $\bar{\xi}^{2,\ell} = (\bar{\xi}^1, \bar{\xi}^2, \ldots, \bar{\xi}^{H-1})$ and solve the obtained model with $x^1 = \bar{x}^1$. Store all the second stage variables $\bar{x}^{2,ev}$.
3. At stage $t$ ($t = 2, \ldots, H - 1$) define a new scenario tree $\mathcal{T}^{t+1, ev}$ with all random elements of stages $t + 1, \ldots, H - 1$ estimated by their expected value $\bar{\xi}^{t+1,\ell} = (\bar{\xi}^1, \bar{\xi}^2, \ldots, \bar{\xi}^{t+1,\ell}, \bar{\xi}^{H-1})$, $t = 2, \ldots, H - 1$ and solve the associated model with

$$x^{(1,t)} = (x^1, x^2, \ldots, x^t) = (\bar{x}^1, \bar{x}_{r,ev}^{2,\ell}, \ldots, \bar{x}_{r,ev}^{t,\ell}) = \bar{x}^{(1,t)}.$$ 

Store all the $t + 1$ stage variables $\bar{x}^{t+1,\ell}_{r,ev}$.
4. Finally, solve the model on the initial scenario tree $\mathcal{T}$ with all the $t$th variables ($t = 1, 2, \ldots, H - 1$) fixed to the stored values $x^{(1,H-1)} = \bar{x}^{(1,H-1)}_{r,ev}$.

We denote the Rolling Horizon Value of the Reference Scenario:

$$RHVRS = E_{\bar{\xi}^{H-1}} \min_{x^H} z(\bar{x}^{(1,H-1)}_{r,ev}, x^H, \bar{\xi}^{H-1}) , \quad (3.37)$$

and Rolling Horizon Value of Stochastic Solution by:

$$RHVSS = RHVRS - RP . \quad (3.38)$$

In a similar way, the Rolling Horizon Expected Skeleton Solution Value $RHESSV$ can be obtained as follows:

1. Solve the expected value problem and store the first stage decision variables $\bar{x}^{1,j}$, $j \in \mathcal{J}^1$ which are at zero or at their lower bound;
2. Define a new scenario tree $\mathcal{T}^{2,ev}$ where all random elements of stages $2, \ldots, H - 1$ are estimated by their expected values and solve the obtained model with $x^{1,j} = \bar{x}^{1,j}$, $j \in \mathcal{J}^1$. Store all the second stage variables which are at zero or at their lower bound $\bar{x}^{2,j}$, $j \in \mathcal{J}^2$.
3. At stage $t$ ($t = 2, \ldots, H - 1$) define a new scenario tree $\mathcal{T}^{t+1, ev}$ with all random elements of stages $t + 1, \ldots, H - 1$ estimated by their expected value and solve the associated model with $x^{(1,t),j} = (x^1, x^2, \ldots, x^t) = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^t)$, $j \in \mathcal{J}^t$. Store all the $t + 1$ stage variables which are at zero or at their lower bound $\bar{x}^{t+1,j}$, $j \in \mathcal{J}^{t+1}$.
4. Finally solve the model on the initial scenario tree $\mathcal{T}$ with all the $j$-components, $j \in \mathcal{J}^t$ at stage $t$ ($t = 1, 2, \ldots, H - 1$) fixed to zero or at their lower bound: $x^{(1,H-1),j} = (x^1, x^2, \ldots, x^{H-1}) = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{(H-1)})$, $j \in \mathcal{J}^t$, ($t = 1, 2, \ldots, H - 1$).

We denote the Rolling Horizon Expected Skeleton Solution Value:

$$RHESSV = E_{\bar{\xi}^{H-1}} \min_{x^H} z(\bar{x}^{(1,H-1)}, x^H, \bar{\xi}^{H-1}) , \quad (3.39)$$
and $\text{Rolling Horizon Loss Using Skeleton Solution}$ by:

$$RHLUSS = RHESSV - RP.$$  \hspace{1cm} (3.40)

Starting from the definition of $\text{MEIV}^t$, we can analogously define the $\text{Rolling Horizon Expected Input Value}$ $\text{RHEIV}$ and $\text{Rolling Horizon Loss of Upgrading the Deterministic Solution}$:

$$RHLUDS = RHEIV - RP.$$  \hspace{1cm} (3.41)

4. Case study: a multistage stochastic optimization model for a single-sink transportation problem. We consider a real case of clinker replenishment in Sicily, provided by the primary Italian cement producer. The problem has been already analyzed in detail in [13]. The logistics system is organized as follows: in Catania there is a warehouse to be replenished by clinker produced by four plants located in Palermo (PA), Agrigento (AG), Cosenza (CS) and Vibo Valentia (VV). The monthly demand of the single customer at Catania as well as the production capacities of the four plants are stochastic.

All the vehicles must be booked in advance from an external transportation company, before the demand and production capacities are revealed. We assume that the transportation company has an unlimited fleet and that only full load shipments are allowed. When the demand and the production capacity are revealed, there is an option to cancel some of the reservations against a cancellation fee. If the quantity delivered from the four suppliers is not enough to satisfy the demand, the residual quantity is purchased from an external company at a higher price $b$. The problem is to determine, for each supplier, the number of vehicles to book at the beginning of the month, in order to minimize the total costs, given by the sum of the transportation costs (including the cancellation fee for vehicles booked but not used) and the costs of the product purchased from the external company.

The notation adopted is the following:

Sets:

$$\mathcal{I} = \{i : i = 1, \ldots, I\} : \text{set of suppliers (AG, CS, PA, VV)};$$

$$\mathcal{J}^t = \{j : j = 1, \ldots, n_t\} : \text{set of ordered nodes of the tree at stage } t = 1, \ldots, H;$$

and $n_t$ is the number of nodes at stage (month) $t$. 
Parameters:

- \( t_i \): unit transportation costs of supplier \( i \in \mathcal{I} \);
- \( b \): buying cost from an external source;
- \( q \): vehicle capacity;
- \( g \): unloading capacity at the customer;
- \( l_0 \): initial inventory level at the customer;
- \( l_{\text{max}} \): storage capacity at the customer;
- \( p_j \): probability of node \( j \in \mathcal{J}^t \), \( t = 1, \ldots, H \);
- \( v_{i,j} \): production capacity of supplier \( i \in \mathcal{I} \) in node \( j \in \mathcal{J}^t \), \( t = 2, \ldots, H \);
- \( d_j \): customer demand at node \( j \in \mathcal{J}^t \), \( t = 2, \ldots, H \);
- \( \alpha \): cancellation fee;
- \( \mathcal{J}^1 = \{0\} \): root of the tree;
- \( a(j) \): ancestor of the node \( j \in \mathcal{J}^t \), \( t = 2, \ldots, H \) in the scenario tree.

Notice that \( b \) is fixed on the basis of the known production and transportation costs of each producers. In our case we suppose \( b > \max_i (t_i + c_i) \) where \( c_i \) is the unit production costs of supplier \( i \in \mathcal{I} \).

Variables:

- \( x_{i,j} \in \mathbb{N} \): number of vehicles booked from supplier \( i \in \mathcal{I} \), \( j \in \mathcal{J}^t \), \( t = 1, \ldots, H - 1 \);
- \( z_{i,j} \in \mathbb{N} \): number of vehicles actually used from \( i \in \mathcal{I} \), \( j \in \mathcal{J}^t \), \( t = 2, \ldots, H \);
- \( y_j \in \mathbb{R} \): product to purchase from an external source in \( j \in \mathcal{J}^t \), \( t = 2, \ldots, H \);
- \( l_j \in \mathbb{R} \): inventory level of the customer at node \( j \):

\[
l_j = l_{a(j)} + q \sum_{i=1}^{H} z_{i,j} + y_j - d_j, \quad j \in \mathcal{J}^t, \ t = 2, \ldots, H; \tag{4.1}
\]

The multistage model can be then formulated as follows:

\[
\min \sum_{t=1}^{H-1} \sum_{j=1}^{n_t} p_j \left[ q \sum_{i=1}^{l} t_i x_{i,j} \right] + \sum_{t=2}^{H} \sum_{j=1}^{n_t} p_j \left[ b y_j - (1 - \alpha) q \sum_{i=1}^{H} l_t (x_{i,a(j)} - z_{i,j}) \right] \tag{4.2}
\]
subject to

\[ q \sum_{i=1}^{I} x_{i,j} \leq g, \quad j \in J^t, \quad t = 1, \ldots, H - 1 \quad (4.3) \]

\[ l_u(j) + q \sum_{i=1}^{I} z_{i,j} + y_j - d_j \geq 0, \quad j \in J^t, \quad t = 2, \ldots, H \quad (4.4) \]

\[ l_u(j) + q \sum_{i=1}^{I} z_{i,j} + y_j - d_j \leq l_{\text{max}}, \quad j \in J^t, \quad t = 2, \ldots, H \quad (4.5) \]

\[ z_{i,j} \leq x_{i,a(j)}, \quad i \in I, \quad j \in J^t, \quad t = 2, \ldots, H \quad (4.6) \]

\[ q z_{i,j} \leq v_{i,j}, \quad i \in I, \quad j \in J^t, \quad t = 2, \ldots, H \quad (4.7) \]

\[ x_{i,j} \in \mathbb{N}, \quad i \in I, \quad j \in J^t, \quad t = 1, \ldots, H - 1 \quad (4.8) \]

\[ y_j \geq 0, \quad j \in J^t, \quad t = 2, \ldots, H \quad (4.9) \]

\[ z_{i,j} \in \mathbb{N}, \quad i \in I, \quad j \in J^t, \quad t = 2, \ldots, H \quad (4.10) \]

The first sum in the objective function (4.2) is the booking costs of the vehicles, while the second sum represents the recourse actions, consisting of buying extra clinker \((y_j)\) and canceling unwanted vehicles. Constraint (4.3) guarantees that the total quantity delivered from the suppliers to the customer is not greater than the customer’s unloading capacity \(g\), inducing thus an upper bound on the total number of vehicles. Constraints (4.4) and (4.5) ensure that the storage levels are between zero and \(l_{\text{max}}\). Constraint (4.6) guarantees that the number of vehicles servicing supplier \(i\) is at most equal to the number booked in advance and (4.7) controls that the quantity of clinker delivered from supplier \(i\) does not exceed its production capacity \(a_{i,j}^t\). Finally, (4.8)–(4.10) define the decision variables of the problem.

4.1. Computation of measures for “a single sink transportation problem”. We compute the performance measures described in Sections 2 and 3 on the single sink transportation problem (4.2)–(4.10). For simplicity we analyze first the two-stage case and then the multi-stage stochastic one by means of a four-stages scenario tree defined by the user, where a stage represents a month and the root the month of January. We refer to [13] for the data used in the simulation. In the two stage case the number of variables and constraints is limited (74 variables and 155 constraints), in the four stage we have 750 variables and 1185 constraints.

4.1.1. Two-stage case. Tables 4.1 shows the optimal number of booked vehicles for each supplier, the total optimal costs and the values (see Table 4.2) assumed by the performance measures in the two-stage case.

We first observe that both the deterministic cases, using the mean \((EV)\) and the worst scenario, underestimate the stochastic optimal cost and the model will always book the exact number of vehicles needed in the next time period (see [13]). The deterministic model sorts the suppliers according to the transportation costs and books much less vehicles than the stochastic one with a resulting cost lower than \(RP\) solution. However, \(EEV\) is much higher than the deterministic cost (\(\text{€} \, 495\,788\) instead of the predicted cost of \(\text{€} \, 294\,898\)) resulting in

\[ \text{VSS} = 495\,788 - 438\,304 = 57\,384, \]

which shows that we can save about 12% of the cost by using the stochastic model, compared to the deterministic one. \(EVRS\) is still higher (\(\text{€} \, 522\,877\) instead of the
predicted cost of €427,374). Notice also that the inequalities of Theorem 2.1

\[ WS = 319,100 \leq RP = 438,304 \leq EEV = 495,788 \]

are satisfied.

Fixing as auxiliary scenario the average scenario we get a value for \( SPEV \) equal to 319,100 which is exactly the wait and see solution \( (WS) \) (see Proposition 3.1), while choosing as auxiliary scenario the worst one \( \xi_{10} \) we get a worst value for \( SPEV \) of 343,626. The series of inequalities (see Proposition 7 in [3]):

\[ WS = 319,100 \leq SPEV = 343,626 \leq RP = 438,304 \]

are satisfied and shows the advantage of a deeper information on the future: when we know all the information, the objective function reaches the minimum value \( (WS) \), in the case of partial information \( (SPEV) \) the objective function is greater or equal to the minimum value while in case of uncertainty we get the highest cost \( (RP) \). For details on the optimal values of the pair subproblems \( \xi_{10} \) and \( \xi_{k} \), \( k = 1, \ldots, 14 \) with respect to the worst scenario \( \xi_{10} \) see Table 4.3. The value of \( EPEV \) is determined by the optimal first stage solution of the pair subproblem \( \xi_{10} \) and \( \xi_{14} \) performed into the stochastic model and it satisfies the inequality in Proposition 3.5:

\[ RP = 438,304 \leq EPEV = 485,875 \leq EVRS = 522,877 \]

In order to understand the reason of the badness of the deterministic solution quantified by the high value of \( VSS = 57,384 \), we compute now the Expected Skeleton Solution Value \( ESSV \), following the skeleton from the deterministic model, not allowing to book vehicles from CS and VV. \( ESSV \) is €462,214, still higher than \( RP \) with a consequent Loss Using the Skeleton Solution of

\[ LUSS = 462,214 - 438,304 = 23,910 \]

which measures the loss by booking vehicles coming only from suppliers AG and PA as suggested by the deterministic model. We can conclude that the deterministic solution is bad because it books the wrong number of vehicles from the wrong suppliers.

The Expected Input Value is computed by taking the number of vehicles booked in the deterministic solution as input in the stochastic model and checking if the solution can be upgraded in a second run. Notice that for all the four suppliers the stochastic solution is higher than in the deterministic one (see Table 4.2) with \( LUDS = 0 \).

The measures defined in [12] allow us to conclude that the deterministic solution does not perform well in a stochastic environment because of the too low number of vehicles booked at the fist stage (736 instead of 1080) just considering AG and PA as possible suppliers. However, the deterministic solution should be considered as a lower bound for the stochastic case.

### 4.1.2. Multi-stage case

In this section we compute the performance measures described in Sections 2 and 3 on the four-stage case of single sink transportation problem. For this purpose, we consider the scenario tree from Figure 4.2: this is a four-stage tree with 5 branches from the root, 5 from each of the second-stage nodes, and three from each of the third-stage nodes, resulting in \( S = 5 \times 5 \times 3 = 75 \) scenarios and 106 nodes. We declare it as a benchmark to evaluate the cost of optimal solutions obtained using the other reduced scenario trees (see Figures 4.1). The results are presented in Table 4.5 and Figure 4.3 and performance measures in Table 4.6. From
Table 4.1

Optimal solutions and comparison measures for the two-stage case of the “single-sink transportation problem”. The table shows optimal number of booked vehicles for each supplier and total optimal costs.

<table>
<thead>
<tr>
<th></th>
<th>AG</th>
<th>CS</th>
<th>PA</th>
<th>VV</th>
<th>Objective value (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>deterministic (mean scenario = (\xi))</td>
<td>206</td>
<td>0</td>
<td>530</td>
<td>0</td>
<td>294 898 = EV</td>
</tr>
<tr>
<td>deterministic (worst scenario = (\xi_{10}))</td>
<td>433</td>
<td>33</td>
<td>366</td>
<td>0</td>
<td>427 374</td>
</tr>
<tr>
<td>stochastic</td>
<td>400</td>
<td>0</td>
<td>563</td>
<td>117</td>
<td>438 304 = RP</td>
</tr>
<tr>
<td></td>
<td>206</td>
<td>0</td>
<td>530</td>
<td>0</td>
<td>495 788 = EEV</td>
</tr>
<tr>
<td></td>
<td>433</td>
<td>33</td>
<td>366</td>
<td>0</td>
<td>522 877 = EVRS (w.r.t. (\xi_{10}))</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0</td>
<td>637</td>
<td>0</td>
<td>462 214 = ESSV</td>
</tr>
<tr>
<td></td>
<td>400</td>
<td>0</td>
<td>563</td>
<td>117</td>
<td>438 304 = EIV</td>
</tr>
<tr>
<td>pair subproblem ((\xi_{10}, \xi_{14}))</td>
<td>300</td>
<td>0</td>
<td>370</td>
<td>110</td>
<td>485 875 = EPEV (w.r.t. (\xi_{10}))</td>
</tr>
</tbody>
</table>

Table 4.2

Performance measures for the two-stage case of the “single-sink transportation problem”.

<table>
<thead>
<tr>
<th>Performance measures</th>
<th>Value (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VSS</td>
<td>57 384</td>
</tr>
<tr>
<td>EVRS – RP</td>
<td>84 573</td>
</tr>
<tr>
<td>LUSS</td>
<td>23 910</td>
</tr>
<tr>
<td>LUDS</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.3

Pair subproblems first stage solutions and total costs with respect to the worst case scenario \(\xi_{10}\).

<table>
<thead>
<tr>
<th>pair subproblem</th>
<th>AG</th>
<th>CS</th>
<th>PA</th>
<th>VV</th>
<th>Objective value (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\xi_{10}, \xi_{1})</td>
<td>303</td>
<td>0</td>
<td>297</td>
<td>0</td>
<td>258 185</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{2})</td>
<td>0</td>
<td>0</td>
<td>757</td>
<td>0</td>
<td>365 796</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{3})</td>
<td>0</td>
<td>97</td>
<td>533</td>
<td>116</td>
<td>427 930</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{4})</td>
<td>56</td>
<td>0</td>
<td>638</td>
<td>0</td>
<td>331 009</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{5})</td>
<td>433</td>
<td>0</td>
<td>334</td>
<td>0</td>
<td>302 193</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{6})</td>
<td>133</td>
<td>0</td>
<td>577</td>
<td>0</td>
<td>325 037</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{7})</td>
<td>136</td>
<td>0</td>
<td>333</td>
<td>281</td>
<td>379 316</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{8})</td>
<td>303</td>
<td>0</td>
<td>126</td>
<td>298</td>
<td>361 825</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{9})</td>
<td>316</td>
<td>0</td>
<td>438</td>
<td>0</td>
<td>313 279</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{10})</td>
<td>433</td>
<td>33</td>
<td>366</td>
<td>0</td>
<td>522 877</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{11})</td>
<td>433</td>
<td>0</td>
<td>237</td>
<td>0</td>
<td>270 515</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{12})</td>
<td>0</td>
<td>170</td>
<td>563</td>
<td>0</td>
<td>447 357</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{13})</td>
<td>40</td>
<td>0</td>
<td>680</td>
<td>0</td>
<td>344 212</td>
</tr>
<tr>
<td>(\xi_{10}, \xi_{14})</td>
<td>300</td>
<td>0</td>
<td>370</td>
<td>110</td>
<td>340 483</td>
</tr>
</tbody>
</table>

Table 4.5 we see that a better description of the stochasticity leads to larger bookings in the first stage. Actually, in the four-stage scenario tree, the total number of booked vehicles is equal to 1260, that is the customer’s unloading capacity. This is due to the low initial inventory level \(l_0 = 2000\) at the customer (the actual case from real data).
Fig. 4.1. Reduced scenario trees respectively considered for the mean scenario model (EV), for the two-stage relaxation (TP) and the computations of the rolling horizon values reported in Tables 4.4 and 4.5 for the single-sink transportation problem.

Fig. 4.2. Four-stages scenario tree considered for the single-sink transportation problem.

The total costs from the two deterministic models, mean scenario (EV) (see Table 4.4) and worst scenario \( \xi_{44} \), the recourse problem (RP) and the two-stage relaxation (TP) (see Table 4.5) are not directly comparable because they refer to different scenario trees. The optimal solutions are then compared on the scenario tree of Figure 4.2 used as a benchmark. First of all we evaluate the optimal solutions of the average scenario model by fixing just the number \( x_{i,j} \) of vehicles booked from supplier \( i \in \mathcal{I} \) in nodes \( j \in \mathcal{J}^t \) until stage \( t (t = 1, 2, 3) \) in the stochastic framework by means of \( EEV^t \):

\[
EEV^1 = 1364343.58 < EEV^2 = 1431896.24 < EEV^3 = 1579825.92 .
\]

The associated chain

\[
VSS^1 = 91267.32 < VSS^2 = 185002.22 < VSS^3 = 299449.62 ,
\]

show the losses by booking the number of vehicles suggested by the deterministic solution (see Table 4.4). The low first stage deterministic booking is compensated at each of second stage nodes of the stochastic framework, by a reservation almost equal to the customer’s unloading capacity and by buying extra clinker at a higher price.

The evaluation of the deterministic solution on a rolling-horizon basis (see Figure 4.1), allows to update the estimations and add more information step by step as measured by:

\[
RHEEV = 1540248.22 < EEV^3 = 1579825.92 .
\]

Notice that by fixing all the decision variables \( x_{i,j} , y_j \) and \( z_{i,j} \) from the average scenario model until the second stage, we get \( EEV^2 = \infty \) and consequently \( EEV^3 = \)
\( EEV^2 \leq EEV^3 \) concluding a badness of the deterministic solution. We will try to understand later by means of \( M\text{LUSS}^t \) and \( M\text{LU DS}^t \) the reason of its infeasibility.

The same considerations can be applied by evaluating the worst scenario model \( \xi_{44} \) in the stochastic framework by means of \( M\text{EVRS}^t \). In particular by fixing just the number \( x_{i,j} \) of vehicles booked from supplier \( i \in \mathcal{I} \) in nodes \( j \in \mathcal{J} \) until stage \( t \), we get

\[
M\text{EVRS}^1 = 1372\,285.48 < M\text{EVRS}^2 = 1433\,501.32 < M\text{EVRS}^3 = 1436\,997.78
\]

The evaluation of the worst case solution on a rolling-horizon basis is measured by

\[
RH\text{VRS} = 1\,535\,476.52
\]

\( M\text{ESSV}^t \) allows the evaluation of the structure of the deterministic solution until stage \( t \). We do not allow to book vehicles from CS in all the three stages and from VV in the root and at stage \( t = 3 \) (see Table 4.4) and we get:

\[
M\text{ESSV}^1 = 1\,299\,327.68 \leq M\text{ESSV}^2 = 1\,301\,017.28 \leq M\text{ESSV}^3 = 1\,404\,215.16
\]

with a consequent chain of measures:

\[
M\text{LUSS}^1 = 26\,282.62 \leq M\text{LUSS}^2 = 27\,863.9 \leq M\text{LUSS}^3 = 131\,140.86
\]

which measure the loss by booking vehicles coming only from the suppliers as suggested by the deterministic model. We can conclude that the deterministic solution \( x_{i,j} \) is bad because it books the wrong number of vehicles from the wrong suppliers already from the first stage.

The evaluation of the deterministic skeleton solution on a rolling-horizon basis, allows to update the estimations as suggested by:

\[
RH\text{ESSV} = 1\,302\,705.36 \leq M\text{ESSV}^3 = 1\,404\,215.16
\]

As before, by fixing at zero also the clinker purchased \( y_j \) and the vehicles actually used \( z_{i,j} \) we get an infeasibility already at the second stage \( (M\text{ESSV}^2 = M\text{ESSV}^3 = \infty) \) concluding a badness of the structure of the full deterministic solution.

We then consider the vehicles booked in the average scenario model as an input in the stochastic setting and we check if the solution can be upgraded. Notice that in the root for all the four suppliers the booked number of vehicles in the stochastic solution is higher than in the deterministic one with \( M\text{LU DS}^1 = 0 \). The condition is no longer satisfied at stage 2 for suppliers PA and VV with \( M\text{LU DS}^2 = 2707.4 \) and at stage 3 for suppliers AG and PA with \( M\text{LU DS}^3 = 27552.22 \). Notice that the chain (3.35) in Proposition 3.8:

\[
M\text{LU DS}^1 = 0 \leq M\text{LU DS}^2 = 2707.4 \leq M\text{LU DS}^3 = 27552.22
\]

holds true. We can conclude that the deterministic solution can be taken as input in the stochastic model only in the first stage.

An alternative approach to the deterministic solution is to solve pairs subproblems of the initial stochastic program with respect to the worst scenario \( \xi_{44} \).

The best pair subproblem is given by the couple \((\xi_{44}, \xi_2)\) with \( \text{MEPEV} = 1\,313\,983.3 \) which satisfies the chain of inequalities (3.16):

\[
RP = 1\,273\,074.3 < \text{MEPEV} = 1\,313\,983.3 < MEVRS^1 = 1\,372\,285.48 < MEVRS^2 = 1\,433\,501.32 < MEVRS^3 = 1\,436\,997.78
\]
This means that the optimal first stage solution of the pair subproblem \((\xi_{44}, \xi_2)\) performs better than the deterministic one (mean or worst scenario), and it should be chosen for large scale problems instead of solving them, in case we have more information on the future.

When the auxiliary scenario does not belong to the scenario tree, \(MSPEV = WS = 1037820\) as proved in Proposition 3.1. If the auxiliary scenario (the worst one) belongs to the scenario tree, then \(WS = 1037820 < 1041627.34 = MSPEV\) (see Proposition 3.2). Notice that the sum (3.7) is zero: if \(k = 1, \ldots, 30\) or \(k = 46, \ldots, 75\), scenarios \(k\) and the auxiliary \(a\), branch at stage \(H = 2\) and (3.7) is not defined, if \(k = 31, \ldots, 43\) scenarios \(k\) and a branch at \(H = 3\) and at stage 2 are both defined on node 3 with \(\hat{x}^2_a(\xi_{44}) = \hat{x}^2_a(\xi_4)\). The same arguments can be applied for scenarios 43 and 46 which branch with scenario 44 at \(H = 4\) and are defined on the same node 20 at stage 3.

If we choose as auxiliary scenario the best one \(\xi_1\) (the one that gives the minimum cost over all the scenarios in the tree) \(WS = 1037820 < MSPEV = 1041627.34\).

Finally, Proposition 3.3 is satisfied being \(RP = 1273074.3 > MSPEV = 1039068.91\). Theorem 3.6 is verified.

Figure 4.3 shows as the values of the new approaches (denoted with \(\times\)) are closer to \(RP\) value than those ones already known in literature.

By the analyzed measures we can conclude that the deterministic model performs bad in the multistage stochastic environment because of the too low number of booked vehicles, already from the first stage. The positive values of \(MLUSS^d\) mean that the badness of the deterministic solution is partially in its structure, booking vehicles from the wrong suppliers. However the deterministic solution should be considered as a lower bound for the first stage stochastic one. The rolling-horizon approach should be also considered as an useful alternative to the standard methods allowing to update the estimation at each time period. A better option than choosing the deterministic solution is also given by the best pair subproblem solution with performance measured by \(MEPEV\), under the assumption of a better information about the future.

Table 4.4

Optimal solution for the deterministic (mean scenario) four-stage case of the “single-sink transportation problem”. The table shows optimal number of booked vehicles (equal to the optimal number of used vehicles at node \(j+1\)) for each supplier, the clinker purchased \(y_j\) at each stage \(j = 1, 2, 3\) and partial optimal costs.

<table>
<thead>
<tr>
<th>node</th>
<th>AG</th>
<th>CS</th>
<th>PA</th>
<th>VV</th>
<th>Costs (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>158</td>
<td>0</td>
<td>647</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>1</td>
<td>264</td>
<td>0</td>
<td>416</td>
<td>143</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>315</td>
<td>0</td>
<td>518</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>

5. Conclusions. The paper extends classical measures to value different approaches and levels of information for two-stage stochastic problems to the multistage case. We generalize bounds of Value of Stochastic Solution \(VSS\) to the multistage case through the Multistage Sum of Pairs of Expected Value \(MSPEV\) and Multistage Expectation of Pairs Expected Value \(MEPEV\) by solving a series of sub-problems more computationally tractable than the initial one under the assumption that a piece of information on the future development of a random variable is available. This extension has been done by introducing the new concept of auxiliary scenario and redefinition.
Table 4.5
Optimal solutions and comparison measures for the four-stage case of the “single-sink transportation problem”.

<table>
<thead>
<tr>
<th></th>
<th>AG</th>
<th>CS</th>
<th>PA</th>
<th>VV</th>
<th>Objective value (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EV (mean scenario = ( \xi ))</td>
<td>158</td>
<td>0</td>
<td>647</td>
<td>0</td>
<td>1000 182</td>
</tr>
<tr>
<td>deterministic (worst scenario = ( \xi_{44} ))</td>
<td>0</td>
<td>299</td>
<td>533</td>
<td>116</td>
<td>1342 803</td>
</tr>
<tr>
<td>RP</td>
<td>389</td>
<td>0</td>
<td>755</td>
<td>116</td>
<td>1273 074.30</td>
</tr>
<tr>
<td>TP</td>
<td>401</td>
<td>0</td>
<td>641</td>
<td>116</td>
<td>1104 279.60</td>
</tr>
<tr>
<td>EEV (w.r.t. ( \xi )) fixing ( x_{i,j} )</td>
<td>158</td>
<td>0</td>
<td>647</td>
<td>0</td>
<td>1364 343.58</td>
</tr>
<tr>
<td>EEV (w.r.t. ( \xi )) fixing all 1(^{st} ) and 2(^{nd} ) stage var.</td>
<td>158</td>
<td>0</td>
<td>647</td>
<td>0</td>
<td>( \infty )</td>
</tr>
<tr>
<td>EEV (w.r.t. ( \xi )) fixing all 1(^{st} ), 2(^{nd} ) and 3(^{rd} ) stage var.</td>
<td>158</td>
<td>0</td>
<td>647</td>
<td>0</td>
<td>( \infty )</td>
</tr>
<tr>
<td>MEV (w.r.t. ( \xi_{44} )) fixing ( x_{i,j} )</td>
<td>0</td>
<td>299</td>
<td>533</td>
<td>116</td>
<td>1372 285.48</td>
</tr>
<tr>
<td>MEV (w.r.t. ( \xi_{44} )) fixing ( x_{i,j} ) and ( y_{j} )</td>
<td>0</td>
<td>299</td>
<td>533</td>
<td>116</td>
<td>( \infty )</td>
</tr>
<tr>
<td>MEV (w.r.t. ( \xi_{44} )) fixing all 1(^{st} ) and 2(^{nd} ) stage var.</td>
<td>0</td>
<td>299</td>
<td>533</td>
<td>116</td>
<td>( \infty )</td>
</tr>
<tr>
<td>W (pair subproblem ( \xi_{44}, \xi_{2} ))</td>
<td>303</td>
<td>0</td>
<td>516</td>
<td>116</td>
<td>1313 983.30</td>
</tr>
<tr>
<td>MPEV (w.r.t. ( \xi_{44} )) lower bound on ( x_{i,j} )</td>
<td>401</td>
<td>0</td>
<td>743</td>
<td>116</td>
<td>1278 053.08</td>
</tr>
<tr>
<td>MPEV (w.r.t. ( \xi_{44} )) lower bound on ( x_{i,j} )</td>
<td>401</td>
<td>0</td>
<td>743</td>
<td>116</td>
<td>( \infty )</td>
</tr>
<tr>
<td>W (pair subproblem ( \xi_{44}, \xi_{2} ))</td>
<td>303</td>
<td>0</td>
<td>516</td>
<td>116</td>
<td>1313 983.30</td>
</tr>
<tr>
<td>MEPEV (w.r.t. ( \xi_{44} )) lower bound on ( x_{i,j} )</td>
<td>401</td>
<td>0</td>
<td>743</td>
<td>116</td>
<td>1278 053.08</td>
</tr>
<tr>
<td>MEPEV (w.r.t. ( \xi_{44} )) lower bound on ( x_{i,j} )</td>
<td>401</td>
<td>0</td>
<td>743</td>
<td>116</td>
<td>( \infty )</td>
</tr>
<tr>
<td>MPEV (w.r.t. ( \xi_{44} )) lower bound on ( x_{i,j} )</td>
<td>401</td>
<td>0</td>
<td>743</td>
<td>116</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

The results show that a better alternative than choosing the deterministic solution is given by the best pair subproblem solution as measured by \( MEPEV \) in case we have more information about the future. We also extend to the multistage case the Expected Value of the Reference Scenario \( MEVRS \) and measures of quality of the expected value solution in terms of structure and upgradeability such as Multistage Loss Using the Skeleton Solution \( MLUSS \) and Multistage Loss of Upgrading the Deterministic Solution \( MLUDS \) and related with...
Table 4.6
Performance measures for the four-stages case of the “single-sink transportation problem”.

<table>
<thead>
<tr>
<th>Performance measures</th>
<th>Value (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( VSS^1 )</td>
<td>91,267.32</td>
</tr>
<tr>
<td>( VSS^2 ) fixing ( x_{i,j} )</td>
<td>185,002.22</td>
</tr>
<tr>
<td>( VSS^2 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( VSS^3 ) fixing ( x_{i,j} )</td>
<td>299,449.62</td>
</tr>
<tr>
<td>( VSS^3 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( MEVRS^1 - RP = MVSS^1 )</td>
<td>103,463.38</td>
</tr>
<tr>
<td>( MEVRS^2 - RP = MVSS^2 ) fixing ( x_{i,j} )</td>
<td>164,122.76</td>
</tr>
<tr>
<td>( MEVRS^2 - RP = MVSS^2 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( MEVRS^3 - RP = MVSS ) fixing ( x_{i,j} )</td>
<td>168,187.02</td>
</tr>
<tr>
<td>( MEVRS^3 - RP = MVSS )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( MLUSS^1 )</td>
<td>26,282.62</td>
</tr>
<tr>
<td>( MLUSS^2 ) fixing ( x_{i,j} )</td>
<td>27,863.90</td>
</tr>
<tr>
<td>( MLUSS^2 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( MLUSS^3 ) fixing ( x_{i,j} )</td>
<td>27,863.90</td>
</tr>
<tr>
<td>( MLUSS^3 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( MLUDS^1 )</td>
<td>0</td>
</tr>
<tr>
<td>( MLUDS^2 ) lower bound on ( x_{i,j} )</td>
<td>2,707.40</td>
</tr>
<tr>
<td>( MLUDS^2 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( MLUDS^3 ) lower bound on ( x_{i,j} )</td>
<td>27,552.22</td>
</tr>
<tr>
<td>( MLUDS^3 )</td>
<td>( \infty )</td>
</tr>
<tr>
<td>( RHVSS ) (w.r.t ( \xi ))</td>
<td>267,173.92</td>
</tr>
<tr>
<td>( RHVSS ) (w.r.t ( \xi_{44} ))</td>
<td>262,402.22</td>
</tr>
<tr>
<td>( RHLUSS )</td>
<td>29,631.06</td>
</tr>
<tr>
<td>( RHLUUDS )</td>
<td>17,530.02</td>
</tr>
</tbody>
</table>

Fig. 4.3. Comparison of objective functions of different approaches as in Table 4.5, reported for increasing values. The black circle denotes the multistage stochastic recourse problem \( RP \), the squares the values of the new performance measures and the grey circles are the values of measures already known in literature.
the standard Value of Stochastic Solution $V_{SS}^t$ at stage $t$. Such measures can help to understand the behavior of the deterministic solution with respect to the stochastic and the reason of its badness/goodness. The above measures are also defined in a rolling horizon framework by means of the Rolling Horizon Value of Stochastic Solution $RHV_{VSS}$, the Rolling Horizon Loss Using Skeleton Solution and Rolling Horizon Loss of Upgrading the Deterministic Solution $RHLUDS$. The results show that they should be considered as an useful alternative to the standard methods allowing to update the estimation at each time period.

Chains of inequalities among the new measures are proved and tested on a stochastic multi-stage single-sink transportation problem. Differences among the values in the chains indicate the distance, at stage $t$, of the proposed approach to the stochastic one and give insight of what is potentially wrong with a solution coming from the deterministic or the approximated method considered.

REFERENCES