Multi-Objective Probabilistically Constrained Programming with Variable Risk: New Models and Applications

Miguel A. Lejeune*   Siqian Shen†

Abstract

We consider a class of multi-objective probabilistically constrained problems **MOPCP** with a joint chance constraint, a multi-row random technology matrix, and a risk parameter (i.e., the reliability level) defined as a decision variable. We propose a Boolean modeling framework and derive a series of new equivalent mixed-integer programming formulations. We demonstrate the computational efficiency of the formulations that contain a small number of binary variables. We provide modeling insights pertaining to the most suitable reformulation, to the trade-off between the conflicting cost/revenue and reliability objectives, and to the scalarization parameter determining the relative importance of the objectives. Finally, we propose several MOPCP variants of multi-portfolio financial optimization models that implement a downside risk measure and can be used in a centralized or decentralized investment context. We study the impact of the model parameters on the portfolios, show, via a cross-validation study, the robustness of the proposed models, and perform a comparative analysis of the optimal investment decisions.

Key words: chance-constrained programming, variable reliability, Boolean programming, risk management, multi-portfolio optimization

1 Introduction

In this paper, we propose a new modeling and solution method for a class of multi-objective stochastic programming problems [31]. The formulation includes a joint probabilistic constraint with a random technology matrix which requires a system of inequalities to hold with some probability level. The probability level, often used to characterize the quality of service or the reliability of a system, is defined as a lower-bounded continuous decision variable. The generic formulation of the Multi-Objective Probabilistically Constrained Problems **MOPCP** reads:

$$\textbf{MOPCP}: \qquad \max \qquad q^\mathsf{T} x + \eta(p) \tag{1}$$
$$\text{subject to} \qquad Ax \geq b \tag{2}$$
$$\mathbb{P}(Tx \leq d) \geq p \tag{3}$$
$$p \geq \underline{p} \tag{4}$$
$$0 \leq x \leq u, \tag{5}$$

where T is an $[r \times |J|]$-matrix and its rows $T_1^\mathsf{T}, \ldots, T_r^\mathsf{T}$ are discretely distributed random vectors not necessarily independent. Each component of T_i is given by $t_{ij}\xi_j$, where t_{ij} is a fixed scalar and ξ_j is a random variable. The notation $|J|$ refers to the cardinality of the set J while u and d are $|J|$- and r-dimensional deterministic vectors. We denote by $\underline{p}$ the lowest acceptable reliability level

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*George Washington University, Washington, DC, USA; email: `mlejeune@gwu.edu`

†University of Michigan, Ann Arbor, MI, USA; email: `siqian@umich.edu`

and by $\mathbb{P}$ a probability measure. We assume that each coefficient t_{ij} is non-negative. As it will be shown after having introduced the reformulation method (see Remark 1), this assumption can be relaxed and our method can be generalized for $t_{ij} \in \mathbb{R}$. The decision variables $x_j, j \in J$ are non-negative and upper-bounded (see (5)). Constraint (2) defines a set of deterministic constraints, and (3) is a multi-row probabilistic constraint with a random technology matrix, ensuring that the r inequalities $T_i^\mathsf{T} x \le d_i$ $(i = 1, \ldots, r)$ hold jointly with a probability at least equal to p. The system of stochastic inequalities $Tx \le d$ in (3) includes $|J|$ inequalities of form $\xi_j x_j \le d_{r-|J|+j}, j \in J$.

The objective function is $q^\mathsf{T} x + \eta(p) : \mathbb{R}^{|J|+1} \to \mathbb{R}$ where $\eta(p)$ is a monotone increasing function in p. The decision variable p is lower-bounded by $\underline{p}$ in (4) and represents the enforced reliability level. It can be interpreted as the reliability of a quality control process [22] or the ready rate service level provided to customers [16]. As the reliability p increases, the probability of losing customers and the related loss of goodwill become lower, resulting in an increase of the overall benefits of a company. On the other hand, a larger p may trigger additional resources required for the company and raise additional operational and management costs. Ang and De Leon [1] advocate the use of a multi-objective chance-constrained formulation to decide whether an existing infrastructure for earthquake protection should be upgraded. They use a weighted objective function accounting for the degree of structural damage caused by earthquake, the cost of upgrading versus potential losses due to damages, and interpret p as a target reliability level for damage control and life safety. In this paper, we assume that the reliability component $\eta(p)$ of the objective function is linear in p, and the objective (1) is specified as $q^T x + ap$, where $a > 0$ is the scalarization parameter that defines a proper balance between the profit and reliability objectives [e.g., 8, 19]. The determination of the form $\eta(p)$ is problem-dependent and is a complicated issue beyond the scope of this study.

The first multi-objective chance-constrained problem, in which cost and reliability are two components of a scalarized objective function, is due to Evers [10]. Sengupta and Portillo-Campbell [28] suggest to define reliability levels as decision variables and propose a multi-objective chance-constrained production scheduling model. Prékopa [23] underlines the difficulty of solving **MOPCP** and suggests a parametric approximation approach. The method requires the recursive solution of a variant of **MOPCP** in which p is fixed and to choose the best solution obtained during the iterative process. Morgan et al. [21] propose a chance-constrained formulation which trades off pumping costs and reliability of an aquifer network. Rengarajan and Morton [24] and Rengarajan et al. [25] analyze the trade-off between the cost of building a network and the probability of an adversarial event causing a network failure, both of which motivate the use of stochastic multi-objective formulations with their ability in comprehending the risk tradeoff in network applications, and contrast this approach with a more traditional one in which the risk (or cost) level is fixed *a priori* and the other objective is optimized.

Additional motivations for studying **MOPCP** stem from the prevalence of various types of hidden costs (e.g., reputation damage) associated with a reliability level p, which are not necessarily taken into account in chance-constrained programs with fixed reliability level. Chapman and Harwood [5] as well as Sullivan and Kida [33] suggest that the risk faced by a company and the one faced by its managers may need separate consideration as manager's reputation might suffer from a highly visible mistake. The function $\eta(p)$ can then be used to capture the manager's reputation objective associated with a reliability level p. The function $\eta(p)$ can also represent a company's long-term reputation or brand name modelled as an increasing function of the quality of service level p delivered to the customers. Under this context, the objective function in **MOPCP** combines a short-term profit $q^\mathsf{T} x$ goal with the long-term brand name objective. Alternatively, $\eta(p)$ gives the flexibility of ignoring the violation of the constraints $Tx \le d$ in (3) with a variable probability level at most equal to $(1-p)$. The impact of the violation depends on the value of p and may significantly impact decision quality, especially in highly uncertain environments (e.g., high-

tech industries including oil drilling and renewable energy investment) where optimal decisions are sensitive with respect to p. Shen [30] investigates similar chance-constrained programs where the associated reliability guarantees are considered as decision variables. Moreover, the paper assumes individual chance constraints with each having a single row in the technology matrix, yielding a much simpler modeling mechanism and solution schemes.

The main contribution of this study is to develop new and exact formulations, as well as a solution method for the NP-hard multi-objective stochastic programming **MOPCP**. The **MOPCP** contains a joint probabilistic constraint in which the components of the multi-row random technology matrix are discretely distributed. A discrete representation of the uncertainty with a set of joint scenarios is frequently employed (see, e.g., [7, 26]), which permits to take into account dependencies among random variables, and can be derived by sampling from some underlying distribution of the uncertainty. While algorithmic methods have been proposed for single-objective stochastic problems including a joint chance constraint with random right-hand sides [e.g., 7, 13, 15, 17], and, most recently, with multi-row random technology matrix [2, 12], no efficient reformulation or solution method has been proposed, to the best of our knowledge, for the class of multi-objective stochastic programming problems considered in this study.

Moreover, our reformulation approach does not require the solution of multiple complex problems approximating **MOPCP** in which p is fixed and iteratively set to (all or) a series of acceptable values. This is in contrast with studies that perform a Pareto analysis of the "efficient frontier" of chance-constrained programming models. Two representative articles [24, 25] are reviewed above, where the authors vary p in (3) and compute the corresponding objective $q^\mathsf{T} x$ subject to (2)–(5) for a fixed p. Mitra et al. [20] use a similar approach to construct the efficient frontier of a chance-constrained supply chain planning problem in which the objectives are cost and demand satisfaction. Greenberg et al. [11] analyze the impact of changing the value of p in chance-constrained programs with recourse variables by constructing the response space frontier using generalized Lagrangian duality. All these studies require the solution of a chance-constrained program for a sufficiently large number of reliability choices, whereas we provide a nonparametric approach that simultaneously determines the optimal values of x and p through the solution of one single problem **MOPCP**.

We extend a recently proposed Boolean modeling framework [12, 15] to derive a series of new mixed-integer programming (MIP) formulations for **MOPCP**. The Boolean method involves the binarization of the probability distribution with a set of so-called cut points. This permits to represent a joint probabilistic constraint as a partially defined Boolean function, and to subsequently model its satisfiability with sets of mixed-integer linear inequalities. In a recent study, Kogan and Lejeune [12] employ the Boolean method to formulate equivalent mixed-integer formulations of a joint chance constraint with random technology matrix and fixed reliability level. By contrast, our study defines the reliability as a decision variable. This further compounds the computational challenge of solving the associated multi-objective problem and requires a significant extension of the Boolean framework [12]. We first introduce binary variables to model the determination of the optimal reliability level. By analyzing the special structure of the optimal solutions of **MOPCP**, we are able to further relax the integrality of the new binary variables and keep new variables and constraints in sizes that are polynomial to the number of cut points and to the number of scenarios considered for the random vector ξ. We also provide computational and modeling insights which cover the following aspects. First, we analyze the interplay between the objectives. Second, we assess the sensitivity of the solution with respect to the parameter defining the relative importance of each objective, which provides guidance to specify the value of this parameter. Additionally, we provide formulations for stochastic multi-portfolio optimization which balance return with downside risk and can be used in a centralized or decentralized investment fashion.

The remainder of the paper is organized as follows. In Section 2, we define the key components

of the Boolean modeling framework first introduced in [12, 15]. Section 3 revisits some fundamental Boolean concepts and the reformulation of the single-objective version of **MOPCP** in which the probability level p is fixed. Section 4 derives new equivalent mixed-integer linear programming (MILP) reformulations of the multi-objective **MOPCP** with variable p. Section 5 compares the computational tractability of the proposed formulations, and presents insights about the trade-off between objectives and the sensitivity of the solution. In Section 6, we conduct a comprehensive analysis of stochastic multi-portfolio models that can be used in decentralized and centralized investment contexts. Section 7 provides concluding remarks and future research directions.

2 Boolean Modeling

For self-containment purposes, we present succinctly the Boolean modeling framework that defines the satisfiability of a joint probabilistic constraint in terms of a partially defined Boolean function (pdBf). The method was first introduced to handle joint probabilistic constraints with dependent random right-hand sides [15]. It was also used in Lejeune [14] to elicit the exhaustive list of p-efficient points [see, e.g., 23], represented as combinatorial e^p-patterns, through solutions of an MILP problem. Most recently, the method was employed for deriving reformulations and exact solutions of stochastic programming problems with joint probabilistic constraints and multi-row random technology matrix [12]. We shall here expand the method for optimizing probabilistically constrained problems with random technology matrix in which the reliability level is a decision variable. The Boolean method involves the construction of the set of recombinations and the binarization of the probability distribution.

2.1 Construction of Set of Recombinations

Let ξ be the $|J|$-dimensional vector of distinct random variables in the matrix T. Further, we denote by Ω the support set of ξ. The set Ω contains all possible realizations of the $|J|$-random vector ξ with distribution function F. A realization k is represented by the $|J|$-numerical vector ω^k.

Definition 1. [14] A realization k is called p-**sufficient** if and only if $F(\omega^k) \geq p$ and is p-**insufficient** if $F(\omega^k) < p$.

A p-sufficient realization defines sufficient conditions for the probabilistic constraint (3) to hold. Let $T(\omega^k)$ represent the image of T when ξ is realized as ω^k. If k is p-sufficient, which implies that $F(\omega^k) \geq p$, we have: $T(\omega^k)x \leq d \Rightarrow \mathbb{P}(Tx \leq d) \geq p$. The dichotomy between p-sufficient and p-insufficient realizations gives a partition of the set Ω into two disjoint sets of p-sufficient $\Omega^+(p) = \left\{k \in \Omega : F(\omega^k) \geq p\right\}$ and p-insufficient $\Omega^-(p) = \Omega \setminus \Omega^+(p)$ realizations.

We generate all points that can be p-sufficient. These are called **recombinations** [12]. Let F_j be the marginal probability distribution of ξ_j. The inequalities

$$F_j(\omega_j^k) \geq p, \ j = 1, \ldots, |J| \tag{6}$$

define necessary conditions for (3) to hold: $\mathbb{P}(Tx \leq d) \geq p \Rightarrow F_j(\omega_j^k) \geq p, j = 1, \ldots, |J|$. The direct product $\bar{\Omega}(p) = C_1(p) \times \ldots \times C_j(p) \times \ldots \times C_{|J|}(p)$ of each set

$$C_j(p) = \left\{\omega_j^k : F_j(\omega_j^k) \geq p, \ k \in \Omega\right\}, \ j = 1, \ldots, |J| \tag{7}$$

provides the set $\bar{\Omega}(p)$ of recombinations containing all points that can possibly be p-sufficient. Each vector ω^k associated to a recombination $k \in \bar{\Omega}(p)$ satisfies the $|J|$ conditions defined by (6). The disjoint sets of p-sufficient and p-insufficient recombinations are respectively denoted by $\bar{\Omega}^+(p) = \{k \in \bar{\Omega}(p) : F(\omega^k) \geq p\}$ and $\bar{\Omega}^-(p) = \{k \in \bar{\Omega}(p) : F(\omega^k) < p\}$.

2.2 Binarization Process

We shall now **binarize** the probability distribution and the recombinations using **cut points**. The concept of cut point is frequently used in Boolean and combinatorial data mining methods [4, 34] to separate points belonging to different classes. The objective of the binarization process is first to represent the joint probabilistic constraint as a partially defined Boolean function, and to subsequently model its satisfiability with sets of mixed-integer linear inequalities derived by using the concept of minorant of a threshold Boolean function. The number of binary variables in the set of inequalities is equal to the number of cut points and is typically order of magnitude smaller than the number of possible realizations for the vector of random variables. Let n_j be the number of cut points associated with ξ_j and $n = \sum_{j \in J} n_j$ be the total number of cut points over all j.

Definition 2. [15] The binarization process is a mapping $\mathbb{R}^{|J|} \to \{0,1\}^n$ of a numerical vector ω^k into a binary vector $\beta^k = \left[\beta_{11}^k, \ldots, \beta_{1n_1}^k, \ldots, \beta_{jl}^k, \ldots, \beta_{jn_j}^k, \ldots \right]$, such that the value of each Boolean component β_{jl}^k is defined with respect to a cut point c_{jl} as follows:

$$\beta_{jl}^k = \begin{cases} 1 & \text{if } \omega_j^k \geq c_{jl} \\ 0 & \text{otherwise} \end{cases} , \tag{8}$$

where c_{jl} denotes the lth cut point associated with ξ_j,

$$l' < l \implies c_{jl'} < c_{jl} , \ j \in J, \ l = 2, \ldots, n_j, l' = 1, \ldots, n_j - 1 . \tag{9}$$

For any given p, the binarization of $\bar{\Omega}(p)$ creates the set $\bar{\Omega}_B(p) \subseteq \{0,1\}^n$ of relevant Boolean vectors, each of which is a binary mapping of a recombination. The cut points are arranged in ascending order in (9). Lemma 1 follows directly from the definition of the binarization process.

Lemma 1. The binarization process (8)–(9) generates a regularized set of Boolean vectors, i.e., for every component $\xi_j, j \in J$, if $c_{jl'} < c_{jl}$ for some $l, \ l' = 1, \ldots, n_j, \ l \neq l'$, then

$$\beta_{jl}^k \leq \beta_{jl'}^k , \ \forall k \in \bar{\Omega}(p) . \tag{10}$$

The binarization process must be based on a consistent set of cut points to preserve the disjointedness between binary projections $\bar{\Omega}_B^+(p)$ and $\bar{\Omega}_B^-(p)$ of $\bar{\Omega}^+(p)$ and $\bar{\Omega}^-(p)$. We employ the sufficient-equivalent consistent set of cut points [12]

$$C^e(p) = \bigcup_{j=1}^{|J|} C_j(p) , \tag{11}$$

with $C_j(p), j \in J$ defined by (7), that guarantees that no pair of p-sufficient and p-insufficient recombinations can have the same binary image. The set of relevant Boolean vectors is partitioned into disjoint sets of p-sufficient $\bar{\Omega}_B^+(p)$ and p-insufficient $\bar{\Omega}_B^-(p)$ relevant Boolean vectors. The binarization based on the sufficient-equivalent set of cut points defined in (11) ensures a one-to-one mapping between vectors ω^k and β^k associated with a recombination k. The binarization permits the derivation of a pdBf representing the satisfiability of the joint probabilistic constraint (2).

Definition 3. Let $\mathcal{T}(g)$ and $\mathcal{F}(g)$ denote the sets of respectively true and false points of a Boolean function. A Boolean function f defined by the pair of disjoint sets $(\mathcal{T}, \mathcal{F}) \subseteq \{0,1\}^n$ is a mapping $f : (\mathcal{T} \bigcup \mathcal{F}) \to \{0,1\}$ such that $f(k) = 1$ (resp., $f(k) = 0$) if k is a true (resp., false) point: $k \in \mathcal{T}$ (resp., $k \in \mathcal{F}$).

Definition 4. For any given p and two disjoint subsets $\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)$: $\bar{\Omega}_B^+(p) \bigcup \bar{\Omega}_B^-(p) = \bar{\Omega}_B(p) \subseteq \{0,1\}^n$, $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$ is a pdBf with sets of true points $\bar{\Omega}_B^+(p)$ and false points $\bar{\Omega}_B^-(p)$.

The set of p-sufficient (resp., p-insufficient) relevant Boolean vectors contains the true (resp., false) points. Theorem 1 provides a characterization of the set of relevant Boolean vectors. We refer to Kogan and Lejeune [12] for a detailed proof.

Theorem 1. Consider the binarization process obtained with C^e. The set $\bar{\Omega}_B(p)$ of relevant Boolean vectors is a binary projection of $\bar{\Omega}(p)$ and is given by

$$\bar{\Omega}_B(p) = \left\{ k : \beta^k \in \{0,1\}^n : \beta_{j1}^k = 1, \ \beta_{jl}^k \le \beta_{jl-1}^k, \ j \in J, \ l = 2, \ldots, n_j \right\}. \tag{12}$$

Example 1. Consider the following example with two decision variables and a four-row random technology matrix to illustrate how the Boolean method reformulates the satisfiability of the probabilistic constraint as a pdBf.

$$\max \quad 3x_1 + 2x_2 + 2p$$

$$\text{subject to} \quad \mathbb{P}\left\{ \begin{array}{l} \xi_1 x_1 + 2\xi_2 x_2 \le 13 \\ 5\xi_1 x_1 + 3\xi_2 x_2 \le 27 \\ \xi_1 x_1 \le 4 \\ \xi_2 x_2 \le 7 \end{array} \right\} \ge p \tag{13}$$

$$x_1, x_2 \ge 0, \ x_1 \le 1.5, \ x_2 \le 2.5$$

$$0.6 \le p \le 1$$

where $t_{11} = 1$, $t_{12} = 2$, $t_{21} = 5$, $t_{22} = 3$, $t_{31} = 4$, $t_{42} = 7$, $d_1 = 13$, $d_2 = 27$, $d_3 = 4$, $d_4 = 7$, random row vectors $T_1 = [\xi_1, 2\xi_2]$, $T_2 = [5\xi_1, 3\xi_2]$, $T_3 = [\xi_1, 0]$, $T_4 = [0, \xi_2]$, and $\eta(p) = 2p$.

Assume that $\xi = [\xi_1, \xi_2]$ has eight joint possible realizations denoted by $\omega^k = [\omega_1^k, \omega_2^k]$, $k = 1, \ldots, 8$. Table 1 displays the probability of each realization ω^k, the joint cumulative probability function F of $\xi = [\xi_1, \xi_2]$ and the marginal cumulative distribution functions F_i of ξ_i, $i = 1, 2$.

Table 1: Realizations ω^k of ξ and cumulative probabilities.

k	ω_1^k	ω_2^k	$\mathbb{P}\{\xi = \omega^k\}$	$F_1(\omega_1^k)$	$F_2(\omega_2^k)$	$F(\omega^k)$
1	2	2	0.5	0.77	0.65	0.5
2	2	4	0.05	0.77	0.78	0.55
3	2	5	0.22	0.77	1	0.77
4	3	2	0.05	0.87	0.65	0.55
5	3	3	0.05	0.87	0.7	0.6
6	4	1	0.04	0.94	0.04	0.04
7	4	4	0.03	0.94	0.78	0.72
8	5	2	0.06	1	0.65	0.65

For $p = 0.6$, the sufficient-equivalent set of cut points is:

$$C^e(0.6) = \{c_{11} = 2; c_{12} = 3; c_{13} = 4; c_{14} = 5; c_{21} = 2; c_{22} = 3; c_{23} = 4; c_{24} = 5\}.$$

Table 7 in Appendix A.1 displays the recombinations, their binary images (relevant Boolean vectors), and their partitioning into $\bar{\Omega}_B^-(0.6)$ and $\bar{\Omega}_B^+(0.6)$.

3 Reformulations for Probabilistically Constrained Programs with Fixed Reliability Level

Prior to deriving equivalent reformulations of **MOPCP** (Section 4), we shall first review some key Boolean concepts (Section 3.1) and model the feasible region of probabilistically constrained problems with multi-row random technology matrix and fixed reliability level (Section 3.2). These concepts and properties play a fundamental role in the derivation of reformulations equivalent to **MOPCP** in which the reliability level p is a decision variable.

3.1 Boolean Function and Minorant

The Boolean programming concepts (see Crama and Hammer [6] for a comprehensive overview of the Boolean programming discipline) presented below are used in Section 3.2 to characterize the feasible set of a probabilistically constrained problem with fixed reliability level.

Definition 5. [6] A function $f : \{0,1\}^n \to \{0,1\}$ is a **threshold Boolean function** if for all $(o_1, \ldots, o_n) \in \{0,1\}^n$, there exists a vector of weights $\lambda \in \mathbb{R}^n$ and a threshold $\theta \in \mathbb{R}$, such that

$$f(o_1, \ldots, o_n) = 1 \quad \text{if and only if} \quad \sum_{l=1}^{n} \lambda_l o_l \geq \theta \ . \tag{14}$$

The hyperplane $\left\{ o \in \{0,1\}^n : \sum_{l=1}^{n} \lambda_l o_l = \theta \right\}$ is a separator for f and the $(n+1)$-tuple (λ, θ) is the **separating structure** for the threshold Boolean function f.

Definition 6. A Boolean function f is a minorant of a pdBf $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$ if $\bar{\Omega}_B^-(p) \subseteq \mathcal{F}(f)$.

We shall use the concept of **tight minorant** to obtain a more precise characterization of the pdBf $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$.

Definition 7. A Boolean function f is a tight minorant of a pdBf $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$ if (i) f is a minorant of $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$ and (ii) the true set $\mathcal{T}(f)$ of f is such that: $\mathcal{T}(f) \bigcap \bar{\Omega}_B^+(p) \neq \emptyset$.

Since a threshold Boolean function is characterized by a separating structure, Definition 7 leads to Lemma 2.

Lemma 2. [12] A threshold Boolean function f defined by the separating structure $(\lambda, |J|)$ is a tight minorant of a pdBf $g(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p))$ if the system of inequalities

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ \beta_{jl}^k \geq |J|, \qquad \text{for at least one } k \in \bar{\Omega}_B^+(p) \tag{15}$$

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ \beta_{jl}^k \leq |J| - 1, \qquad k \in \bar{\Omega}_B^-(p) \tag{16}$$

has a feasible solution.

At least one condition must hold for each component of the random vector ξ in the chance constraint (3). The minimal number of non-zero weights in a threshold tight minorant of $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$ is therefore $|J|$, which validates the replacement of θ in (14) by $(|J|-1)$ in the right-hand side of (16). Importantly, Kogan and Lejeune [12] have shown that there exists a threshold tight minorant, and hence a feasible solution for (15)–(16), for any $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$ defined as in Definition 4.

3.2 Equivalent Reformulations

In this section, we assume that the probability level p in constraint (3) is fixed to a number in $(0, 1]$ and present several reformulations for the single-objective version of **MOPCP** with fixed p.

3.2.1 An Equivalent Mixed-Integer Nonlinear Formulation

We first derive an equivalent mixed-integer bilinear reformulation of the probabilistic constraint (3) when p is fixed. We provide the proof of the equivalence in Appendix B.

Theorem 2. Given $p \in (0, 1]$, every feasible solution of the system of inequalities

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl}\ \beta_{jl}^k \le |J| - 1, \qquad\qquad k \in \bar{\Omega}_B^-(p) \tag{17}$$

$$\sum_{l=1}^{n_j} \lambda_{jl} = 1, \qquad\qquad j \in J \tag{18}$$

$$\lambda_{jl} \in \{0, 1\}, \qquad\qquad j \in J,\ l = 1, \dots, n_j \tag{19}$$

defines a threshold tight minorant f with integral separating structure $(\lambda, |J|) \in \{0,1\}^n \times \mathbb{Z}_+$. Every point k such that $\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl}\ \beta_{jl}^k = |J|$ belongs to $\bar{\Omega}_B^+(p)$.

Theorem 2 shows that we do not need to include the disjunction (15) to generate a tight minorant of $g\left(\bar{\Omega}_B^+(p), \bar{\Omega}_B^-(p)\right)$. This permits to derive a quadratic mixed-integer set representing the feasible region of (3) with fixed p.

Theorem 3. The feasible set defined by the probabilistic constraint (3) with multi-row random technology matrix and fixed reliability level p is equivalent to the feasible set defined by the system **S1** of mixed-integer quadratic inequalities

$$\textbf{S1:} \qquad \begin{array}{c} (17)\ ;\ (18)\ ;\ (19) \\[6pt] \sum_{j \in J} t_{ij} x_j \left(\sum_{l=1}^{n_j} \lambda_{jl}\ c_{jl} \right) \le d_i, \quad i = 1, \dots, r\ . \end{array} \tag{20}$$

Proof. We show the equivalence between **S1** and the chance constraint (3) by verifying two facts, indicated as **Result (i)** and **Result (ii)**.
Result (i): Any solution feasible for **S1** is feasible for (3). Due to constraints (17), (18) and (19), it follows from Theorem 2 that any $\hat{\lambda} \in \{\lambda : (17), (18), (19)\}$ defines a separating structure $(\hat{\lambda}, |J|)$ of a threshold tight minorant f. Hence, we define

$$G = \left\{ k \in \bar{\Omega}_B(p) : \sum_{j \in J} \sum_{l=1}^{n_j} \hat{\lambda}_{jl}\ \beta_{jl}^k = |J| \right\},$$

and we have that $G \subseteq \bar{\Omega}_B^+(p)$. Let $L = \left\{ (j, l) : \hat{\lambda}_{jl} = 1, j \in J, l = 1, \dots, n_j \right\}$. For any $k \in G$, the binarization process (8) implies that

$$\omega_j^k \ge c_{jl},\ (j, l) \in L. \tag{21}$$

Further, we have from the definition of the sufficient-equivalent set of cut points (11) that $\exists\, k' \in G$ such that:

$$\omega_j^{k'} = c_{jl},\ (j, l) \in L\ . \tag{22}$$

Constraints (18) and (19) ensure that exactly one term $\hat{\lambda}_{jl}c_{jl}$ in each summation $\sum_{l=1}^{n_j} \lambda_{jl}c_{jl}$ of constraint (20) is non-zero and equal to $c_{jl}, (j,l) \in L$. Hence, (22) can be rewritten as

$$\omega_j^{k'} = c_{jl} = \sum_{i=1}^{n_j} \lambda_{jl} \; c_{jl}, \; (j,l) \in L. \tag{23}$$

Recall that every component of T is: $t_{ij} \; \xi_j, i = 1, \ldots, r, j \in J$:

$$\mathbb{P}(Tx \leq d) \; = \; \mathbb{P}\left(\sum_{j \in J} t_{ij} \; \xi_j \; x_j \leq d_i, \; i = 1, \ldots, r \right). \tag{24}$$

Given that x_j and t_{ij} are non-negative, and $k' \in G \subseteq \bar{\Omega}^+(p)$, we have $\mathbb{P}\left(\xi \leq \omega^{k'} \right) \geq p$ and thus $\mathbb{P}\left(\sum_{j \in J} t_{ij} \; \xi_j \; x_j \leq \sum_{j \in J} t_{ij} \; \omega_j^{k'} \; x_j, \; i = 1, \ldots, r \right) \geq p$. This, combined with (24), implies that

$$\sum_{j \in J} t_{ij} \; \omega_j^{k'} \; x_j \leq d_i, \; i = 1, \ldots, r \quad \Rightarrow \quad \mathbb{P}\left(\sum_{j \in J} t_{ij} \; \xi_j \; x_j \leq d_i, \; i = 1, \ldots, r \right) \geq p. \tag{25}$$

which, together with (23), further implies that

$$\sum_{j \in J} t_{ij} \; x_j \left(\sum_{l=1}^{n_j} \lambda_{jl} \; c_{jl} \right) \leq d_i, \; i = 1, \ldots, r \quad \Rightarrow \quad \mathbb{P}(Tx \leq d) \geq p. \tag{26}$$

Result (ii): Any solution feasible for (3) is feasible for **S1**. For any k'' such that $\mathbb{P}(\xi \leq \omega^{k''}) \geq p$, the definition of the sufficient-equivalent set of cut points implies that there exists $k' \in \bar{\Omega}_B^+(p)$ such that $\omega^{k'} \leq \omega^{k''}$ and $\omega_j^{k'} = \bigvee\limits_{l=1,\ldots,n_j} c_{jl}, \; j \in J$. Let $l_j^* = \arg \max\limits_{l} \{c_{jl} : c_{jl} = \omega_j^{k'}\}, \; j \in J$. What is left to prove is that the vector λ'

$$\lambda'_{jl} = \begin{cases} 1 & \text{if } l = l_j^* \\ 0 & \text{otherwise} \end{cases} , \; j \in J \tag{27}$$

is feasible for **S1**. Clearly, λ' is feasible for (18), (19) and (20). From (27), we have λ' feasible for (17) if

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda'_{jl} \; \beta_{jl}^k = \sum_{j \in J} \lambda'_{jl_j^*} \; \beta_{jl_j^*}^k = \sum_{j \in J} \beta_{jl_j^*}^k \leq |J| - 1, \; k \in \bar{\Omega}_B^-(p). \tag{28}$$

The feasibility of the above constraints is ensured if $\beta_{jl_j^*}^k = 0$ for at least one $j \in J$, $\forall k \in \bar{\Omega}_B^-(p)$.

For any $k' \in \bar{\Omega}_B^+(p)$, there is no $k \in \bar{\Omega}_B^-(p)$ such that $\beta^k \geq \beta^{k'}$ (see Definition (11) for the sufficient-equivalent set of cut points). This implies that

$$(\beta_{j1}^k, \ldots, \beta_{jn_j}^k) < (\beta_{j1}^{k'}, \ldots, \beta_{jn_j}^{k'}) \text{ for at least one } j, \; \forall k \in \bar{\Omega}_B^-(p).$$

Let $h \in J$ be an index such that $(\beta_{h1}^k, \ldots, \beta_{hn_h}^k) < (\beta_{h1}^{k'}, \ldots, \beta_{hn_h}^{k'})$ for an arbitrary $k \in \bar{\Omega}_B^-(p)$. Since $l_h^* = \arg \max\limits_{l} \beta_{hl}^{k'} = 1$, we have for any $l > l_h^*$ that $\beta_{hl}^k = \beta_{hl}^{k'} = 0$. Thus, vectors $\beta^{k'}$ and β^k differ only in the first l_h^* components, and we have

$$(\beta_{h1}^k, \ldots, \beta_{hl_h^*}^k) < (\beta_{h1}^{k'}, \ldots, \beta_{hl_h^*}^{k'}).$$

The regularization property (see Lemma 1) indicates that this relationship can only be true if $\beta_{hl_h^*}^k = 0 < \beta_{hl_h^*}^{k'} = 1$. This shows that, for any $k \in \bar{\Omega}_B^-(p)$, $\beta_{jl_j^*}^k = 0$ for at least one j, which results in $\sum_{j \in J} \beta_{jl_j^*}^k$ to be bounded from above by $(|J| - 1)$ for each $k \in \bar{\Omega}_B^-(p)$ and implies that λ' is feasible for (28) and thus (17). This provides the result that we set out to prove. $\qquad\square$

Remark 1. Result (i) in the proof of Theorem 3 uses the assumption made in Section 1 that each coefficient t_{ij} is non-negative. This assumption is non-restrictive and was made to ease the notations. Indeed, if one (or more) of the coefficients t_{ij} is negative, we rewrite $t_{ij} = t_{ij}^+ - t_{ij}^-$ with t_{ij}^+, $t_{ij}^- \geq 0$, introduce a random vector $\tilde{\xi} = -\xi$, and then rewrite all entries in Tx as $t_{ij}^+ x_j \xi_j + t_{ij}^- x_j \tilde{\xi}_j$ in (3). The proof of Theorem 3 can be derived the same way on that basis.

In Appendix A.2, we provide the reformulation yielded by Theorem 3 for the probabilistic constraint in Example 1.

3.2.2 Equivalent Mixed-Integer Linear Programming Reformulations

We now propose several linearization approaches for the bilinear constraints (20) in **S1**.

We first linearize the bilinear terms $x_j \lambda_{jl}$ in (20) by using the inequalities proposed by Mc-Cormick [18]. Note that each contains a binary variable λ_{jl} and a non-negative continuous variable x_j. Define $y_{jl} \equiv x_j \lambda_{jl}$, $j \in J$, $l = 1, \ldots, n_j$, and enforce the relationship with the inequalities

$$y_{jl} \leq u_j \lambda_{jl} \tag{29}$$
$$y_{jl} \leq x_j \tag{30}$$
$$y_{jl} \geq x_j - u_j(1 - \lambda_{jl}) \tag{31}$$
$$y_{jl} \geq 0, \tag{32}$$

where u_j is the upper bound of variable $x_j, j \in J$. Constraints (29) and (32) together force y_{jl} to take value 0 if $\lambda_{jl} = 0$, whereas (30) and (31) imply that $y_{jl} = x_j \times 1 = x_j$ if $\lambda_{jl} = 1$.

Lemma 3. The feasible set of the probabilistic constraint (3) with a fixed p and the feasible set defined by the mixed-integer quadratic inequalities in **S1** can be equivalently represented by

$$\textbf{S2:} \quad \begin{array}{l} (17) \ ; \ (18) \ ; \ (19) \ ; \ (29) \ ; \ (30); \ (31) \ ; \ (32) \\[2mm] \sum_{j \in J} t_{ij} \left(\sum_{l=1}^{n_j} y_{jl} \ c_{jl} \right) \leq d_i, \qquad\qquad i = 1, \ldots, r. \end{array} \tag{33}$$

Note that **S2** only contains mixed-integer linear inequalities. Furthermore, we can reduce the number of binary integer variables in **S2** by noting that $\sum_{l=1}^{n_j} \lambda_{jl} = 1$. Thus, for every $j \in J$, only one of the variables y_{jl} is equal to x_j while all the other y_{jl} are equal to zero. Lemma 4 provides an alternative MILP formulation that is equivalent to **S2**.

Lemma 4. Given a fixed reliability level p, the feasible sets defined by constraint (3), system **S1**, and system **S2** are equivalent to the following system of mixed-integer inequalities:

$$\textbf{S3:} \quad \begin{array}{l} (17) \ ; \ (18) \ ; \ (19) \\[2mm] \sum_{j \in J} t_{ij} z_j \leq d_i \ , \qquad\qquad i = 1, \ldots, r \\[2mm] z_j \geq c_{jl} x_j - M_{jl}(1 - \lambda_{jl}) \ , \quad j \in J, \ l = 1, \ldots, n_j \\[2mm] z_j \leq c_{jl} x_j + M_{jl}(1 - \lambda_{jl}) \ , \quad j \in J, \ l = 1, \ldots, n_j, \end{array} \tag{34} \tag{35} \tag{36}$$

where $M_{jl} = c_{jl}(\bar{x}_j - \underline{x}_j)$ with $\bar{x}_j$ and $\underline{x}_j$ being the (sharpened) upper and lower bounds for x_j. For all $j \in J$, $l = 1, \ldots, n_j$, we have:

- if $c_{jn_j} > c_{j1} > 0$, then

$$\overline{x}_j = \min\left\{u_j, \ \max_{i:t_{ij}\neq 0}\left\{\frac{d_i}{t_{ij}c_{j1}} : d_i > 0\right\}\right\}, \text{ and } \underline{x}_j = \max\left\{0, \ \min_{i:t_{ij}\neq 0}\left\{\frac{d_i}{t_{ij}c_{jn_j}} : d_i > 0\right\}\right\}.$$

- if $0 > c_{jn_j} > c_{j1}$, then

$$\overline{x}_j = \min\left\{u_j, \ \max_{i:t_{ij}\neq 0}\left\{\frac{d_i}{t_{ij}c_{jn_j}} : d_i < 0\right\}\right\}, \text{ and } \underline{x}_j = \max\left\{0, \ \min_{i:t_{ij}\neq 0}\left\{\frac{d_i}{t_{ij}c_{j1}} : d_i < 0\right\}\right\}.$$

- if $c_{jn_j} > 0 > c_{j1}$, then

$$\overline{x}_j = \min\left\{u_j, \ \max_{i:t_{ij}\neq 0}\left\{\frac{d_i}{t_{ij}c_{jn_j}} : d_i > 0\right\}\right\}, \text{ and } \underline{x}_j = \max\left\{0, \ \min_{i:t_{ij}\neq 0}\left\{\frac{d_i}{t_{ij}c_{j1}} : d_i < 0\right\}\right\}.$$

Proof. Let $z_j = \sum_{l=1}^{n_j} y_{jl}c_{jl}, j \in J$. Due to constraints (18) (i.e., $\sum_{l=1}^{n_j} \lambda_{jl} = 1, \ j \in J$), exactly one variable $\lambda_{jl}, \ l = 1, \ldots, n_j$ can take value 1 for each $j \in J$. Thus, for all $j \in J$

$$z_j = \sum_{l=1}^{n_j} y_{jl} \ c_{jl} = \sum_{l=1}^{n_j}(x_j\lambda_{jl})c_{jl}$$

takes the value of $x_j c_{jl}$ for some $l \in \{1, \ldots, n_j\}$ if the corresponding $\lambda_{jl} = 1$. As a result,

$$z_j \geq c_{jl}x_j - c_{jl}u_j(1 - \lambda_{jl}) , \quad j \in J, \ l = 1, \ldots, n_j \tag{37}$$
$$z_j \leq c_{jl}x_j + c_{jl}u_j(1 - \lambda_{jl}) , \quad j \in J, \ l = 1, \ldots, n_j, \tag{38}$$

enforce $z_j = x_j c_{jl}$ for some l when $\lambda_{jl} = 1$, and are not binding when the solution does not pick the corresponding cut point (i.e., when $\lambda_{jl} = 0$).

We strengthen the valid inequalities (37)–(38) by reducing the value of the big-M coefficients u_j and setting each to $(\overline{x}_j - \underline{x}_j), j \in J$. Recall that the cut points are sorted out in ascending order (see (9)). To establish the validity of the given upper and lower bounds of $x_j, j \in J$, we distinguish three cases: *(i)* $c_{jn_j} > c_{j1} > 0$, *(ii)* $0 > c_{jn_j} > c_{j1}$, and *(iii)* $c_{jn_j} > 0 > c_{j1}$. We substitute $z_j, \ j \in J$ for $\sum_{l=1}^{n_j} y_{jl} \ c_{jl}$ in (33), and replace the McCormick inequalities (29)–(32) by (35) and (36). This completes the proof. $\qquad\square$

4 Reformulations of MOPCP with Variable Reliability Level

In this section, we shall use the results presented in Theorem 3, Lemma 3, and Lemma 4 to derive reformulations (Sections 4.2–4.3) equivalent to the multi-objective probabilistic problem **MOPCP** in which the reliability level p is a decision variable. The concept of special ordered set of type 1 (SOS1) variables is employed in Section 4.4 for that purpose.

4.1 Backbone Formulation for MOPCP

Using the system of inequalities **S3**, we propose in Lemma 5 (directly implied by Lemma 4) an MILP formulation **MIO**, whose solutions are feasible for **MOPCP** with variable p. Note that instead of **S3**, we could rely upon **S1** and **S2**.

Lemma 5. Any solution (x, λ, z) feasible for the mixed-integer optimization problem

$$\textbf{MIO}: \max \qquad q^{\mathsf{T}} x$$
$$\text{subject to} \quad (2) \ ; \ (5) \ ; \ (18) - (19) \ ; \ (34) - (36)$$
$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k \leq |J| - 1, \qquad k \in \bar{\Omega}_B^-(\underline{p}) \qquad (39)$$

is feasible for **MOPCP**.

Any solution feasible for **MIO** has a reliability level at least equal to $\underline{p}$ and is therefore feasible for **MOPCP**. We shall now use Lemma 4 to obtain equivalent reformulations for **MOPCP** in which p is a decision variable. Problem **MIO** will serve as the backbone formulation in which additional constraints (see Sections 4.2–4.4) are introduced.

4.2 Equivalent Formulation I for MOPCP

A few comments are in order before proceeding to the reformulation of **MOPCP**. First, the sufficient-equivalent set of cut points $C^e(p)$ corresponding to the reliability level $\underline{p}$ is a superset of $C^e(p)$ for any $p \geq \underline{p}$. Second, each set $\bar{\Omega}_B^-(p)$, $p \geq \underline{p}$ of p-insufficient recombinations is a superset of the set $\bar{\Omega}^-(\underline{p})$. Finally, there exists a finite number of acceptable reliability levels $p \geq \underline{p}$. This number can be calculated up-front and is upper bounded by $|\bar{\Omega}^+(\underline{p})|$. We denote by $\tilde{\Omega}_B^-(\underline{p}) \subseteq \bar{\Omega}_B^-(\underline{p})$ the set of non-dominated p-insufficient recombinations that is obtained through the construction of a partial order over the set of p-insufficient recombinations.

Definition 8. Consider the set $\bar{\Omega}_B^-(\underline{p})$ of p-insufficient recombinations. For any $k, \ k' \in \bar{\Omega}_B^-(\underline{p})$, a partial order $\preceq$ defined on $\bar{\Omega}_B^-(\underline{p})$

$$k \preceq k' \quad \Leftrightarrow \quad \beta^k \leq \beta^{k'} .$$

is consistent with problem **MIO**.

The above partial order is implied by the componentwise inequality $\leq$ in the space of p-insufficient recombinations. If the above relationship holds, the p-insufficient recombination k is said to be dominated by $k' : k \in \tilde{\Omega}_B^-(\underline{p})$.

Let S be the set of acceptable reliability levels $h_s : h_s \geq \underline{p}, \ s = 1, \ldots, |S|$. Assume w.l.o.g. that the levels h_s are sorted in an ascending order with $\underline{p} = h_1$. Let $H_s = \{k : \mathbb{P}(\xi \leq \omega^k) = h_s\}, \ s \in S$ denote the set of h_s-sufficient recombinations with cumulative probability equal to h_s. We set $H_0 = \tilde{\Omega}_B^-(\underline{p})$. Let α_s be a binary variable associated with each set H_s. Each variable $\alpha_s, s \in S$ is forced to take value 0 if the solution of the optimization problem has a reliability level strictly smaller than h_s, and can take value 1 otherwise. Since the lowest acceptable reliability level is $\underline{p} = h_1$, we fix α_1 to 1. The above notations and decisions variables permit to rewrite $p = \sum_{s \in S}(h_s - h_{s-1})\alpha_s$ in terms of h_s and α_s, with $h_0 = 0$. As a result, $p = h_s$ if $\alpha_i = 1, i = 1, \ldots, s$ and $\alpha_i = 0, i = s+1, \ldots, |S|$.

Theorem 4. The MILP formulation

$$\textbf{EMP1}: \max \qquad q^{\mathsf{T}} x + a \sum_{s \in S}(h_s - h_{s-1})\alpha_s \qquad (40)$$
$$\text{subject to} \quad (2) \ ; \ (5) \ ; \ (18) - (19) \ ; \ (34) - (36)$$
$$\alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k \leq |J| \qquad k \in H_t, \ t < s \in S \qquad (41)$$

$$\alpha_1 = 1 \tag{42}$$

$$0 \le \alpha_s \le 1 \qquad\qquad s \in S, \ s \ne 1 \tag{43}$$

is equivalent to **MOPCP**.

Proof. Lemme 5 indicates that the feasible set of **MOPCP** is defined by constraints (2), (5), (18), (19), (34)–(36), and (39). Therefore, we only need to demonstrate that the new constraints (41)–(43) inserted in **EMP1** enforce (39) without cutting any feasible solution for **MOPCP**, and permit to capture the reliability level corresponding to a solution (x, z, λ, α) feasible for **EMP1**.
(i) Theorem 2 indicates that for any arbitrary vector $\tilde{\lambda}$ feasible for (18)–(19) and (39), there exists at least one $k \in \bar{\Omega}_B^+(\underline{p})$, such that: $\sum_{j \in J} \sum_{l=1}^{n_j} \tilde{\lambda}_{jl} \ \beta_{jl}^k = |J|$. The construction of the set of sufficient-equivalent cut points further implies that for any feasible solution (and the corresponding integral separation structure $(\tilde{\lambda}, |J|)$), there exists $k^* \in \bar{\Omega}_B^+(\underline{p})$ such that we have for each $j \in J$:

$$\beta_{jl}^{k^*} = 1 \ , \ l = 1, \dots, l_j^*, \ \text{and} \ \beta_{jl}^{k^*} = 0, \ l > l_j^* \ \text{with} \ l_j^* \ \text{such that:} \ \tilde{\lambda}_{jl_j^*} = 1. \tag{44}$$

Given that $\mathbb{P}(\xi \le \omega^{k^*}) = h_s$, $k^* \in H_s \subseteq \bar{\Omega}_B^+(\underline{p})$, the reliability level that can be achieved with $\tilde{\lambda}$ is h_s. It follows from Theorem 2 that

$$\sum_{j \in J} \sum_{l=1}^{n_j} \tilde{\lambda}_{jl} \ \beta_{jl}^k \le |J| - 1, \ k \in \bar{\Omega}_B^-(h_s) \ , \tag{45}$$

and therefore that α_s can take value 1 in each of the inequalities:

$$\alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \tilde{\lambda}_{jl} \ \beta_{jl}^k \le |J|, \ k \in \bar{\Omega}_B^-(h_s) \ . \tag{46}$$

Note that the variable α_s will always take value 1 when allowed, since **EMP1** is a maximization problem and all coefficients $(h_s - h_{s-1})$ of the variables α_s in (40) are non-negative.
(ii) In order for the variable α_s to be an indicator of whether the reliability level h_s is reached ($\alpha_s = 1$) or not ($\alpha_s = 0$), we must also verify the solution $\tilde{\lambda}$ defined in (44) forces $\alpha_r = 0, \forall r \ge s$. For $r = s$, one can immediately see that k^* (see (44)) is such that $\sum_{j \in J} \sum_{l=1}^{n_j} \tilde{\lambda}_{jl} \ \beta_{jl}^{k^*} = |J|$, which implies that α_s must be equal to 0. For $r > s$, $k^* \in \bar{\Omega}_B^-(h_r)$, and for any $r > s$, the inequality

$$\alpha_r + \sum_{j \in J} \sum_{l=1}^{n_j} \tilde{\lambda}_{jl} \ \beta_{jl}^{k^*} \le |J| \ , \tag{47}$$

forces α_r to take value 0 for each $r > s$. This is what was set out to prove and justifies the substitution of (45) and (43) for (39).
This result, along with part *(i)* in which it is shown that each variable α will always take value 1 if allowed, explains why the variables α do not have to be defined as binary ones and can be defined on $[0, 1]$ as in (43). It follows that the second term of $a \sum_{s \in S}(h_s - h_{s-1})\alpha_s$ in (40) defines the reliability level corresponding to a feasible solution $(x, z, \lambda, \gamma, \alpha)$.
(iii) Finally, we identify a number of redundant constraints in (45). The preorder $k \preceq k' \Leftrightarrow \beta^k \le \beta^{k'}$ in Definition 8 implies that

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ \beta_{jl}^k \le \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ \beta_{jl}^{k'} \quad \text{for any } \lambda_{jl} \in \{0, 1\} \ . \tag{48}$$

This implies that if the k'-inequality (46) holds, then the k-inequality (46) is satisfied too:

$$\alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k'} \leq |J| \Rightarrow \alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k} \leq |J| \text{ for any feasible } \lambda \ .$$

Thus, each constraint (46) associated with a dominated $k \in \bar{\Omega}_B^-(h_s)$ is redundant, and we can replace the set of constraints (46) by the more parsimonious one (41). This completes the proof. $\qquad\square$

The **EMP1** model has two most remarkable features. First, the number of binary variables in **EMP1** is not an increasing function of the number of scenarios and is typically much smaller than the number of scenarios. Second, the number of binary variables is equal to the number of cut points used in the binarization process. This means that the MILP reformulations of the multi-objective stochastic **MOPCP** have the same number of binary variables as the MILP reformulations of the single-objective (i.e., p fixed) chance-constrained problem studied in [12].

4.3   Equivalent Formulation II for MOPCP

In this section, we introduce a new MILP formulation in which we introduce precedence constraints and further preprocess the of set covering constraints (41).

Theorem 5. The MILP problem

$$\textbf{EMP2}: \quad \max \qquad q^\mathsf{T} x + a \sum_{s \in S : s \geq 1} (h_s - h_{s-1}) \alpha_s$$

$$\text{subject to} \quad (2) \ ; \ (5) \ ; \ (18) - (19) \ ; \ (34) - (36) \ ; \ (42) - (43)$$

$$\alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ \beta_{jl}^{k} \leq |J| \qquad\qquad k \in H_{s-1}, \ s \in S \quad (49)$$

$$\alpha_s \leq \alpha_{s-1} \qquad\qquad s \in S \qquad\qquad (50)$$

is equivalent to **MOPCP**.

Proof. To establish the equivalence between **EMP1** and **EMP2**, we show that the constraints (41) are implied by the constraints (49)–(50) in **EMP2**.
First, we observe that constraints (41) are implied by their subset

$$\alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ \beta_{jl}^{k} \leq |J|, \ k \in H_t, \ t < s \in S \qquad\qquad (51)$$

when $s = 1$ and thus $k \in H_0 = \tilde{\Omega}_B^-(p)$. For any $s \in S$ and $t < s$, the recombinations in H_t are h_s-insufficient, and thus belong to $\bar{\Omega}_B^-(h_s)$. Therefore, if $\alpha_s = 1$ is feasible for (51), we recover $\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq |J| - 1$, for all $k \in \bar{\Omega}_B^-(h_s)$, and thus λ is feasible for (18)–(19), and (51). It follows from Theorem 3 that every solution $(\mathbf{x}, \lambda, \mathbf{z})$ feasible for (18)–(19), (34)–(36), and (51) satisfies $\mathbb{P}(Tx \leq d) \geq \mathbf{h_s}$.
Next, consider a feasible solution (x, z, λ, α) for **EMP2** with $\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k*} = |J|$ for some $k^* \in H_{s^*}$ with binary image β_{il}^{k*}, $j \in J$, $l = 1, \ldots, n_j$ defined by (44). For any $k \in H_{s^*-1}$, we have $\beta_{jl}^k < \beta_{jl}^{k*}$ for at least one component (j, l). Therefore, we have

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k < \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k*} = |J| \qquad \forall k \in H_{s^*-1},$$

14

and (49) allows $\alpha_{s^*} = 1$. The precedence constraints (50) enforce $\alpha_t = 1$, $t = 1, \ldots, s^* - 1$. Finally, consider the constraints (51) ($s \in S$ and $t < s - 1$) that are not included in (49). Note that for any $k_t \in H_t$ with $h_t < h_{s-1}$, one can find at least one $k \in H_{s-1}$ such that $\beta_{jl}^{k_t} \leq \beta_{jl}^k$ and $\beta_{j'l'}^{k_t} < \beta_{j'l'}^k$ for at least one (j', l'). Therefore, we have

$$\sum_{j \in J} \sum_{l=1}^{n_j} \beta_{jl}^{k_t} < \sum_{j \in J} \sum_{l=1}^{n_j} \beta_{jl}^k \;\Rightarrow\; \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k_t} \leq \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k$$

since the variables λ_{jl} are non-negative. By adding α_s to both sides, we obtain

$$\alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^{k_t} \leq \alpha_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \beta_{jl}^k \leq |J| \qquad \forall k_t \in H_t, \; t < s - 1, \; s \in S$$

which shows that the constraints (51) are implied by the constraints (49). This demonstrates the equivalence between **EMP2** and **EMP1** and completes the proof. $\square$

In Example 1, there are nine distinct acceptable probability levels in the interval $[60\%, 100\%]$. We provide in Appendix A.3 the **EMP2** formulation for Example 1. Note that the elimination of the implied constraints in (41) and their replacement by (49)–(50) reduce significantly the number of constraints in the MILP formulation. In Section 5, we demonstrate the computational efficacy of **EMP2** with a set of randomly generated **MOPCP** instances.

4.4 Equivalent Formulations with SOS1 Variables

In this section, we use the concept of Special Ordered Set Variables of Type One (SOS1) to provide equivalent reformulations to those presented in the previous sub-sections. Let $\pi_s, s \in S$ denote a binary variable taking value 1 if the attained probability level is h_s and 0 otherwise.

Theorem 6. The MILP problem

$$\textbf{EMP1} - \textbf{SOS1}: \quad \max \qquad q^\mathsf{T} x + a \sum_{s=1}^{|S|} h_s \pi_s$$

$$\text{subject to} \quad (2) \; ; \; (5) \; ; \; (18) - (19) \; ; \; (34) - (36)$$

$$\pi_s + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \; \beta_{jl}^k \leq |J| \qquad k \in H_t, \; t < s \in S \qquad (52)$$

$$\sum_{s=1}^{|S|} \pi_s = 1 \qquad\qquad\qquad (53)$$

$$0 \leq \pi_s \leq 1 \qquad\qquad s \in S. \qquad (54)$$

is equivalent to **MOPCP**.

Proof. Recall that set $H_s = \{k : \mathbb{P}(\xi \leq \omega^k) = h_s\}$, $s = 1, \ldots, |S|$ and $H_0 = \tilde{\Omega}_B^-(\underline{p})$. Constraints (53) with binary requirements $\pi_s \in \{0, 1\}, s \in S$ indicate that we can only select one value among $\{h_1, \ldots, h_S\}$ as the reliability level p. Moreover, we can relax the binary constraints and include constraints (54) by noting that exactly one $\pi_s, s \in S$ takes value one at optimum. This is due to the fact that we maximize the objective value in **EMP1-SOS1** and the coefficients $h_s, s \in S$ are strictly positive.

Suppose that $\pi_{s'} = 1$ for some s', then for any $t < s'$ and any $k \in H_t$, (52) is enforced as

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k \leq |J| - 1.$$

Clearly, we have $\bigcup_{t < s_1} H_t \subseteq \bigcup_{t < s_2} H_t$, for any $s_1 < s_2 \in S$. Thus any $k \in H_t$, $t < s \leq s'$ that is included in an inequality for π_s, is also included in an inequality for $\pi_{s'}$. For other $k \in H_t$ where $t < s$ but $s > s'$, constraints (52) are not binding since we only require $\mathbb{P}(Tx \leq d) \geq h_{s'}$ as (3) in **MOPCP**. The rest of the proof is similar to the proof for Theorem 4. $\square$

Following procedures and arguments similar to those presented in Section 4.3, we provide another equivalent **MOPCP** reformulation by replacing the variables $\alpha_s, s \in S$ in **EMP2** with the SOS1 variables $\pi_s, \ s \in S$, yielding

$$\textbf{EMP2} - \textbf{SOS1}: \quad \max \qquad q^{\mathsf{T}} x + a \sum_{s=1}^{|S|} h_s \pi_s$$

$$\text{subject to} \quad (2) \ ; \ (5) \ ; \ (18) - (19) \ ; \ (34) - (36) \ ; \ (53) - (54)$$

$$\sum_{t \geq s} \pi_t + \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k \leq |J| \qquad\qquad k \in H_{s-1}, \ s \in S$$

5 Computational Tractability and Model Sensitivity

As each variable α_s always takes an integer value in the optimal solution of **EMP1**, one can solve the reformulation **EMP1** of **MOPCP** more efficiently and precisely as compared to either directly tackling formulation (1)–(5) or using parametric methods that heuristically vary the inputs of the reliability variable p. Problem **EMP2** further improves the formulation **EMP1** by eliminating many redundant constraints and variables, leading to a significant reduction in the CPU time. The sizes of all our MILP formulations depend on the number r of rows and $|J|$ of columns in $Tx \leq d$, as well as the number of cut points (i.e., $|C_j(\underline{p})|, j \in J$). The latter does not only affect the number of binary variables λ_{jl}, but more importantly, bounds the number of reliability thresholds, i.e., $|S| \leq |\bar{\Omega}^+(\underline{p})| < |\bar{\Omega}(\underline{p})|$ that is further bounded by $|C_1(\underline{p})| \times \cdots \times |C_{|J|}(\underline{p})|$. Different from reformulations based on efficient points, reformulations based on p-sufficient points have much fewer number of variables and constraints (both being bounded by $|S|$). In practice, decision makers often trade off among relatively high reliability levels (e.g., $p \geq 95\%$). As a result, the number of realizations in Ω that can violate the joint chance constraint is quite small, since $|C_j(\underline{p})| \leq \lfloor(1 - \underline{p})|\Omega|\rfloor, j \in J$, of which the latter is a very small value for large p and linear in $|\Omega|$. Therefore, the number $|S|$ of threshold values is bounded by a polynomial function $\left(\lfloor(1 - \underline{p})|\Omega|\rfloor\right)^{|J|}$ where $|J|$ is fixed.

In this section, we evaluate the computational performance of **EMP1**, **EMP2**, **EMP1-SOS1**, and **EMP2-SOS1** on randomly generated **MOPCP** instances. The goals are *(i)* to demonstrate the computational tractability of the developed formulations and *(ii)* to derive insights about the impact of changes of relative weights attributed to the multiple objective items on the optimal revenue and reliability levels.

5.1 Experimental Design

We omit the deterministic constraints $Ax \geq b$. All decision variables are nonnegative without specified upper bounds. The components of vectors q and d take integer values randomly generated from the intervals $[10, 50]$ and $[40, 100]$, with interval $[75\%, 100\%]$ for selecting an optimal reliability

variable p. We generate a $[3 \times 5]$-dimensional T matrix with coefficients $t_{ij}, i = 1, \ldots, 3, \ j = 1, \ldots, 5$ drawn from a Bernoulli distribution with 70% "successful" rate. We generate $|\Omega| = 1500$ equally likely realizations of the random variable $\xi = [\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]^T$, with each realization $\omega^k, \ k \in \Omega$ being the nearest integer of a random number sampled from a normal distribution $\mathcal{N}(2j + \lceil 10/j \rceil, \lceil 5/j \rceil), j = 1, \ldots, 5$. Note that the described parameter settings could generate trivial instances, in which the chance constraint (3) cannot be satisfied unless all $x = 0$. We select ten nontrivial instances for each combination of parameter settings. All problems are solved with CPLEX 12.5.1 via ILOG Concert Technology with C++, and the computations are performed on a HP Workstation Z200 Windows 7 machine with Intel(R) Xeon(R) CPU 2.80 GHz, and 8GB memory. We use the valid inequalities described in Appendix C.

5.2   Comparison of CPU Times

We set the parameter a in $\eta(p) = ap$ as: $a = 30\% \left(\sum_{i=1}^5 q_i \right)$. Table 2 shows the average, minimum, and maximum CPU seconds needed to attain the optimum with each formulation.

Table 2: CPU seconds for each **MOPCP** formulation.

Formulation	CPU Time (in seconds)		
	Minimum	Average	Maximum
EMP1	112.56	321.55	531.50
EMP2	1.29	10.53	26.68
EMP1-SOS1	178.25	369.78	610.40
EMP2-SOS1	1.78	13.35	29.26

Formulation **EMP1**, in which the integrality restrictions on variables α are relaxed, solves all instances within ten minutes. We achieve significant average, minimum and maximum CPU time reduction when using **EMP2** in which the redundant constraints of form (51) are removed. This illustrates the computational benefits obtained by preprocessing and eliminating the redundant linear constraints and by reducing the number of binary decision variables. **EMP1-SOS1** and **EMP2-SOS1** do not further reduce solution times, and even take slightly more time than their counterparts **EMP1** and **EMP2**. As it is shown to be the most efficient formulation, we will employ formulation **EMP2** in Section 6 for the financial problems.

5.3   Sensitivity to Scalarization Parameter a

We also analyze the relative importance of the two objectives, i.e., revenue and reliability, by testing various values for the parameter $a = \tau \ \sum_{i=1}^5 q_i$ in function $\eta(p) = ap$. We
Table 3 illustrates the results for an arbitrarily picked instance. We denote by (x^*, p^*) the optimal solution of **EMP2** and by $q^T x^*$ the optimal objective value. We also report the change in the revenue objective function $(q^T x^*)$ following a 1% change of the optimal reliability level p^*.

Table 3: Sensitivity analysis with respect to $a = \tau \ \sum_{i=1}^5 q_i$.

τ	$[0.01, 0.15)$	$[0.15, 0.27)$	$[0.27, 0.31)$	$[0.31, 0.47)$	$[0.47, 0.63)$	$[0.63, 0.90)$	$[0.90, 1]$
Optimal p^*	75%	80%	82%	91%	96%	98%	100%
$q^T x^*$	217.31	214.72	212.27	209.06	207.45	204.18	198.94
$\dfrac{\text{change in } q^T x^*}{1\% \text{ change in } p^*}$	N/A	-0.52	-1.23	-0.36	-0.32	-1.64	-2.62

Table 3 shows seven value intervals of τ where the optimal reliability level p^* remains the same in each interval. As an result, the optimal solution (x^*, p^*) and the revenue objective component $q^T x^*$ are both unchanged corresponding to each interval. We present two key observations. First, the ratios of revenue changes to the changes of optimal reliability levels (indicated in the last row of Table 3) fluctuate as we increase τ. Similar fluctuating patterns are observed in the other nine instances we tested, for which there are generally between 5 and 7 ranges of values for τ leading to a different optimal value of reliability p^*. In all instances, the optimal revenue $q^T x^*$ decreases sharply when the worst-case scenario is considered and a 100% reliability level is enforced. There exist significant tradeoff fluctuations between revenue and reliability for various values of τ, emphasizing the importance of studying and solving problems of form **MOPCP**. Second, when p^* is relatively low (close to 75%), a 1% change of p^* results in a small reduction in the optimal revenue $q^T x^*$. The reduction becomes more significant as p^* increases (i.e., $p^* \in [80\%, 85\%]$), but becomes small again as p^* incraeses and takves value in $[85\%, 95\%]$. The revenue reduction rate ultimately becomes very large when the optimal reliability p^* is required to be close to 100%. These results highlight the sensitivity of the revenue to the level of reliability p^*. The revenue is, without surprise, a decreasing function of the reliability level. More interestingly, the amount by which the revenue decreases due to a 1%-change in reliability appears to vary tremendously at different reliability levels and to be a highly non-linear and non-monotone function of the optimal p^*. The models proposed in this study permit to identify the "sensitivity zones" of the optimal reliability level on the revenue. Additionally, the sensitivity study provides guidance to set the value of the scalarization parameter a and to specify the minimal required reliability level $\underline{p}$ in problems of form **MOPCP**.

6 Centralized versus Decentralized Multi-Portfolio Optimization

6.1 Problem Description and Formulations

We consider the multi-portfolio problem faced by an investment company that manages several funds specialized in diverse asset classes (e.g., bonds, small caps), industrial sectors (e.g., financial, consumer staples), or geographical areas (e.g., Japan, Eurozone, North America). We propose two types of stochastic multi-portfolio models based on a downside risk measure and study their use in two types of investment management approaches, i.e., centralized versus decentralized investment.

We first introduce the notations. The asset universe includes $|J|$ financial securities. Each security $j \in J$ has a stochastic return ξ_j with mean return μ_j and belongs to one or several of the r^* asset classes i. We denote by R_i the set of assets in sub-portfolio i: $\bigcup_{i=1,\ldots,r^*} R_i = J$. The asset classes are not required to be disjoint, as a security can be included in more than one specialized funds (e.g., a security can be both a large cap and a consumer staples). The notation x_{ij} is the fraction of capital invested in security j by fund i.

With a **centralized investment management** [29] approach, the investment company employs a single generalist manager with a balanced mandate across all asset classes. In the first multi-portfolio model **SMPC1**, the downside risk measure requires that the loss or negative return of a sub-portfolio i does not exceed a prescribed loss level d_i with a certain probability p. A distinct value can be assigned to each d_i depending on the risk-return profile of the corresponding asset class i. The objective function is a weighted summation of the overall expected return of the portfolio of funds and the probability level p with which each sub-portfolio avoids losing more than the specified loss level. The probability level p represents the risk tolerance and the parameter a can be viewed as the marginal rate of substitution of risk or reliability for expected return. Problem **SMPC1** is mean-reliability multi-portfolio investment model, in which expected return is traded off with the probability at which losses exceeding a specified amount can be prevented:

$$\textbf{SMPC1}: \max \qquad \sum_{j \in J} \left(\mu_j \sum_{i=1}^{r^*} x_{ij} \right) + ap \qquad (55)$$

$$\text{subject to} \quad \mathbb{P} \left(\begin{array}{l} - \sum_{j \in R_i} \xi_j x_{ij} \le d_i, \ i = 1, \ldots, r^* \\ -\xi_j x_{ij} \le d'_j, \ j \in J \end{array} \right) \ge p \qquad (56)$$

$$\sum_{j \in R_i} x_{ij} = q_i, \ i = 1, \ldots, r^* \qquad (57)$$

$$0 \le \sum_{i=1}^{r^*} x_{ij} \le u_j, \ j \in J \qquad (58)$$

$$x_{ij} = 0, \ j \notin R_i, i = 1, \ldots, r^* \qquad (59)$$

$$\underline{p} \le p \le 1 \qquad (60)$$

$$x_{ij} \ge 0, \ j \in J, i = 1, \ldots, r^* \qquad (61)$$

The parameter q_i is the fixed fraction of capital allocated to fund i, and (57) requires a proportion q_i of the entire capital to be invested in i. By setting $\sum_{i=1}^{r^*} q_i = 1$, the constraints (57) guarantee that the entirety of the capital is invested. The downside risk constraint (56) stipulates that the loss incurred by each sub-portfolio i must be at most equal to the loss threshold d_i with probability level at least p. The second set of inequalities in (56) impose upper-bounds d'_j on the loss associated with each individual asset j. As the same generalist manager handles all funds, the downside risk constraints takes the form of a joint probabilistic constraint. The constraints (59) do not permit sub-portfolios to have a position in a security not belonging to the asset class they handle. The concentration constraints (58) do not allow that the sum of the positions (across all funds) in security j exceeds an upper bound u_j and can be enforced as the management of each sub-portfolio is entrusted to the same manager. Constraint (60) defines the lowest acceptable $\underline{p}$ probability level while constraints (61) preclude short-selling.

In the second model **SMPC2**, the loss level d_i of each sub-portfolio is a decision variable instead of a fixed parameter. Instead of the expected return, the objective function now includes a component accounting for the weighted loss level $\sum_{i=1}^{r^*} q_i d_i$ that must not be exceeded with a probability level p. The formulation reads:

$$\textbf{SMPC2}: \max \quad - \sum_{i=1}^{r^*} q_i d_i + ap \qquad (62)$$

$$\text{subject to} \quad (56) - (61),$$

with decisions variables $x_{ij}, \ i = 1, \ldots, r^*, \ j \in R_i$, $d_i, \ i = 1, \ldots, r^*$, and p.

With a **decentralized investment management** approach [29], the investment company relies upon multiple managers, each with a specialist mandate within a particular asset class. Collecting information about specific assets or asset classes and capitalizing on such informational advantage require highly specialized skills. This explains why fund companies can replace generalist balanced managers with managers specialized in a single asset class with the expectation that they will outperform the generalists. Decentralized investment management can even be a necessity for investment companies. In fact, some large clients, such as pension funds, international organizations or educational endowments, sometimes require the capital allocated to a particular asset class to be partitioned among several sub-funds or asset managers [35]. As a consequence, investment firms regularly employ multiple specialized managers (even within the same asset class) in an attempt to diversify investment strategies and reduce diseconomies of scale as funds grow larger [3]. Each specialized fund manager acts on a myopic basis and controls their fund independently of the others.

Consequently, the joint probabilistic constraint (56) is replaced by r^* joint chance constraints of smaller dimensionality, and p is replaced by the reliability levels $p_i, i = 1, \ldots, r^*$ representing each the probability that the loss due to each individually considered sub-portfolio i is at most equal to d_i. In the decentralized investment setting, the multi-portfolio multi-manager model is:

$$\textbf{SMPD1}: \max \qquad \sum_{j \in J} \left(\mu_j \sum_{i=1}^{r^*} x_{ij} \right) + a \sum_{i=1}^{r^*} p_i \tag{63}$$

$$\text{subject to} \quad \mathbb{P} \left(\begin{array}{c} -\sum_{j \in R_i} \xi_j x_{ij} \leq d_i \\ -\xi_j x_{ij} \leq d'_j, \ j \in R_i \end{array} \right) \geq p_i \ , \ i = 1, \ldots, r^* \tag{64}$$

$$0 \leq x_{ij} \leq u_j \ , \ j \in J, i = 1, \ldots, r^* \tag{65}$$

$$\underline{p} \leq p_i \leq 1 \ , \ i = 1, \ldots, r^* \tag{66}$$

$$(57) - (59)$$

and replaces the formulation **SMPC1** for the centralized investment approach.

We also analyze a second model for the decentralized investment context in which the loss levels are decision variables $d_i, i = 1, \ldots, r^*$:

$$\textbf{SMPD2}: \max \qquad -\sum_{i=1}^{r^*} q_i d_i + a \sum_{i=1}^{r^*} p_i \tag{67}$$

$$(57) - (59) \ ; \ (64) - (66) \ .$$

The decentralized model **SMPD2** is the counterpart of **SMPC2**.

6.2 Data and Experimental Design

To illustrate our approach, we consider three sub-portfolios or funds, respectively focused on equities, fixed-income assets, and international securities. Table 8 in Appendix D lists the securities selected for possible inclusion in each sub-portfolio. The selection of the funds is based on the following criteria. Each fund has a 4- or 5-star Morningstar rating, belongs to Morningstar's top ten in its fund category, and has a track-record of at least ten years. For each fund, we collect the monthly returns (without missing data) from January 1993 to April 2013. For cross-validation purposes, we split the data into two disjoint subsets. The training set is used to derive the optimal portfolio allocation, while the data the testing set are not used to derive the optimal investment strategy and are reserved to analyze the robustness of the models and cross-validate the results.

We create seventy-two problem instances: 48 for the formulations **SMPC1** and **SMPD1** and 24 for the **SMPC2** and **SMPD2**. In each, the minimal reliability level $\underline{p}$ is set to 95% and the upper bound u_j invested in a security j is set to 20%. The problem instances differ in terms of the weight a balancing the relative importance of the objectives, the proportion of capital $q_i, i = 1, \ldots, r$ allocated to each sub-portfolio, and the maximum loss level $d_i, i = 1, \ldots, r$ tolerated for each sub-portfolio. Appendix D details the construction of the instances. Each problem instance is formulated with the **EMP2** model which is the the one providing the best solutions times (see Section 5.2). The instances are modeled with AMPL and solved with the Cplex 12.5.1 solver.

6.3 Modeling and Managerial Insights

The numerical experiments conducted in this section have three main objectives. The first one is to assess the applicability and the computational tractability of our reformulation approach to the proposed stochastic multi-portfolio optimization problems. The second one pertains to

the derivation of managerial insights related to: *(i)* the effect of the different parameters on the composition of the sub-portfolios, and *(ii)* the impact of the decentralized and centralized financial investment approaches on the results obtained with the proposed downside risk multi-portfolio optimization models. The third one is to verify the robustness of the models. To that effect, we carry out a cross-validation analysis to check whether the results obtained on the testing set (data not used to determine the investment strategy) are aligned (in terms of return and reliability) with what is expected from the investment policy built on the basis of the data in the training set.

6.3.1 Solution Insights

For each of the seventy-two problem instances, optimality can be proven within 175 seconds (see Table 4). The optimal solution of the instances associated with the centralized investment approach (i.e., formulations **SMPC1** and **SMPC2**) takes about 133 seconds on average, while the solution of the instances associated with the decentralized investment approach (i.e., formulations **SMPD1** and **SMPD2**) is faster and takes less than one second (i.e., 0.65 second) on average. This is due to the significantly lower number of constraints included in the deterministic reformulations of **SMPD1** and **SMPD2**. Note also that the formulations **SMPC1** and **SMPD1** in which each $d_i, i = 1, \ldots, r$ takes a fixed value can be solved faster than their counterparts **SMPC2** and **SMPD2** in which $d_i, i = 1, \ldots, r^*$ are decision variables.

Table 4: Computational times in CPU seconds.

	SMPC1	SMPC2	SMPD1	SMPD2
Average Time	119.15	161.82	0.07	0.13
Standard Deviation	3.88	8.19	0.03	0.02
Smallest Time	114.89	153.99	0.05	0.09
Largest Time	128.81	181.59	0.19	0.16

6.3.2 Parametric Sensitivity Analysis

To evaluate the impact of the parameters a, q, and d (for **SMPC1** and **SMPD1** only), we consider the trios of problem instances that differ only in terms of one single parameter.

The parameter a defines the relative importance of the two objectives (i.e., reliability and return) and has in our experiments a marginal impact on the composition of the optimal portfolios. Among the twenty-four triplets of instances that differ only in terms of the value attributed to a, we observe a difference in the composition of the optimal portfolio in only seven out of the 24 cases. The composition of the portfolios is particularly stable with respect to a for the centralized problem instances. For those, in only two of the sixteen triplets of instances do we have a difference in the optimal portfolio when a is modified. Increasing the importance of an objective by up to 20% does not appear to be sufficient to have a strong impact on the composition of the portfolio.

The parameters $q_i, i = 1, \ldots, r^*$ define the proportion of capital allocated to each sub-portfolio i and have a clear impact on the composition of the portfolio, and on the return and reliability objectives. In almost all cases, the lowest reliability level is obtained when the proportion of capital allocated to the equity sub-portfolio is the largest (i.e., 40%). Interestingly, with the **SMPC1** decentralized investment model, the expected return that can be achieved is the largest when the proportion of capital allocated to the equity sub-portfolio is the largest.

The parameters $d_i, i = 1, \ldots, r$ determine the loss that must not be exceeded with a certain probability level. The optimal expected return for **SMPC1** and **SMPD1** with fixed parameter d_i is the lowest when the values assigned to d_i are the largest (except in three instances) and is a

consequence of the trade-off between mean return and downside risk. The optimal objective value of **SMPC1** and **SMPD1** decreases monotonically with the values taken by d_i. In Tables 5 and 6, we refer to the average, smallest, and largest values with the abbreviations AV, SM, and LA.

Table 5: Expected return as a function of maximal authorized loss level.

	Loss Level			Expected Return			Objective Value		
	AV	SM	LA	AV	SM	LA	AV	SM	LA
	d_1	d_2	d_3						
SMPC1	0.05	0.03	0.04	0.110	0.078	0.143	0.1089	0.1081	0.1103
	0.035	0.02	0.03	0.092	0.045	0.130	0.1086	0.1025	0.1122
SMPD1	0.05	0.03	0.04	0.119	0.083	0.155	0.0115	0.0108	0.0119
	0.035	0.02	0.03	0.101	0.064	0.138	0.0107	0.0102	0.0113

For both the centralized **SMPC1** and decentralized **SMPD1** models, the parameter setting leading to the largest objective value is obtained when the weight a is equal to a_3 and gives a priority (20% more important) to the return objective, when the allowable loss levels are the highest, and when the capital is equally split between the three sub-portfolios.

6.3.3 Comparison of Centralized and Decentralized Models

As above-mentioned, the stochastic multi-portfolio optimization models **SMPD1** and **SMPD2** for the decentralized investment approach are easier to solve than their counterparts **SMPC1** and **SMPC2** for the centralized investment approach, since they contain less binary variables and a significantly smaller number of constraints.

In each problem instance, the decentralized approach models generate a higher objective value. Indeed, the optimal objective value of **SMPD1** (resp., **SMPD2**) is always at least equal to the optimal objective value for **SMPC1** (resp., **SMPC2**). This highlights that the decentralized approach models **SMPD1** and **SMPD2** are less constraining than the centralized approach models **SMPC1** and **SMPC2**. In **SMPC1** and **SMPC2**, the optimization is carried out with respect to the joint probabilistic constraint (56) which is more constraining than the set of constraints (56) included in **SMPD1** and **SMPD2** ((56) $\Rightarrow$ (64)). Also, the concentration constraints (58) in **SMPC1** and **SMPC2** are more restrictive than the ones (65) in **SMPD1** and **SMPD2**. The decentralization of the investment decisions does not permit to impose the same concentration limitations (i.e., at most 20% in each security), since some securities can be included in several sub-portfolios managed by different managers.

For forty-five of the forty-eight decentralized problem instances **SMPD1** and **SMPD2**, the optimal portfolio includes at least one position exceeding 20%. With **SMPD1**, the position in the Virtus Emerging Markets Opportunities (resp., Invesco Asia Pacific Growth and Henderson European Focus) fund exceeds the 20% threshold 50% (resp., 12.5% and 45.8%) of the time. With **SMPD2**, the position in the TCW Emerging Markets Income (resp., Henderson European Focus) fund exceeds 20% in 2/3 (resp., all) of the instances. This shows that the decentralization approach can lead to concentration and loss of diversification issues as individual managers have hardly any incentive to take into account the correlation of their fund returns with these of other managers [3]. Elton and Gruber [9] acknowledge this issue but note that it is possible to overcome, to some extent, the lack of coordination between managers and the ensuing loss of diversification by imposing rules to asset managers and crafting appropriate managerial incentive contracts. Schleifer [27] adds that the recourse to multiple managers can induce a "yardstick competition" and that the resulting higher effort levels of the managers can further improve their performance.

The return obtained with **SMPD1** (resp., **SMPD2**) is equal or above the expected return obtained with **SMPC1** (resp., **SMPC2**) for 58.33% (resp., all) the problem instances. Table 5 shows that the average, lowest, and largest return levels for **SMPD1** exceeds those obtained with **SMPC1**. The objective value obtained with **SMPD1** (resp., **SMPD2**) is strictly larger than the one obtained with **SMPC1** (resp., **SMPC2**) for each problem instance. Clearly, the compositions of the decentralized optimal sub-portfolios differ from these of the centralized optimal sub-portfolios. This corroborates what Sharpe [29] observed for the mean-variance models, namely that the centralized optimal solution to the mean-variance problem is different from the optimal linear combination of mean-variance efficient portfolios for each specialized manager (see also [35]).

The entire portfolio's average (resp., minimum and maximum) loss $\sum_{i=1}^{3} q_i d_i$ taken over all instances amounts to 1.9% (resp., 1.7% and 2.1%) and 1.8% (resp., 1.5% and 2%) for **SMPC2** and **SMPD2**, respectively (Table 6). The average, minimal, and maximal loss levels are lower for and **SMPD2** than they are for and **SMPC2** for each sub-portfolio and for the entire portfolio.

Table 6: Optimal loss levels.

Models	Entire Portfolio			Sub-Portfolios								
				Equity			Fixed-Income			International Securities		
	AV	SM	LA	AV	SM	LA	AV	SM	LA	AV	SM	LA
SMPC2	0.019	0.017	0.021	0.028	0.025	0.034	0.011	0.010	0.014	0.016	0.022	0.013
SMPD2	0.018	0.015	0.020	0.028	0.025	0.034	0.009	0.008	0.010	0.016	0.022	0.013

6.3.4 Cross-Validation and Robustness Analysis

For each problem instance, we use for the testing period the asset allocation derived by considering the training data only and observe the average return of the portfolio over the testing period and its reliability. The proposed models cross-validate well as shown by the following statistics. The reliability requirement is satisfied for all the seventy-two problem instances over the testing period. The proportion of times that the loss of the portfolio is below the prescribed maximal loss level is below 5%. For twenty-one (resp., eighteen) of the twenty-four problem instances **SMPD1** (resp., **SMPC1**), the average return over the testing period exceeds the one over the training period. Similarly, for all (resp., eight of) the twelve problem instances **SMPD2** (resp., **SMPC2**), the average return over the testing period exceeds the one over the training period.

7   Conclusions and Future Research

We propose a new modeling and solution method for a class of multi-objective stochastic programming problems. The problems include a joint probabilistic constraint with multi-row random technology matrix which requires a system of inequalities to hold with some probability level. This latter level is adjustable and defined as a decision variable. The weighted linear objective function includes a revenue and a reliability component and is a monotone increasing function of the probability level p. This latter can be used to represent the reputation of a manager, the brand name of a firm, the quality of its service, or the reliability of a network. The scalarization parameter determines the relative importance of the objectives and its value is specified by the decision maker. This parameter can be viewed as the marginal rate of substitution of reliability for revenue. This type of problems permits to capture the trade-offs between revenue and risk or reliability.

The contributions of this study are manifold. On the modeling and algorithmic side, we propose new and exact formulations as well as solution methods for the class of NP-hard **MOPCP**s. While

a variety of algorithmic methods exist for probabilistically constrained problems, to the best of our knowledge, no exact efficient reformulation or solution method is available for the class of problems studied here. We derive four MILP formulations equivalent to **MOPCP**. For that purpose, we develop significant extensions to the work of optimizing risk variables associated with single-row chance constraints [30], and to the Boolean modeling framework allowing for the solution of single-objective chance-constrained problems with random right-hand sides [15] and random technology matrix [12]. A key feature of the approach is that the number of binary variables in the MILP formulations does not depend directly on the number of scenarios. Indeed, the number of binary variables actually is equal to the number of cut points used in the binarization process. This means that the MILP reformulations of the **MOPCP** have the same number of binary variables as the reformulations of the single-objective chance-constrained problems studied in [12].

This study provides a method to solve exactly multi-objective probabilistically constrained problems of form **MOPCP** and to investigate the trade-offs between objectives. This contribution is particularly meaningful as problems of form **MOPCP** are applicable to multiple types of problems (see, e.g., network reliability [25], disaster management [1], water management [21], production scheduling [28]), and industrial sectors.

We have also executed a computational study which provides modeling insights about the efficiency of the reformulations and about the solution sensitivity with respect to the relative importance of the objectives. The computational times differ greatly among the proposed MILP reformulations. The study provides guidance to select the most appropriate formulation and to specify the value of the scalarization parameter defining the relative importance of the several pursued goals. The results show first that the revenues are a decreasing function of the reliability level. Second, the rate of substitution of revenue for reliability varies strongly with and is a non-linear and non-monotone function of the optimal reliability level p^*.

Finally, we propose several stochastic multi-portfolio financial optimization models which balance return with downside risk and which can be used in a centralized fashion (i.e., one generalist manager has a balanced mandate across asset classes) or a decentralized one (i.e., specialized managers each have a mandate for a particular asset class). The tests show: *(i)* the applicability of the models and the ability to solve them efficiently; *(ii)* the effect of the different parameters (tolerated loss levels, scalarization parameter, capital allocated to each sub-portfolio) on the composition of the sub-portfolios; *(iii)* the possible pros and cons of the decentralized and centralized financial investment approaches; *(iv)* the robustness of the proposed models. To our knowledge, this study is the first to propose stochastic multi-objective, multi-portfolio optimization problems.

In the future, we plan to consider different forms (other than linear) for the function $\eta(p)$ depending on the probability level and to develop specific and scalable algorithms for the resulting reformulations. We are also interested in deriving families to valid inequalities that will speed up the solution process and in applying these formulations to specific problems, such as the distribution of perishable products in supply chains and the design of emergency response systems.

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APPENDIX

A   Formulations for Example 1

A.1   Binarization Process and Recombinations for Example 1

Table 7 displays the recombinations, their binary images (relevant Boolean vectors), and their partitioning into $\bar{\Omega}_B^-(0.6)$ and $\bar{\Omega}_B^+(0.6)$. The recombinations $k=$ 1–5 and 7–10 correspond to realizations of ξ, while the recombinations $k=$ 11–17 do not coincide with any realization of the random vector. Realization $k=$ 6 is dropped since it does not meet the criterion stated in (6).

Table 7: Recombinations and relevant Boolean vectors.

k	Recombinations		Relevant Boolean Vectors								$g(k)$	Boolean sets
	ω_1^k	ω_2^k	β_{11}^k	β_{12}^k	β_{13}^k	β_{14}^k	β_{21}^k	β_{22}^k	β_{23}^k	β_{24}^k		
1	2	2	1	0	0	0	1	0	0	0	0	
2	2	4	1	0	0	0	1	1	1	0	0	
4	3	2	1	1	0	0	1	0	0	0	0	$\bar{\Omega}_B^-(0.6)$
9	2	3	1	0	0	0	1	1	0	0	0	
12	4	2	1	1	1	0	1	0	0	0	0	
3	2	5	1	0	0	0	1	1	1	1	1	
5	3	3	1	1	0	0	1	1	0	0	1	
7	4	4	1	1	1	0	1	1	1	0	1	
8	5	2	1	1	1	1	1	0	0	0	1	
10	3	4	1	1	0	0	1	1	1	0	1	
11	3	5	1	1	0	0	1	1	1	1	1	$\bar{\Omega}_B^+(0.6)$
13	4	3	1	1	1	0	1	1	0	0	1	
14	4	5	1	1	1	0	1	1	1	1	1	
15	5	3	1	1	1	1	1	1	0	0	1	
16	5	4	1	1	1	1	1	1	1	0	1	
17	5	5	1	1	1	1	1	1	1	1	1	

	Cut Points							
	c_{11}	c_{12}	c_{13}	c_{14}	c_{21}	c_{22}	c_{23}	c_{24}
	2	3	4	5	2	3	4	5

A.2 Feasibility Set S1 for Example 1

Recall that $k = 1, 2, 4, 9, 12$ belong to $\bar{\Omega}_B(0.6)$ (see Table 7). The system **S1** of inequalities reads:

$$\lambda_{11} + \lambda_{21} \leq 1 \qquad \text{(17) for } k = 1$$
$$\lambda_{11} + \lambda_{21} + \lambda_{22} + \lambda_{23} \leq 1 \qquad \text{(17) for } k = 2$$
$$\lambda_{11} + \lambda_{12} + \lambda_{21} \leq 1 \qquad \text{(17) for } k = 4$$
$$\lambda_{11} + \lambda_{21} + \lambda_{22} \leq 1 \qquad \text{(17) for } k = 9$$
$$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{21} \leq 1 \qquad \text{(17) for } k = 12$$
$$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1 \qquad \text{(18) for } j = 1$$
$$\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1 \qquad \text{(18) for } j = 2$$
$$x_1(2\lambda_{11} + 3\lambda_{12} + 4\lambda_{13} + 5\lambda_{14}) + 2x_2(2\lambda_{21} + 3\lambda_{22} + 4\lambda_{23} + 5\lambda_{24}) \leq 13 \qquad \text{(20) for row } i = 1$$
$$5x_1(2\lambda_{11} + 3\lambda_{12} + 4\lambda_{13} + 5\lambda_{14}) + 3x_2(2\lambda_{21} + 3\lambda_{22} + 4\lambda_{23} + 5\lambda_{24}) \leq 27 \qquad \text{(20) for row } i = 2$$
$$x_1(2\lambda_{11} + 3\lambda_{12} + 4\lambda_{13} + 5\lambda_{14}) \leq 4 \qquad \text{(20) for row } i = 3$$
$$x_2(2\lambda_{21} + 3\lambda_{22} + 4\lambda_{23} + 5\lambda_{24}) \leq 7 \qquad \text{(20) for row } i = 4$$

where all the λ variables are binary (see (19)), $0 \leq x_1 \leq 1.5$, and $0 \leq x_2 \leq 2.5$.

A.3 Formulation EMP2 for Example 1

After calculating the cumulative probabilities $F(\omega^k)$ for all recombinations in Table 7 of Example 1, we obtain nine distinct probability levels $h_s \in [60\%, 100\%]$: $h_1 = 0.6$, $h_2 = 0.64$, $h_3 = 0.65$, $h_4 = 0.7$, $h_5 = 0.72$, $h_6 = 0.78$, $h_7 = 0.87$, $h_8 = 0.94$, and $h_9 = 1$. We associate a variable α_s with each reliability level h_s. We ignore constraints (36) as they are not binding for Example 1 due to the maximization nature of the objective. We provide below the **EMP2** formulation for Example 1.

$$\max \quad 3x_1 + 2x_2$$
$$+1.2\alpha_1 + 0.08\alpha_2 + 0.02\alpha_3 + 0.1\alpha_4 + 0.04\alpha_5$$
$$+0.12\alpha_6 + 0.18\alpha_7 + 0.14\alpha_8 + 0.12\alpha_9$$

subject to

(18) for $j = 1$	$\lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} = 1$
(18) for $j = 2$	$\lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} = 1$
(34) for $i = 1$	$z_1 + 2z_2 \leq 13$
(34) for $i = 2$	$5z_1 + 3z_2 \leq 27$
(34) for $i = 3$	$z_1 \leq 4$
(34) for $i = 4$	$z_2 \leq 7$
(35) for $j = 1, l = 1$	$2x_1 + 13\lambda_{11} - z_1 \leq 13$
(35) for $j = 1, l = 2$	$3x_1 + 19.5\lambda_{12} - z_1 \leq 19.5$
(35) for $j = 1, l = 3$	$4x_1 + 26\lambda_{13} - z_1 \leq 26$
(35) for $j = 1, l = 4$	$5x_1 + 32.5\lambda_{14} - z_1 \leq 32.5$
(35) for $j = 2, l = 1$	$2x_2 + 9\lambda_{21} - z_2 \leq 9$
(35) for $j = 2, l = 2$	$3x_2 + 13.5\lambda_{22} - z_2 \leq 13.5$
(35) for $j = 2, l = 3$	$4x_2 + 18\lambda_{23} - z_2 \leq 18$
(35) for $j = 2, l = 4$	$5x_2 + 22.5\lambda_{24} - z_2 \leq 22.5$

$$
\begin{array}{lll}
(49) \text{ for } s=2, k=5 & \alpha_2 + \lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} \le 2 \\
(49) \text{ for } s=3, k=13 & \alpha_3 + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{21} + \lambda_{22} \le 2 \\
(49) \text{ for } s=4, k=8 & \alpha_4 + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{21} \le 2 \\
(49) \text{ for } s=4, k=10 & \alpha_4 + \lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} + \lambda_{23} \le 2 \\
(49) \text{ for } s=5, k=15 & \alpha_5 + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{21} + \lambda_{22} \le 2 \\
(49) \text{ for } s=6, k=7 & \alpha_6 + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{21} + \lambda_{22} + \lambda_{23} \le 2 \\
(49) \text{ for } s=7, k=16 & \alpha_7 + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{14} + \lambda_{21} + \lambda_{22} + \lambda_{23} \le 2 \\
(49) \text{ for } s=8, k=11 & \alpha_8 + \lambda_{11} + \lambda_{12} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} \le 2 \\
(49) \text{ for } s=9, k=14 & \alpha_9 + \lambda_{11} + \lambda_{12} + \lambda_{13} + \lambda_{21} + \lambda_{22} + \lambda_{23} + \lambda_{24} \le 2 \\
(50) & \alpha_1 = 1, \ \alpha_1 \ge \cdots \ge \alpha_9,
\end{array}
$$

in which the the original bounds $0 \le x_1 \le 1.5$, $0 \le x_2 \le 2.5$, $0 \le \alpha_s \le 1$, $s = 2, \ldots, 9$, and the binary restrictions $\lambda_{jl} \in \{0,1\}$, $l = 1, \ldots, n_j$, $j \in J$ must be added.

The optimal solution for the problem is: $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\alpha_s = 0$, $s = 4, \ldots, 9$, $x_1 = 0.48$, $x_2 = 2.5$, $\lambda_{14} = 1$, $\lambda_{21} = 1$, and all the other λ_{jl} variables take value 0. The integral separation structure of the threshold minorant defined by the optimal solution corresponds to the p-sufficient recombination $(\omega_1, \omega_2) = (5, 2)$. The optimal value of the objective function is 7.74 with a reliability level equal to 65%.

B   Proof of Theorem 2

Proof. Let k' be the "extreme" recombination such that $\beta_{jl}^{k'} = 1, l = 1, \ldots, n_j, j \in J$. Clearly, $F(\omega^{k'}) = 1$ and $k' \in \bar{\Omega}_B^+(p)$. For any λ satisfying (17)–(19), we have

$$
\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^{k'} = \sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \ge \sum_{j \in J} 1 = |J|.
$$

which induces (15) and make it redundant.

Consider an arbitrary separating structure $(\lambda, |J|)$ feasible for (17)–(19). All possible Boolean vectors belong either to the set $\bar{\Omega}_B^+(p)$ of relevant p-sufficient Boolean vectors, or the set $\bar{\Omega}_B^-(p)$ of relevant p-insufficient Boolean vectors, or the set of $\left(\Omega_B^-(p) \setminus \bar{\Omega}_B^-(p)\right)$ of non-relevant p-insufficient realizations. Constraint (17) prevents any $k \in \bar{\Omega}_B^-(p)$ from satisfying (68):

$$
\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k \ge |J|. \tag{68}
$$

Any $k \in \left(\Omega_B^-(p) \setminus \bar{\Omega}_B^-(p)\right)$ is such that $\beta_{j1}^k = 0$ for at least one $j \in J$. Combining this result with the regularization property (see (10) in Lemma 1), we have:

$$
\beta_{jl}^k = 0, l = 1, \ldots, n_j \text{ for at least one } j \in J, \ \forall k \in \left(\Omega_B^-(p) \setminus \bar{\Omega}_B^-(p)\right).
$$

Therefore,

$$
\sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k = 0, \text{ for at least one } j \in J, \ \forall k \in \left(\Omega_B^-(p) \setminus \bar{\Omega}_B^-(p)\right),
$$

implying that

$$\sum_{j \in J} \sum_{l=1}^{n_j} \lambda_{jl} \, \beta_{jl}^k \leq |J| - 1, \ k \in \left(\Omega_B^-(p) \setminus \bar{\Omega}_B^-(p) \right),$$

and thus that (68) does not hold for any $k \in \left(\Omega_B^-(p) \setminus \bar{\Omega}_B^-(p) \right)$. Hence, any k for which (68) holds should belong to $\bar{\Omega}_B^+(p)$. $\qquad\square$

C   Valid Inequalities for the MILP Formulations

We present here a family of valid inequalities that can be used to strengthen the mixed-integer inequalities (35) in formulations **EMP1**, **EMP2**, **EMP1-SOS1** and **EMP2-SOS1**.

Proposition 1. Consider the set

$$G_j = \left\{ (x_j, z_j, \lambda) \in \mathbb{R}_+^2 \times \{0,1\}^{n_j} : z_j \geq c_{jl} \, x_j - M_{jl}(1 - \lambda_{jl}), \ l = 1, \ldots, n_j \, ; \sum_{j=1}^{n_j} \lambda_{jl} = 1 \right\} . \quad (69)$$

The inequalities

$$z_j \geq c_{jl} \, x_j - M_{jl}(1 - v_{jl}), \ l = 1, \ldots, n_j \, , \qquad (70)$$

with

$$v_{jl} = \sum_{r=l}^{n_j} \lambda_{jr} \, , \ l = 1, \ldots, n_j \, , \qquad (71)$$

are valid for G_j and are tighter than the inequalities (35).

Valid inequalities similar to (70) and (71) are derived by Kogan and Lejeune [12] for a single-objective stochastic programming problem of type **MOPCP** with fixed reliability level p. We refer the reader to [12] for the proof of Proposition 1.

D   Financial Data and Construction of Problem Instances

We build three sub-portfolios or funds including respectively equities, fixed-income assets, and international securities. The equity sub-portfolio comprises seven securities: a growth large cap domestic equity mutual fund, a value large cap domestic equity mutual fund, a growth small cap domestic equity mutual fund, a value small cap domestic equity mutual fund, an Asian equity fund, a European equity fund, and an emerging market equity fund. The fixed-income sub-portfolio also includes seven securities: a high yield bond fund, a multi-sector bond fund, an emerging markets bond fund, a government bond fund, a world bond fund, an inflation-protected bond fund, and a high yield muni fund. The international securities sub-portfolio includes five funds covering the emerging markets, Europe, and Asia.

We create 72 problem instances. The forty-eight problem instances constructed for the models **SMPC1** and **SMPD1** differ in terms of the weight a balancing the relative importance of the objectives, the proportion of capital $q_i, i = 1, \ldots, r$ allocated to each sub-portfolio, and the maximum loss level $d_i, i = 1, \ldots, r$ tolerated for each sub-portfolio.

We consider three values for the parameter a. The first value a_1 is set to give an equal weight to the two objectives (i.e., expected return and reliability). A standard procedure in the multi-objective literature [32] is to define a_1 as follows:

$$a_1 = \frac{R^U - R^L}{P^U - P^L}$$

Sub-Portfolio	Category	Security
Equities	Domestic Growth Large Cap Equity	Wells Fargo Advantage Growth
	Domestic Value Large Cap Equity	SunAmerica Focused Dividend
	Domestic Growth Small Cap Equity	T. Rowe Price New Horizons
	Domestic Value Small Cap Equity	Undiscovered Mgrs Behavioral Value Inst
	European Equity	Henderson European Focus
	Pacific/Asia Equity	Invesco Asia Pacific Growth
	Emerging Markets Equity	Virtus Emerging Markets Opportunities
Fixed-Income	High Yield Bond	Delaware Pooled High-Yield Bond
	Multisector Bond	Loomis Sayles Fixed Income
	Emerging Markets Bond	TCW Emerging Markets Income
	Government Bond	PIMCO Long-Term US Government Instl
	World Bond	Templeton Global Bond Adv
	Inflation-Protected Bond	PIMCO Real Return Asset Instl
	High Yield Muni	Ivy Municipal High Income
International	European Equity	Henderson European Focus
	Pacific/Asia Equity	Invesco Asia Pacific Growth
	Emerging Markets Equity	Virtus Emerging Markets Opportunities
	Emerging Markets Bond	TCW Emerging Markets Income
	World Bond	Templeton Global Bond Adv

Table 8: Asset universe.

where R^U (resp., R^L) is an upper (resp., lower) bound on the maximal (resp., minimal) value of the expected return. Similarly, P^U (resp., P^L) is an upper (resp., lower) bound on the maximal (resp., minimal) value of the reliability p. The second (resp., third) value $a_2 = 1.2a_1$ (resp., $a_3 = 0.8a_1$) assigned to a gives 20% more importance to the reliability (resp., expected return) objective. We consider four combinations of values for the parameters q_i determining the allocation of capital between sub-portfolios. First, we allocate an equal amount of capital to each sub-portfolio: $q_i = 1/3, i = 1, 2, 3$. In the other three settings, 40% of the capital is allocated to one sub-portfolio $i^* : q_{i^*} = 40\%$, while the remaining part is split equally between the other two sub-portfolios: $q_i = 30\%, \ i \neq i^*$. As for the maximal allowable loss d_i for each sub-portfolio, we consider two sets of values. In the first one, the equity (resp., fixed-income and international securities) sub-portfolio is required to generate a loss lower than 3% (resp., 1.5% and 2%), while the maximal loss associated to the equity (resp., fixed-income and international securities) sub-portfolio is equal to 4% (resp., 2.5% and 3%) in the second parameter setting.

The twenty-four problem instances constructed for the models **SMPC2** and **SMPD2** differ in terms of the weight and the proportion of capital $q_i, i = 1, \dots, r^*$ allocated to each sub-portfolio. Indeed, the portfolio loss level $d_i, i = 1, \dots, r^*$ that can be obtained with a certain reliability is a decision variable in formulations **SMPC2** and **SMPD2**. The considered values for the parameters a and q are the same as those employed in the **SMPC1** and **SMPD1** problem instances.