Risk Measures for Vector-Valued Returns

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Abstract

Portfolios, which are exposed to different currencies, have separate and different returns in each individual currency and are thus vector-valued in a natural way.

This paper investigates the natural domain of these risk measures. A Banach space is presented, for which the risk measure is continuous, and which reflects the vector-valued outcomes of the corresponding risk measures from mathematical finance. We develop its key properties and describe the corresponding duality theory. We finally outline extensions of this space, which are along classical L^p spaces.

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1 Introduction

Classical risk measures have been extended in several directions. Rudloff et al., Molchanov and other authors extend the concept to *set*-valued risk functionals in the papers [11, 25, 2, 16]. Jouini at al. [19] and Burgert and Rüschendorf [4] follow a different approach by extending the concept of risk measures to *vector*-valued random variables (cf. also Kabanov [20]). Risk measures on $\mathbb{R}^d$-valued random variables are naturally present in many real life situations. An example is given by considering a portfolio, which has exposures in d (say) different currencies, where each individual portfolio is exposed to uncertainty and subject to individual considerations on risk. A further example is a consolidated financial statement of an internationally operating company, for example an insurance company. Ekeland and Schachermayer [10] consider the domain space L^∞ for these risk measures. The first multivariate generalization of a Kusuoka representation for risk measures on vector-valued random variables is provided by Ekeland et al. [9] on L^2 (notice the difference to multi-valued risk objectives, cf. [14]).

Svindland et al. [36, 12, 21], but many other authors as well mention and consider different domain spaces for risk measures (for example Orlicz spaces, cf. Cheridito [5] or Bellini [3]). Here we consider domain spaces, for which the risk measure is continuous, and extend this basic setting in two different directions.

In a first extension we employ elementary representations of risk measures and use them to define a Banach space to carry the investment strategies and risk measures in a natural way. This

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setting allows a generalization to vector-valued ($\mathbb{R}^d$, or more general Banach space-valued) random variables. The corresponding space has properties similar to L^1 and thus generalizes the before mentioned domain spaces. Its dual is fully described in case that the state space enjoys the Radon–Nikodým property.

The second part of the paper extends the new space by picking up differences or similarities between L^1 and L^p spaces. The duality theory for these spaces essentially differs from the initial space. Again, these spaces are large enough to carry risk measure in a natural way, and—above all—the risk measure is continuous on the spaces described.

Outline. The following section (Section 2) provides the mathematical setting and elaborates initial details on the relation between the space and the continuity properties of the risk functional. The Banach space L^p_σ to carry a risk measures for vector-valued random variables is introduced in Section 3. It is demonstrated that risk functionals are continuous with respect to the norm of the space introduced. The following Section 4 elaborates the dual of L^1_σ ($p=1$), while Section 5 establishes the dual of L^p_σ for $p>1$.

The new space is larger than L^∞, but not an L^p space in general. The spaces are related to rearrangement spaces introduced by G. Lorentz in [24, 23] (following earlier results obtained by Halperin [15], cf. Pick et al. [31]).

2 Mathematical setting and motivation

We consider a probability space $(\Omega,\mathcal{F},P)$ and denote the *distribution function* (cdf) of a $\mathbb{R}$-valued random variable Y by

$$F_Y(q) := P\left(Y\le q\right) = P\left(\{\omega:\ Y(\omega)\le q\}\right).$$

The *generalized inverse* is the nondecreasing and lower semi-continuous function

$$F_Y^{-1}(\alpha) := \inf\left\{q:\ P\left(Y\le q\right)\ge\alpha\right\},$$

also called the quantile or conditional Value-at-Risk. The Average Value-at-Risk is

$$\mathsf{AV@R}_\alpha(Y) := \frac{1}{1-\alpha}\int_\alpha^1 F_Y^{-1}(\alpha) = \min_{q\in\mathbb{R}}\ q+\frac{1}{1-\alpha}\mathbb{E}(Y-q)_+ \tag{1}$$

(cf. Rockafellar and Uryasev [32] and Pflug [27] for the latter equality).

With $(X,\|\cdot\|)$ we denote a separable Banach space. Let $Y\colon\Omega\to(X,\|\cdot\|)$ be a strongly measurable random variable. We write $\|Y\|$ for the $[0,\infty)$-valued random variable

$$\|Y\|:\omega\mapsto\|Y(\omega)\|.$$

$L^p(X)$ stands for the Banach space (the Lebesgue–Bochner space) of all (equivalence classes of) X-valued, Bochner integrable random variables Y with finite p-mean norm,

$$\|Y\|_p := \left(\mathbb{E}\,\|Y\|^p\right)^{1/p} = \left(\int_\Omega \|Y(\omega)\|^p\,P(\mathrm{d}\omega)\right)^{1/p} < \infty \qquad (1\le p<\infty).$$

The p-mean norm can be expressed by the quantile and its generalized inverse by

$$\|Y\|_p = \left(\int_0^1 F_{\|Y\|}^{-1}(u)^p\mathrm{d}u\right)^{1/p} = \left(\int_0^1 p\,t^{p-1}\left(1-F_{\|Y\|}(t)\right)\mathrm{d}t\right)^{1/p} \qquad (1\le p<\infty). \tag{2}$$

For $\mathbb{R}$-valued random variables, i.e., $(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, the space $L^p(X) =: L^p$ represents the usual Lebesgue space.

This paper introduces a new Banach space on vector-valued, strongly measurable random variables by weighting the quantiles in a different way than (2). The results obtained extend and generalize characterizations obtained in Pichler [29], where only real valued random variables and $p = 1$ are considered in a context of insurance. Here, we characterize the dual space and prove that the new space does not have a pre-dual.

Remark 1. We shall assume throughout the paper that the probability space is rich enough to carry a uniform distribution.[1] If this is not the case, then one may replace Ω by $\tilde{\Omega} := \Omega \times [0,1]$ with the product measure $\tilde{P}(A \times B) := P(A) \cdot \text{Lebesgue measure}(B)$. Every random variable Y on Ω extends to $\tilde{\Omega}$ by $\tilde{Y}(\omega, u) := Y(\omega)$, and $U(\omega, u) := u$ is a uniform random variable, as $\tilde{P}(U \leq u) = \tilde{P}(\Omega \times [0,u]) = u$.

With an $\mathbb{R}$-valued random variable Y one may further associate its *generalized quantile* transform

$$F(y,u) := (1-u) \cdot \lim_{y' \uparrow y} F_Y^{-1}(y') + u \cdot F_Y^{-1}(y).$$

The random variable $F(Y, U)$ is uniformly distributed again, and $F(Y, U)$ is coupled in a comonotone way with Y (cf. Pflug and Römisch [28]).

The relation to risk measures and their continuity properties. A Kusuoka representation (cf. Kusuoka [22]) for risk measures based on $\mathbb{R}^d$-valued random variables is extracted in Ekeland and Schachermayer [10, Theorem 1.7]. The risk functional identified in the "regular case" in [10] for the homogeneous risk functional on random vectors is

$$\rho_Z(Y) := \sup \left\{ \mathbb{E} \langle Z, Y' \rangle : Y' \sim Y \right\}, \tag{3}$$

where $Y \sim Y'$ indicates that Y and Y' enjoy the same law in $\mathbb{R}^d$.[2] ρ_Z is called the *maximal correlation risk measure in direction* Z.

The linear form in (3) is $\mathbb{E} \langle Z, Y \rangle$, where $\langle Z, Y \rangle(\omega) = \sum_{i=1}^d Z_i(\omega) Y_i(\omega)$ is the inner product and $Z \geq 0$ is normalized to satisfy

$$\mathbb{E}\left[\sum_{i=1}^d |Z_i|\right] = 1,$$

that is,

$$1 = \mathbb{E}\, \|Z\|_{\ell_1^d} = \int_0^1 F^{-1}_{\|Z\|_{\ell_1^d}}(u) \mathrm{d}u.$$

The rearrangement inequality (cf. Hardy et al. [17]) provides an upper bound for the linear form by

$$|\mathbb{E} \langle Z, Y \rangle| \leq \mathbb{E}\, \|Z\|^* \cdot \|Y\| \leq \mathbb{E}\, K \cdot \|Z\|_{\ell_1^d} \cdot \|Y\| \leq K \cdot \int_0^1 F^{-1}_{\|Z\|_{\ell_1^d}}(u) F^{-1}_{\|Y\|}(u) \mathrm{d}u, \tag{4}$$

where the norms $\|\cdot\|$ on $\mathbb{R}^d$ and $\|\cdot\|^*$ are dual to each other. $K > 0$ is the constant linking the norms by $\|\cdot\|^* \leq K \cdot \|\cdot\|_{\ell_1^d}$ on (the dual of) $\mathbb{R}^d$. Without loss of generality we may (and will) assume that $K = 1$ (otherwise, consider the equivalent norm $\|\cdot\|' := K \|\cdot\|$ instead of $\|\cdot\|$ on $\mathbb{R}^d$).

[1] U is uniform, if $P(U \leq u) = u$ for all $u \in [0,1]$.

[2] That is, $P(Y_1 \leq y_1, \dots Y_d \leq y_d) = P\left(Y_1' \leq y_1, \dots Y_d' \leq y_d\right)$ for all $(y_1, \dots y_d) \in \mathbb{R}^d$.

The maximal correlation risk measure (3) employs the linear form $\mathbb{E}\langle Z,Y\rangle$, which satisfies the bounds (4). This motivates fixing the function

$$\sigma(\cdot):=F^{-1}_{\|Z\|_{\ell_1^d}}(\cdot) \tag{5}$$

and to endow a space with the form

$$\|Y\|_\sigma:=\int_0^1\sigma(u)F^{-1}_{\|Y\|}(u)\mathrm{d}u.$$

It turns out that $\|\cdot\|_\sigma$ is a norm (Proposition 6 below) and the maximal correlation risk measure is continuous with respect to the norm. We collect these observations in the following definition and proposition.

Definition 2 (Weighting function)**.** A nondecreasing, nonnegative function $\sigma\colon[0,1)\to[0,\infty)$, which is normalized by $\int_0^1\sigma(u)\mathrm{d}u=1$, is called a *distortion function* (in the literature occasionally also *spectrum*, cf. Acerbi [1]).

The following proposition establishes the continuity relation.

Proposition 3. *ρ_Z, the maximal correlation risk measure in direction Z defined in (3) is Lipschitz continuous with respect to the norm $\|\cdot\|_\sigma$, that is,*

$$|\rho_Z(Y)-\rho_Z(Y')|\le\|Y-Y'\|_\sigma\,,$$

where σ is defined in (5) and provided that ρ_Z is finite valued on Y and Y'.

Proof. Note first that ρ_Z is subadditive (convex), and thus

$$\rho_Z(Y)=\rho_Z(Y'+Y-Y')\le\rho_Z(Y')+\rho_Z(Y-Y').$$

Chose $Z'\sim Z$ such that $\rho_Z(Y-Y')=\mathbb{E}\langle Z',Y-Y'\rangle$. Repeating the sequence of inequalities in (4) reveals that

$$\begin{aligned}\rho_Z(Y)&\le\rho_Z(Y')+\mathbb{E}\langle Z',Y-Y'\rangle\\&\le\rho_Z(Y')+\int_0^1\sigma(u)F^{-1}_{\|Y-Y'\|}(u)\mathrm{d}u\le\rho_Z(Y')+\|Y-Y'\|_\sigma\,.\end{aligned}$$

The assertion is immediate by interchanging the roles of Y and Y'. ☐

3 The vector-valued Banach space $L^p_\sigma(X)$

The risk measure ρ_Z introduced in the introduction is continuous with respect to $\|\cdot\|_\sigma$. However, the proper space has not been specified. This section introduces the space and the norm in a more general setting. Basic properties of the space are elaborated.

Definition 4. For a distortion function σ and a random variable Y with outcomes in the Banach space $(X,\|\cdot\|)$ define

$$\|Y\|_{\sigma,p}:=\sup_{U\text{ uniform}}\left(\mathbb{E}\,\sigma(U)\,\|Y\|^p\right)^{1/p}, \tag{6}$$

where the supremum is among all uniform random variables U. In line with p-measurable random variables we denote by $L^p_\sigma(X)$ the space of equivalence classes from

$$\left\{Y:\Omega\to X \text{ strongly measurable and } \|Y\|_{\sigma,p}<\infty\right\},$$

where random variables are identified which cannot be distinguished by the probability measure P.

Remark 5. We shall use the abbreviations $\|Y\|_\sigma := \|Y\|_{\sigma,1}$ as well as $L_\sigma(X) := L^1_\sigma(X)$.

Basic characterization of $L^p_\sigma(X)$. The space $L^p_\sigma(X)$, equipped with the norm $\|\cdot\|_{\sigma,p}$, is indeed a Banach space (see Theorem 11 below). We elaborate important relations and comparisons with the norm of $L^p(X)$ spaces first.

Proposition 6. *$\|\cdot\|_{\sigma,p}$ is a seminorm on $L^p_\sigma(X)$ whenever $1\le p<\infty$ and it holds that*

$$\|Y\|_{\sigma,p} = \left(\int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \mathrm{d}u\right)^{1/p}. \tag{7}$$

Moreover, the supremum in (6) is attained.

Proof. It is evident that $\|Y\|_{\sigma,p}$, as defined in (6), is positively homogeneous. As for the triangle inequality notice that

$$\begin{aligned}
\left(\mathbb{E}\,\sigma(U)\,\|Y_1+Y_2\|^p\right)^{1/p} &\le \left(\mathbb{E}\,\sigma(U)\,(\|Y_1\|+\|Y_2\|)^p\right)^{1/p}\\
&= \left(\mathbb{E}\left(\sigma(U)^{1/p}\,\|Y_1\| + \sigma(U)^{1/p}\,\|Y_2\|\right)^p\right)^{1/p}\\
&= \left\|\sigma(U)^{1/p}\,\|Y_1\| + \sigma(U)^{1/p}\,\|Y_2\|\right\|_p\\
&\le \left\|\sigma(U)^{1/p}\,\|Y_1\|\right\|_p + \left\|\sigma(U)^{1/p}\,\|Y_2\|\right\|_p\\
&= \left(\mathbb{E}\big(\sigma(U)^{1/p}\,\|Y_1\|^p\big)\right)^{1/p} + \left(\mathbb{E}\big(\sigma(U)^{1/p}\,\|Y_2\|^p\big)\right)^{1/p}
\end{aligned}$$

by Minkowski's inequality. By passing to the supremum it follows that

$$\|Y_1+Y_2\|_{\sigma,p} \le \|Y_1\|_{\sigma,p} + \|Y_2\|_{\sigma,p}\,,$$

the triangle inequality.

To accept (7) let U be coupled in a comonotone way with $\|Y\|$ (which exists according to Remark 1). It follows from the rearrangement inequality that the supremum in (6) is attained for U and further that

$$\mathbb{E}\,\sigma(U)\,\|Y\|^p = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \mathrm{d}u,$$

the assertion. □

Remark 7. Clearly, $\|Y\|_{\sigma,p} = \|Y\|_p$ and $L^p_\sigma(X) = L^p(X)$ for the (trivial) distortion function $\sigma(\cdot) = 1$.

Example 8. It follows directly from (7) that the norm of an indicator function of a set E is

$$\|\mathbb{1}_E\|_{\sigma,p} = \left(\int_{1-P(E)}^1 \sigma(u)\mathrm{d}u\right)^{1/p} = S\big(1-P(E)\big)^{1/p},$$

where

$$S(\alpha) := \int_\alpha^1 \sigma(u)\mathrm{d}u. \tag{8}$$

Notice that $P(E) \le P(E)^{1/p} \le \|\mathbb{1}_E\|_{\sigma,p} \le 1$.

Equation (2) in the introduction provides a possibility to compute the norm directly, without involving its generalized inverse. The following corollary generalizes the formula for the norm $\|\cdot\|_{\sigma,p}$.

Corollary 9. *The seminorm $\|\cdot\|_{\sigma,p}$ can be expressed in terms of the cdf $F_{\|Y\|}$ directly (without involving its generalized inverse $F_{\|Y\|}^{-1}$) as*

$$\|Y\|_{\sigma,p}^p = \int_0^\infty p\, y^{p-1} \cdot S\left(F_{\|Y\|}(y)\right) \mathrm{d}y.$$

Proof. By Riemann–Stieltjes integration by parts and change of variables it holds that

$$\begin{aligned}
\|Y\|_{\sigma,p}^p &= \int_0^1 \sigma(u) F_{\|Y\|}^{-1}(u)^p \mathrm{d}u = -\int_0^1 F_{\|Y\|}^{-1}(u)^p \,\mathrm{d}S(u) \\
&= -F_{\|Y\|}^{-1}(u)^p \cdot S(u)\Big|_{u=0}^1 + \int_0^1 S(u)\,\mathrm{d}F_{\|Y\|}^{-1}(u)^p \\
&= 0 + \int_0^\infty S\big(F_{\|Y\|}(u)\big)\mathrm{d}u^p = \int_0^\infty p u^{p-1}\, S\big(F_{\|Y\|}(u)\big)\mathrm{d}u,
\end{aligned}$$

as $\|Y\| \ge 0$. This is the assertion as announded. □

Comparison of norms. The following inequalities relate the norms of $L_\sigma^p(X)$ and $L^p(X)$. They turn out to be useful to establish completeness of the linear space $L_\sigma^p(X)$.

Proposition 10. *For $p < p'$ it holds that*

$$\|Y\|_1 \le \|Y\|_p \le \|Y\|_{\sigma,p} \le \|Y\|_{\sigma,p'} \tag{9}$$

and hence $L^p(X) \supseteq L_\sigma^p(X) \supseteq L_\sigma^{p'}(X)$. Further

$$\|Y\|_{\sigma,p} \le \|\sigma\|_q^{1/p} \cdot \|Y\|_{p'},$$

where $q = \frac{p'}{p'-p}$ and $\|\sigma\|_q := \left(\int_0^1 \sigma(u)^q \mathrm{d}u\right)^{1/q}$.

Proof. The function $\sigma(\cdot)$ and the function $F_{\|Y\|}^{-1}(\cdot)$ are nondecreasing and nonnegative. It follows from the continuous version of Chebyshev's sum inequality (cf. Hardy et al. [17]) that

$$\|Y\|_{\sigma,p}^p = \int_0^1 \sigma(u) F_{\|Y\|}^{-1}(u)^p \mathrm{d}u \ge \int_0^1 \sigma(u)\mathrm{d}u \cdot \int_0^1 F_{\|Y\|}^{-1}(u)^p \mathrm{d}u = \|Y\|_p^p,$$

which is the second inequality. Further it holds by Hölder's inequality (note that $\frac{1}{q}+\frac{1}{p'/p}=1$) that

$$\begin{aligned}\|Y\|_{\sigma,p}^p &= \int_0^1 \sigma(u)^{\frac{1}{q}}\sigma(u)^{\frac{1}{p'/p}}F_{\|Y\|}^{-1}(u)^p\mathrm{d}u\\ &\le \left(\int_0^1\sigma(u)\mathrm{d}u\right)^{1/q}\cdot\left(\int_0^1\sigma(u)F_{\|Y\|}^{-1}(u)^{p'}\mathrm{d}u\right)^{p/p'} = \|Y\|_{\sigma,p'}^p\end{aligned}$$

and

$$\begin{aligned}\|Y\|_{\sigma,p}^p &= \int_0^1 \sigma(u)F_{\|Y\|}^{-1}(u)^p\mathrm{d}u\\ &\le \left(\int_0^1\sigma(u)^q\mathrm{d}u\right)^{1/q}\cdot\left(\int_0^1 F_{\|Y\|}^{-1}(u)^{p\frac{p'}{p}}\mathrm{d}u\right)^{p/p'} = \|\sigma\|_q\cdot\|Y\|_{p'}^p,\end{aligned}$$

from which the assertions are immediate. □

Theorem 11. *The pair* $\left(L_\sigma^p(X),\|\cdot\|_{\sigma,p}\right)$ *is a Banach space.*

Proof. Suppose that the sequence Y_k, $k=1,2,\dots$ is a Cauchy sequence. Then it follows from (9) that Y_k is Cauchy with respect to $L^1(X)$. As $L^1(X)$ is complete there is a limit $Y\in L^1(X)$ with $\|Y_k-Y\|_1\to 0$, as $k\to\infty$. It remains to be shown that $Y\in L_\sigma^p(X)$.

From convergence in $L^1(X)$ it follows further that $\|Y_k\|$ converges in distribution, and thus that $F_{\|Y_k\|}^{-1}(\alpha)\to F_{\|Y\|}^{-1}(\alpha)$ at every point of continuity (cf. van der Vaart [37, Lemma 21.2]). As Y_k is Cauchy with respect to $\|\cdot\|_{\sigma,p}$ there is $k^*\in\mathbb{N}$ such that $\|Y_{k^*}-Y_k\|_{\sigma,p}<\varepsilon$ for all $k>k^*$, and it follows that $\|Y_k\|_{\sigma,p}\le\|Y_{k^*}-Y_k\|_{\sigma,p}+\|Y_{k^*}\|_{\sigma,p}<\|Y_{k^*}\|_{\sigma,p}+\varepsilon$. Now

$$\begin{aligned}\|Y\|_{\sigma,p}^p &= \int_0^1\sigma(u)F_{\|Y\|}^{-1}(u)^p\mathrm{d}u=\int_0^1\sigma(u)\liminf_{k\to\infty}F_{\|Y_k\|}^{-1}(u)^p\mathrm{d}u\\ &\le \liminf_{k\to\infty}\int_0^1\sigma(u)F_{\|Y_k\|}^{-1}(u)^p\mathrm{d}u=\liminf_{k\to\infty}\|Y_k\|_{\sigma,p}^p<\left(\|Y_{k^*}\|_{\sigma,p}+\varepsilon\right)^p<\infty\end{aligned}$$

by Fatou's inequality. Hence, $Y\in L_\sigma^p(X)$ and $L_\sigma^p(X)$ thus is complete. □

Proposition 12. *Simple functions, and* $L^\infty(X)$ *are dense in* $L_\sigma^p(X)$. *Even more,* $L^\infty(X)\subseteq L_\sigma^p(X)$ *and*

$$\|Y\|_{\sigma,p}\le\|Y\|_\infty$$

whenever $Y\in L^\infty(X)$ *and* $p<\infty$.

Proof. As for the second assertion note that $0\le F_{\|Y\|}^{-1}(\cdot)\le\|Y\|_\infty$, and thus

$$\|Y\|_{\sigma,p}^p=\int_0^1\sigma(u)F_{\|Y\|}^{-1}(u)^p\mathrm{d}u\le\int_0^1\sigma(u)\|Y\|_\infty^p\,\mathrm{d}u=\|Y\|_\infty^p.$$

For $Y\in L_\sigma^p(X)$ choose $u_\varepsilon<1$ such that $\int_{u_\varepsilon}^1\sigma(u)F_{\|Y\|}^{-1}(u)^p\mathrm{d}u<\varepsilon^p$. Further, as X is separable, one may find a sequence $\{x_i\colon i=1,\dots\}\subset X$ such that $\bigcup_{i=1}B_{\varepsilon/2}(x_i)\supseteq X$. Let $E_i\subseteq B_{\varepsilon/2}(x_i)$ be disjoint sets with $X=\bigcup_{i=1}^\infty E_i$, and find n such that $\sum_{i=1}^n P(E_i)>1-\varepsilon$. Define $x_i:=$

$\mathbb{E}(Y|\,Y\in E_i)$. By rearranging the enumeration, one may assume that $\|x_i\|\le\|x_{i+1}\|$ for all $i\le n$. Finally define $Y_\varepsilon:=\sum_{i=1}^n \mathbb{1}_{E_i}x_i$. Note, that $\|Y-Y_\varepsilon\|\le\varepsilon$ on $\bigcup_{i=1}^n E_i$, and $P\left(\|Y-Y_\varepsilon\|\ge\varepsilon\right)\le\varepsilon$. Hence

$$\|Y-Y_\varepsilon\|_{\sigma,p}^p\le\int_0^1 F_{\|Y-Y_\varepsilon\|}^{-1}(u)^p\sigma(u)\mathrm{d}u\le\int_0^{u_\varepsilon}\varepsilon^p\sigma(u)\mathrm{d}u+\int_{u_\varepsilon}^1 F_{\|Y\|}^{-1}(u)^p\sigma(u)\mathrm{d}u<2\varepsilon^p. \tag{10}$$

The assertion follows, as Y_ε is a simple function and the right side of (10) can be made arbitrarily small. ☐

The essential relation to the risk measures introduced in the introductory Section 2 is the following proposition on continuity. This is an immediate consequence of Proposition 3 and (9) (with $p=1$).

Proposition 13. *The risk measure ρ_Z, considered on the space $L_\sigma^p(X)$, is (Lipschitz) continuous*

$$|\rho_Z(Y)-\rho_Z(Y')|\le\|Y-Y'\|_{\sigma,p}$$

for every $p\ge1$.

For the sake of completeness we mention the following statement.

Theorem 14. *The space $L_\sigma^p(X)$ is not a Hilbert space, unless $p=2$ and $\sigma(\cdot)=1$.*

Proof. It is straight forward to verify that the random variables $Y_1=c\cdot\mathbb{1}_E$ and $Y_2=c'\cdot\mathbb{1}_{E^\complement}$ with $0<P(E)<1$ violate the parallelogram law. ☐

4 Duality theory for $L_\sigma(X)$ $(p=1)$

We shall establish first that the space L_σ does not have a pre-dual space and hence is not reflexive. This is the same result as for L^1 (although the same proof does not apply for L_σ).

The second part of this section introduces Bochner integrable random variables. This is essential to establish the dual of $L_\sigma(X)$, which involves the Radon–Nikodým property of the state space X.

4.1 A pre-dual does not exist

The following statement establishes non-existence of a pre-dual space. Perhaps it is interesting to note that the following proof works without particular knowledge about the dual space of $L_\sigma(X)$.

Theorem 15. *The Banach space $\left(L_\sigma^1(X),\|\cdot\|_\sigma\right)$ does not have a pre-dual: there does not exist a Banach space $(E,\|\cdot\|)$, say, such that its topological dual space is $(E,\|\cdot\|)^*=\left(L_\sigma^1(X),\|\cdot\|_\sigma\right)$.*

Proof. Suppose that $\left(L_\sigma^1,\|\cdot\|_\sigma\right)$ were the dual of $(E,\|\cdot\|)$. Then, by Alaoglu's Theorem (cf. Wojtaszczyk [39]), the unit ball of $L_\sigma^1(X)$ is weakly* compact. Further, by the Krein–Milman Theorem, the closed unit ball equals the closure of the convex hull of its extreme points. However, we shall demonstrate now that there is a random variable in the unit ball of $L_\sigma^1(X)$ which is not contained in the closure of its extreme points.

Suppose that $Y \in L^1_\sigma(X)$ with $\|Y\|_\sigma = 1$. Define

$$Y'(\omega) := \begin{cases} \frac{\|Y(\omega)\|-1}{\|Y(\omega)\|} Y(\omega) & \text{if } \|Y(\omega)\| \geq 1, \\ 0 & \text{else} \end{cases}$$

and set $\lambda := \|Y'\|_\sigma$. As $\|Y'(\omega)\| < \|Y(\omega)\|$ (except on $\{\|Y(\cdot)\| = 0\}$) it follows that $0 \leq \lambda = \|Y'\|_\sigma < \|Y\|_\sigma = 1$. Note, that

$$F^{-1}_{\|Y'\|}(\cdot) = \max\left\{F^{-1}_{\|Y\|}(\cdot) - 1,\, 0\right\} \text{ and } F^{-1}_{\|Y-Y'\|}(\cdot) = \min\left\{F^{-1}_{\|Y\|}(\cdot),\, 1\right\},$$

and both functions are nondecreasing. Further it holds that $F^{-1}_{\|Y\|} = F^{-1}_{\|Y'\|} + F^{-1}_{\|Y-Y'\|}$, such that

$$\begin{aligned} 1 = \|Y\|_\sigma &= \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)\mathrm{d}u = \int_0^1 \sigma(u)\left(F^{-1}_{\|Y'\|}(u) + F^{-1}_{\|Y-Y'\|}(u)\right)\mathrm{d}u \\ &= \int_0^1 \sigma(u) F^{-1}_{\|Y'\|}(u)\mathrm{d}u + \int_0^1 \sigma(u) F^{-1}_{\|Y-Y'\|}(u)\mathrm{d}u \\ &= \|Y'\|_\sigma + \|Y-Y'\|_\sigma = \lambda \left\|\frac{1}{\lambda}Y'\right\|_\sigma + (1-\lambda)\left\|\frac{1}{1-\lambda}(Y-Y')\right\|_\sigma, \end{aligned}$$

provided that $\lambda > 0$. As $\left\|\frac{1}{\lambda}Y'\right\|_\sigma = 1$, it follows that $\left\|\frac{1}{1-\lambda}(Y-Y')\right\|_\sigma = 1$ and further that $Y = \lambda \cdot \frac{1}{\lambda}Y' + (1-\lambda)\cdot\frac{1}{1-\lambda}(Y-Y')$ is the convex combination of two different random variables with norm 1, such that Y is not an extreme point in the unit ball of L_σ.

Every extreme point in the unit sphere of $L_\sigma(X)$ thus satisfies $\lambda = 0$, i.e., $\|Y(\omega)\| \leq 1$ for almost all $\omega \in \Omega$. Further, any convex combination of extremal points satisfies $\|Y(\omega)\| \leq 1$ for almost all $\omega \in \Omega$, and every point Y from the closure of the convex hull of extreme points satisfies $\|Y(\omega)\| \leq 1$ as well.

Choose $u \in (0,1)$ such that $\sigma(u) > 0$, a measurable set $A \in \mathcal{F}$ with $1-u < P(A) < 1$ and define the random variable $Y := \mathbb{1}_A \cdot \frac{1}{\|x\|\cdot\int_{1-P(A)}^1 \sigma(u)\mathrm{d}u} x$ (where $0 \neq x \in X$). Then $\|Y\|_\sigma = 1$, but Y is not in the closure of the convex combination of extreme points, as $\|Y(\omega)\| > 1$ for $\omega \in A$. Hence, the unit ball of L_σ is strictly larger than the closure of the convex hull of its extreme points. This completes the proof. □

The following is an immediate consequence.

Corollary 16. *The Banach space* $\left(L^1_\sigma(X), \|\cdot\|_\sigma\right)$ *is not reflexive.*

We shall outline below that the dual of $\left(L^1_\sigma(X), \|\cdot\|_\sigma\right)$ is not separable.

4.2 The dual of $L_\sigma(X)$

The duality theory of $L_\sigma(X)$ involves the dual of the state space X. Here we establish the relevant space first and relate it to $L_\sigma(X)$ in a second step.

Definition 17. For a random variable Z with values in $(X, \|\cdot\|)$ define

$$\|Z\|^*_\sigma := \sup_{\alpha<1} \frac{\mathsf{AV@R}_\alpha(\|Z\|)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} \tag{11}$$

and the set of all (equivalence classes of) X-valued Bochner integrable functions

$$L_\sigma^*(X) := \left\{Z:\Omega\to X:\ \|Z\|_\sigma^* < \infty\right\}.$$

Remark 18 (Stochastic dominance of second order). Risk functionals are convex functionals and hence enjoy a representation involving the convex conjugate according the Fenchel–Moreau theorem (cf. Ruszczyński et al. [35]). The convex conjugate of risk functionals on various spaces involves second order stochastic dominance relations. This is elaborated in the literature, cf. for example Shapiro [34], Föllmer and Schied [13]or Pichler [30].

The definition of $\|\cdot\|_\sigma^*$ reflects the duality of risk functionals. Indeed, the supremum (11) can be restated as

$$\|Z\|_\sigma^* = \inf\left\{\eta>0:\mathsf{AV@R}_\alpha(\|Z\|)\le \eta\cdot\int_\alpha^1\sigma(u)\mathrm{d}u \text{ for all } \alpha<1\right\}.$$

This equivalent formulation involves the statement

$$\mathsf{AV@R}_\alpha(\|Z\|)\le \mathsf{AV@R}_\alpha\big(\eta\,\sigma(U)\big), \tag{12}$$

where U is coupled in a comonotone way with $\|Z\|$. Following Ogryczak and Ruszczyński [26], (12) is equivalent to saying that $\|Z\|$ is dominated by $\|Z\|_\sigma^*\cdot\sigma(U)$ in second stochastic order.[3]

Remark 19. By the choice $\alpha=0$ in (11) it follows that

$$\|Z\|_\sigma^* \ge \mathsf{AV@R}_0\left(\|Z\|\right) = \int_0^1 F_{\|Z\|}^{-1}(u)\mathrm{d}u = \mathbb{E}\,\|Z\| = \|Z\|_1, \tag{13}$$

such that $L^1(X)\supseteq L_\sigma^*(X)$ for every σ.

Theorem 20. *$\left(L_\sigma^*(X),\|\cdot\|_\sigma^*\right)$ is a Banach space.*

Proof. The norm $\|\cdot\|_\sigma^*$ is positively homogeneous, as the Average Value-at-Risk, $\mathsf{AV@R}$, is positively homogeneous. The triangle inequality is satisfied, because of the triangle inequality in the state space $(X,\|\cdot\|)$, and as

$$\mathsf{AV@R}_\alpha\left(\|Z_1+Z_2\|\right)\le \mathsf{AV@R}_\alpha\left(\|Z_1\|+\|Z_2\|\right)\le \mathsf{AV@R}_\alpha\left(\|Z_1\|\right)+\mathsf{AV@R}_\alpha\left(\|Z_2\|\right)$$

by monotonicity and subadditivity of the Average Value-at-Risk.

It remains to be shown that $L_\sigma^*(X)$ is complete with respect to the norm $\|\cdot\|_\sigma^*$. To this end let Z_k be a Cauchy sequence. Then there is an index k^* such that $\|Z_k\|_\sigma^*\le\|Z_{k^*}\|_\sigma^*+\|Z_k-Z_{k^*}\|_\sigma^*<\|Z_{k^*}\|_\sigma^*+\varepsilon$. Hence, $\mathsf{AV@R}_\alpha\left(\|Z_k\|\right)\le\frac{\|Z_{k^*}\|_\sigma^*+\varepsilon}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u$.

By (13), the sequence Z_k is a Cauchy sequence for the norm $\|\cdot\|_1$ as well, and by completeness of $L^1(X)$ it follows that there is a limit $Z\in L^1(X)$. As in the proof of Theorem 11 the sequence Z_k converges in distribution, and thus $F_{\|Z_k\|}^{-1}(\alpha)\to F_{\|Z\|}^{-1}(\alpha)$.

By Fatou's inequality again,

$$\begin{aligned}\mathsf{AV@R}_\alpha\left(\|Z\|\right) &= \frac{1}{1-\alpha}\int_\alpha^1 F_{\|Z\|}^{-1}(u)\sigma(u)\mathrm{d}u = \frac{1}{1-\alpha}\int_\alpha^1 \liminf_{k\to\infty}F_{\|Z_k\|}^{-1}(u)\sigma(u)\mathrm{d}u\\ &= \liminf_{k\to\infty}\frac{1}{1-\alpha}\int_\alpha^1 F_{\|Z_k\|}^{-1}(u)\sigma(u)\mathrm{d}u\\ &= \liminf_{k\to\infty}\mathsf{AV@R}_\alpha\left(\|Z_k\|\right)\le\frac{\|Z_{k^*}\|_\sigma^*+\varepsilon}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u.\end{aligned}$$

[3] Cf. Dentcheva et al. [6, 7, 33] for stochastic dominance of second order.

It follows that $\|Z\|_\sigma^* \le \|Z_{k^*}\|_\sigma^* + \varepsilon < \infty$ and $Z \in L_\sigma^*(X)$, thus $L_\sigma^*(X)$ is complete. □

Bochner integrable functions. We consider vector measures $\mu : \mathcal{F} \to X$ which are finitely additive ($\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ whenever E_1 and E_2 are disjoint members of $\mathcal{F}$). The *variation* of a (finitely additive) vector measure μ is

$$\|\mu\|(E) := \sup_\pi \sum_{E_i \in \pi} \|\mu(E_i)\|,$$

where the supremum is among all partitions $\pi = \{E_1, \dots E_n\}$ of E into a *finite* number of pairwise disjoints members of $\mathcal{F}$. For measures μ with bounded variation ($\|\mu\|(\Omega) < \infty$) the variation $\|\mu\| : \mathcal{F} \to \mathbb{R}$ is a (finitely additive) measure.

Before we proceed to identify the dual let us recall the following fact for Bochner integrable functions. Consider $f \in L^1(X)$, which is Bochner integrable by definition with norm $\|f\|_1 = \int \|f\| \,\mathrm{d}P$. The vector-valued measure induced by f is

$$\mu_f(E) := \int_E f \mathrm{d}P.$$

For the Bochner integrable function f it holds further that $\int_E \|f\| \,\mathrm{d}P = \|\mu_f\|(E)$, such that in particular $\|f\|_1 = \|\mu_f\|(\Omega)$ (cf. Diestel et al. [8, Theorem 4, p. 46]).

Further, by the Hahn–Banach theorem, it holds that $\|\mu_f(E')\| = \sup_{\|x^*\| \le 1} x^*\big(\mu_f(E')\big)$, such that

$$\int_E \|f\| \,\mathrm{d}P = \|\mu_f\|(E) = \sup_{\pi,\, \|x_i^*\| \le 1} \sum_{i=1}^n x_i^*\big(\mu_f(E_i)\big),$$

where $x_i^* \in X^*$ and $\pi = \{E_1, \dots E_n\}$ is a finite partition of E again.

Recall now that $x_i^*\big(\mu_f(E_i)\big) = x_i^*\left(\int_{E_i} f \mathrm{d}P\right) = \int_{E_i} x_i^*(f) \mathrm{d}P$ by Hille's theorem, such that

$$\int_E \|f\| \,\mathrm{d}P = \sup_{\pi,\, \|x_i^*\| \le 1} \int \left\langle \sum_{i=1}^n \mathbb{1}_{E_i} x_i^*,\, f \right\rangle \mathrm{d}P.$$

If the state space of f is a dual space itself, $f \in L^1(X^*)$, then it is obvious by the same reasoning that

$$\int_E \|f\| \,\mathrm{d}P = \sup_{\pi,\, \|x_i\| \le 1} \int \left\langle f, \sum_{i=1}^n \mathbb{1}_{E_i} x_i \right\rangle \mathrm{d}P, \tag{14}$$

where x_i are chosen in the unit ball of the space X.

We are ready to prove the following embedding.

Theorem 21. *Let $(X, \|\cdot\|)$ be a Banach space with separable dual $(X, \|\cdot\|)^* =: \big(X^*, \|\cdot\|^*\big)$. Then $\big(L_\sigma^*(X^*), \|\cdot\|_\sigma^*\big)$ is isometric to a subspace of $\big(L_\sigma(X), \|\cdot\|_\sigma\big)^*$.*

Proof. Let $Z \in L_\sigma^*(X^*)$ be a random variable with values $Z(\omega) \in X^*$ and consider the linear form

$$\ell_Z(Y) := \mathbb{E}\langle Z, Y\rangle = \int_\Omega \langle Z, Y\rangle \,\mathrm{d}P = \int_\Omega \langle Z(\omega), Y(\omega)\rangle \, P(\mathrm{d}\omega).$$

We demonstrate first that $\|\ell_Z\| = \|Z\|_\sigma^*$.

From the Hardy–Littlewood inequality (cf. [17] and (4)) it follows that

$$\ell_Z(Y) = \mathbb{E}\left\langle Z, Y\right\rangle \leq \mathbb{E}\left\|Z\right\|\left\|Y\right\| \leq \int_0^1 F_{\|Z\|}^{-1}(u) F_{\|Y\|}^{-1}(u)\mathrm{d}u.$$

Define $G(\alpha) := \int_\alpha^1 F_{\|Z\|}^{-1}(u)\mathrm{d}u$, observe that $G(0) = \mathbb{E}\left\|Z\right\| = \left\|Z\right\|_1$, then

$$\begin{aligned}
\ell_Z(Y) &\leq -\int_0^1 F_{\|Y\|}^{-1}(u)\,\mathrm{d}G(u) = -\left. F_{\|Y\|}^{-1}(u) G(u)\right|_{u=0}^1 + \int_0^1 G(u)\,\mathrm{d}F_{\|Y\|}^{-1}(u) \\
&= F_{\|Y\|}^{-1}(0)\,\left\|Z\right\|_1 + \int_0^1 G(u)\,\mathrm{d}F_{\|Y\|}^{-1}(u).
\end{aligned}$$

Note now that $G(\alpha) = \int_\alpha^1 F_{\|Z\|}^{-1}(u)\mathrm{d}u \leq \|Z\|_\sigma^* \cdot \int_\alpha^1 \sigma(u)\mathrm{d}u = \|Z\|_\sigma^* \cdot S(\alpha)$ and the integrator $F_{\|Y\|}^{-1}(\cdot)$ is nondecreasing ($S(\alpha) = \int_\alpha^1 \sigma(u)\mathrm{d}u$, cf. (8)). Hence,

$$\ell_Z(Y) \leq F_{\|Y\|}^{-1}(0)\,\left\|Z\right\|_1 + \|Z\|_\sigma^* \int_0^1 S(u)\,\mathrm{d}F_{\|Y\|}^{-1}(u).$$

By Riemann–Stieltjes integration by parts thus,

$$\begin{aligned}
\ell_Z(Y) &\leq F_{\|Y\|}^{-1}(0)\,\left\|Z\right\|_1 + \|Z\|_\sigma^* \left. F_{\|Y\|}^{-1}(u) S(u)\right|_{u=0}^1 - \|Z\|_\sigma^* \int_0^1 F_{\|Y\|}^{-1}(u)\,\mathrm{d}S(u) \\
&= F_{\|Y\|}^{-1}(0)\left(\left\|Z\right\|_1 - \|Z\|_\sigma^*\right) + \|Z\|_\sigma^* \cdot \int_0^1 \sigma(u) F_{\|Y\|}^{-1}(u)\mathrm{d}u \\
&\leq \|Z\|_\sigma^* \cdot \left\|Y\right\|_\sigma,
\end{aligned}$$

as $\left\|Z\right\|_1 - \|Z\|_\sigma^* \leq 0$ by (13), and as $F_{\|Y\|}^{-1}(0) = \operatorname{ess\,inf}\left\|Y\right\| \geq 0$. It follows that $\|\ell_Z\| \leq \|Z\|_\sigma^*$.

For the converse inequality find $\alpha < 1$ such that $\frac{\mathsf{AV@R}_\alpha(\|Z\|)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} \geq \|Z\|_\sigma^* - \varepsilon$. Find a set E such that

$$\left\{\left\|Z\right\| > F_{\|Z\|}^{-1}(\alpha)\right\} \subseteq E \subseteq \left\{\left\|Z\right\| \geq F_{\|Z\|}^{-1}(\alpha)\right\} \text{ and } P(E) = 1-\alpha. \tag{15}$$

It holds that $\mathsf{AV@R}_\alpha(\|Z\|) = \frac{1}{1-\alpha}\int_E \left\|Z\right\|\mathrm{d}P$. Let $\pi = \{E_1, \dots E_n\}$ be a partition of E and x_i in the unit sphere of X be chosen such that

$$\int_E \left\|Z\right\|\mathrm{d}P < \int \left\langle Z, \sum_{i=1}^n \mathbb{1}_{E_i} x_i\right\rangle \mathrm{d}P + \varepsilon, \tag{16}$$

which is possible by (14).

Define $Y := \sum_{i=1}^n \mathbb{1}_{E_i} x_i$ and observe that

$$\left\|Y\right\|_\sigma = \int_0^1 \sigma(u) F_{\|Y\|}^{-1}(u)\mathrm{d}u = \int_\alpha^1 \sigma(u)\mathrm{d}u \leq 1.$$

From (16) and (15) it follows that

$$\begin{aligned}
\mathbb{E}\left\langle Z, Y\right\rangle + \varepsilon &\geq (1-\alpha)\mathsf{AV@R}_\alpha(\|Z\|) = \frac{\mathsf{AV@R}_\alpha(\|Z\|)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u}\left\|Y\right\|_\sigma \\
&\geq \left(\|Z\|_\sigma^* - \varepsilon\right)\left\|Y\right\|_\sigma \geq \|Z\|_\sigma^*\left\|Y\right\|_\sigma - \varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ is chosen arbitrarily it follows that $\|\ell_Z\| \geq \|Z\|_\sigma^*$, that is $\|\ell_Z\| = \|Z\|_\sigma^*$, which is the assertion. □

Theorem 22. *Let X^* have the Radon–Nikodým property. Then the dual $L_\sigma(X)^*$ can be identified with $L_\sigma^*(X^*)$.*

Proof. Let $\ell \in L_\sigma(X)^*$ be fixed. Define the X^*-valued measure

$$\mu(E)(x) := \ell\left(\mathbb{1}_E \cdot x\right) \text{ for } E \in \mathcal{F}.$$

Let x_i be in the closed unit ball of X and E_i be a finite partition. Then

$$\begin{aligned}\left|\sum_{i=1}^n \mu(E_i)(x_i)\right| &= \left|\ell\left(\sum_{i=1}^n x_i \mathbb{1}_{E_i}\right)\right| \\ &\leq \|\ell\| \cdot \left\|\sum_{i=1}^n x_i \mathbb{1}_{E_i}\right\|_\sigma \leq \|\ell\|.\end{aligned}$$

Taking the supremum with respect to all tessellations E_i and x_i in the unit ball shows that the X^*-valued measure μ is of bounded variation. By the Radon–Nikodým property of X^* there is a measurable $Z\colon \Omega \to X^*$, such that

$$\mu(E) = \int_E Z \mathrm{d}P = \int \mathbb{1}_E Z \mathrm{d}P.$$

Explicitly, $\ell(Y) = \sum_{i=1}^n \int_{E_i} Z(x_i)\mathrm{d}P = \int \langle Z, Y\rangle \mathrm{d}P$ for simple functions $Y = \sum_{i=1}^n \mathbb{1}_{E_i} x_i$.

It remains to verify that $Z \in L_\sigma(X^*)$ and $\ell(\cdot) = \mathbb{E}\langle Z, \cdot\rangle$. To this end define $E_n := \{\|Z\| \leq n\}$. The functional

$$\ell_n(Y) := \int_{E_n} \langle Z, Y\rangle \mathrm{d}P$$

is a bounded linear functional, as $Z \cdot \mathbb{1}_{E_n}$ is bounded. As $\ell_n(\cdot) = \ell(\cdot)$ for all simple functions supported by E_n, it follows that $\ell_n(\cdot) = \ell(\cdot)$ for *all* random variables $Y \in L_\sigma(X)$ supported by E_n. That is,

$$\ell\left(Y \cdot \mathbb{1}_{E_n}\right) = \int \langle Z \cdot \mathbb{1}_{E_n}, Y\rangle \mathrm{d}P$$

and $\|Z \cdot \mathbb{1}_{E_n}\|_\sigma^* \leq \|\ell\|$. The monotone convergence theorem implies that $\|Z\|_\sigma^* \leq \|\ell\|$, such that $Z \in L_\sigma^*(X^*)$. Finally it follows that

$$\ell(Y) = \lim_{n\to\infty} \int \langle Z \cdot \mathbb{1}_{E_n}, Y\rangle \mathrm{d}P = \int \langle Z, Y\rangle \mathrm{d}P$$

for all $Y \in L_\sigma(X)$ by continuity and the inequalities already established, which establishes that $\ell(\cdot) = \mathbb{E}\langle Z, \cdot\rangle$ for all $Y \in L_\sigma(X)$. This completes the proof. □

Theorem 23. *The space $\left(L_\sigma^*(X^*), \|\cdot\|_\sigma^*\right)$ is not separable.*

check this!

Proof. Consider the random variables $Y_A := \sigma(U)\mathbb{1}_A x$ for some A with $0 < P(A) < 1$ and $Y_B := \sigma(U)\mathbb{1}_B x$ with $A \cap B = \emptyset$ and some fixed $x \in X$. Then $\|Y_A - Y_B\| = \|Y_A + Y_B\| > \|Y_A\| =: r$.

Suppose that C is a countable, dense subset of L_σ^*. all sets functions □

5 Duality of the spaces L^p_σ for $p>1$

The dual space L^p_σ, $p>1$, is of different nature than for $p=1$, which was outlined in the previous section (Section 4). However, similar to L^p, the spaces L^p_σ are reflexive. But although reflexive, it turns out that the duals of L^p_σ are not L^q_σ spaces, except for the classical (Lebesgue) case with $\sigma(\cdot)=1$.

We develop the duality theory for the state space $\mathbb{R}$. The results extend to the general state spaces $(X,\|\cdot\|)$, but this extension is in line with the previous section.

5.1 The dual of L^p_σ

Definition 24. We say that Z' σ-dominates Z (in symbols $Z' {}_\sigma\!\succcurlyeq Z$) if there is a uniform random variable U such that

$$\mathsf{AV@R}_\alpha\big(\sigma(U)Z'\big)\ge \mathsf{AV@R}_\alpha(Z)\text{ for all }\alpha<1. \tag{17}$$

Further we define the mapping

$$\|Z\|^*_{\sigma,q}:=\inf\Big\{\|Z'\|_{\sigma,q}: Z' {}_\sigma\!\succcurlyeq |Z|\Big\} \tag{18}$$

and the set (of equivalence classes of) $L^{q*}_\sigma:=\Big\{Z:\|Z\|^*_{\sigma,q}<\infty\Big\}$.

Remark 25. By Hoeffding's Lemma (cf. [18]) one may assume that Z' and U are coupled in a comonotone way in (17). This establishes the equivalence

$$Z' {}_\sigma\!\succcurlyeq Z,\text{ iff }\int_\alpha^1\sigma(u)F^{-1}_{Z'}(u)\mathrm{d}u\ge\int_\alpha^1 F^{-1}_{|Z|}(u)\mathrm{d}u\text{ for all }\alpha<1.$$

Proposition 26. *It holds that*

$$\|Z\|_1\le\|Z\|^*_{\sigma,q}\le\|Z\|_{\sigma,q}$$

and thus $L^1\supseteq L^{q*}_\sigma\supseteq L^q_\sigma$.

Proof. Suppose that Z' is feasible for (18), $Z' {}_\sigma\!\succcurlyeq |Z|$. Then

$$\|Z\|_1=\int_0^1F^{-1}_{|Z|}(u)\mathrm{d}u\le\int_0^1\sigma(u)F^{-1}_{Z'}(u)\mathrm{d}u=\|Z'\|_{\sigma,1}\le\|Z'\|_{\sigma,q}$$

by choosing $\alpha=1$ in (18) and by (9), from which the first inequality is immediate.

The second inequality follows from $Z {}_\sigma\!\succcurlyeq Z$. Indeed, by Chebyshev's sum inequality it holds that

$$\begin{aligned}\mathsf{AV@R}_\alpha\big(\sigma(U)Z\big)&=\frac{1}{1-\alpha}\int_\alpha^1\sigma(u)F_Z^{-1}(u)\mathrm{d}u\\&=\int_0^1\sigma\big(\alpha+u(1-\alpha)\big)F_Z^{-1}\big(\alpha+u(1-\alpha)\big)\mathrm{d}u\\&\ge\int_0^1\sigma\big(\alpha+u(1-\alpha)\big)\mathrm{d}u\cdot\int_0^1F_Z^{-1}\big(\alpha+u(1-\alpha)\big)\mathrm{d}u\\&=\frac{1}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u\cdot\frac{1}{1-\alpha}\int_\alpha^1F_Z^{-1}(u)\mathrm{d}u\ge\mathsf{AV@R}_\alpha(Z).\end{aligned}$$

□

Theorem 27. $\left(L_\sigma^{q*}, \|\cdot\|_{\sigma,q}^*\right)$ *is a Banach space for* $1 \le q < \infty$*. In particular,* $\|\cdot\|_{\sigma,q}^*$ *is a norm for which the triangle inequality holds true.*

Remark 28. It is worth noting that the definition of the norm (18) is an *infimum* over *convex* functions $\|\cdot\|_{\sigma,q}$. In general the *supremum* over convex functions is convex, but not necessarily the infimum. However, convexity of the norm $\|\cdot\|_{\sigma,q}^*$ is immediate from Theorem 27.

Proof of Theorem 27. Positive homogeneity of $\|\cdot\|_{\sigma,q}^*$ is evident by positive homogeneity of the norm $\|\cdot\|_{\sigma,q}$ and the Average Value-at-Risk.

As for the triangle inequality let Z_1 and Z_2 be fixed. Without loss of generality we assume that $Z_1 \ge 0$ and $Z_2 \ge 0$.

(i) Assume in a first step that Z_1 and Z_2 are coupled in a comonotone way. Choose Z_1', Z_2' and U such that

$$\mathsf{AV@R}_\alpha\left(\sigma(U)Z_i'\right) \ge \mathsf{AV@R}_\alpha(|Z_i|) \text{ for all } \alpha < 1 \text{ and } i = 1, 2.$$

Then

$$\begin{aligned}\mathsf{AV@R}_\alpha\left(\sigma(U)(Z_1' + Z_2')\right) &= \mathsf{AV@R}_\alpha\left(\sigma(U)Z_1'\right) + \mathsf{AV@R}_\alpha\left(\sigma(U)Z_2'\right)\\ &\ge \mathsf{AV@R}_\alpha\left(Z_1\right) + \mathsf{AV@R}_\alpha\left(Z_2\right)\\ &\ge \mathsf{AV@R}_\alpha\left(Z_1 + Z_2\right)\end{aligned}$$

by subadditivity and monotonicity, such that $Z' := Z_1' + Z_2'$ is feasible for $Z := Z_1 + Z_2$. It holds that $\|Z_1' + Z_2'\|_{\sigma,q} \le \|Z_1'\|_{\sigma,q} + \|Z_2'\|_{\sigma,q}$. The triangle inequality of the norm $\|\cdot\|_{\sigma,q}^*$,

$$\|Z_1 + Z_2\|_{\sigma,q}^* \le \|Z_1\|_{\sigma,q}^* + \|Z_2\|_{\sigma,q}^*,$$

follows by passing to the infimum.

(ii) If Z_1 and Z_2 are not coupled in a comonotone way, then there is a copy $\tilde{Z}_1$ of Z_1 such that $\tilde{Z}_1$ and Z_2 are comonotone. As the Average Value-at-Risk is comonotone additive and subadditive it follows that

$$\begin{aligned}\mathsf{AV@R}_\alpha\left(\tilde{Z}_1 + Z_2\right) &= \mathsf{AV@R}_\alpha\left(\tilde{Z}_1\right) + \mathsf{AV@R}_\alpha\left(Z_2\right)\\ &= \mathsf{AV@R}_\alpha\left(Z_1\right) + \mathsf{AV@R}_\alpha\left(Z_2\right)\\ &\ge \mathsf{AV@R}_\alpha\left(Z_1 + Z_2\right).\end{aligned}$$

Hence, in view of (17) it follows that

$$\|Z_1 + Z_2\|_{\sigma,q}^* \le \left\|\tilde{Z}_1 + Z_2\right\|_{\sigma,q}^* \le \left\|\tilde{Z}_1\right\|_{\sigma,q}^* + \|Z_2\|_{\sigma,q}^* = \|Z_1\|_{\sigma,q}^* + \|Z_2\|_{\sigma,q}^*.$$

To accept that L_σ^{q*} is complete note that Proposition 26 implies that every $\|\cdot\|_{\sigma,q}^*$-Cauchy sequence is a Cauchy sequence for the norm $\|\cdot\|_1$ as well. Hence the limit exists and the remainder of the proof is along the lines of the proof of Theorem 20. □

Proposition 29. *Simple functions (and thus* L^∞*) are dense in* L_σ^{q*}*, whenever* $q < \infty$*.*

Proof. Let $\mathfrak{F}$ contain all *finite* sigma algebras $\mathcal{F}$ for which the measure P is defined. Note that $(\mathfrak{F},\subseteq)$ is a filter, and the proof of Proposition 12 actually demonstrates that

$$\left\|\mathbb{E}\left(Y|\mathcal{F}\right)-Y\right\|_{\sigma,p}\underset{\mathfrak{F}}{\to}0$$

whenever $\mathcal{F}\in\mathfrak{F}$ increases.

Recall first that $\mathsf{AV@R}_\alpha\big(\mathbb{E}(Y|\mathcal{F})\big)\leq\mathsf{AV@R}_\alpha(Y)$. Indeed, it follows from the conditional Jensen inequality (cf. Williams [38, Section 34]) that $\big(\mathbb{E}(Y|\mathcal{F})-q\big)_+\leq\mathbb{E}\big((Y-q)_+|\mathcal{F}\big)$, and hence, using (1),

$$\begin{aligned}\mathsf{AV@R}_\alpha\big(\mathbb{E}(Y|\mathcal{F})\big) &= \min_{q\in\mathbb{R}} q+\frac{1}{1-\alpha}\mathbb{E}\big(\mathbb{E}(Y|\mathcal{F})-q\big)_+\leq\min_{q\in\mathbb{R}} q+\frac{1}{1-\alpha}\mathbb{E}\mathbb{E}\big((Y-q)_+|\mathcal{F}\big)\\ &= \min_{q\in\mathbb{R}} q+\frac{1}{1-\alpha}\mathbb{E}\big((Y-q)_+|\mathcal{F}\big)=\mathsf{AV@R}_\alpha(Y).\end{aligned}$$

Suppose that $Z'\,{}_\sigma\!\succcurlyeq Z$. It follows that

$$\int_\alpha^1\sigma(u)F_{Z'}^{-1}(u)\mathrm{d}u\geq\int_\alpha^1F_Z^{-1}(u)\mathrm{d}u\geq\int_\alpha^1F_{\mathbb{E}(Z|\mathcal{F})}^{-1}(u)\mathrm{d}u$$

for every $\alpha\leq1$, that is $Z'\,{}_\sigma\!\succcurlyeq\mathbb{E}\left(Z|\mathcal{F}\right)$ and thus $\left\|\mathbb{E}\left(Z|\mathcal{F}\right)\right\|_{\sigma,q}^*\leq\left\|Z\right\|_{\sigma,q}^*$. The assertion follows as $\{\mathbb{E}\left(Z|\mathcal{F}\right):\,\mathcal{F}\in\mathfrak{F}\}$ is arbitrarily close to Z in the norm $\left\|\cdot\right\|_{\sigma,q}$ by Proposition 12. □

The following relation is crucial to obtain the duality result for L_σ^p.

Proposition 30. *It holds that*

$$\mathbb{E}\,YZ\leq\left\|Y\right\|_{\sigma,p}\cdot\left\|Z\right\|_{\sigma,q}^*\tag{19}$$

whenever $Y\in L_\sigma^p$ *and* $Z\in L_\sigma^{q*}$, $1<p<\infty$ *and the exponents are conjugate,* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Note first that $\mathbb{E}YZ\leq\mathbb{E}\left|Y\right|\left|Z\right|$, without loss of generality we thus assume that $Y\geq0$ and $Z\geq0$.

Chose U and Z' with $\mathsf{AV@R}_\alpha\big(\sigma(U)Z'\big)\geq\mathsf{AV@R}_\alpha(Z)$ for all $\alpha<1$, and recall that $\int_\alpha^1\sigma(u)F_{Z'}^{-1}(u)\mathrm{d}u\geq\int_\alpha^1F_Z^{-1}(u)\mathrm{d}u$. Then

$$\begin{aligned}\mathbb{E}YZ &\leq \int_0^1F_Y^{-1}(u)F_Z^{-1}(u)\mathrm{d}u=-\int_0^1F_Y^{-1}(u)\,\mathrm{d}\int_u^1F_Z^{-1}(t)\mathrm{d}t\\ &= -F_Y^{-1}(u)\int_u^1F_Z^{-1}(t)\mathrm{d}t\Big|_{u=0}^1+\int_0^1\int_u^1F_Z^{-1}(t)\mathrm{d}t\,\mathrm{d}F_Y^{-1}(u)\\ &= F_Y^{-1}(0)\cdot\mathbb{E}Z+\int_0^1\int_u^1F_Z^{-1}(t)\mathrm{d}t\,\mathrm{d}F_Y^{-1}(u).\end{aligned}$$

Observe now that $Z'\,{}_\sigma\!\succcurlyeq Z$ and F_Y^{-1} is nondecreasing, thus

$$\mathbb{E}YZ\leq F_Y(0)\cdot\mathbb{E}Z+\int_0^1\int_u^1\sigma(t)F_{Z'}^{-1}(t)\mathrm{d}t\,\mathrm{d}F_Y^{-1}(u).$$

By Riemann–Stieltjes integration again it follows that

$$\begin{aligned}\mathbb{E}YZ &\leq F_Y(0)\cdot\mathbb{E}Z + F_Y^{-1}(u)\int_u^1 \sigma(t)F_{Z'}^{-1}(t)\mathrm{d}t\Big|_{u=0}^1 - \int_0^1 F_Y^{-1}(u)\mathrm{d}\int_u^1 \sigma(t)F_{Z'}^{-1}(t)\mathrm{d}t\\ &= F_Y(0)\left(\mathbb{E}Z - \mathbb{E}\sigma(U)Z'\right) + \int_0^1 F_Y^{-1}(u)\sigma(u)F_{Z'}^{-1}(u)\mathrm{d}u.\end{aligned}$$

By choosing $\alpha=0$ in (17) it follows that $\mathbb{E}Z\leq\mathbb{E}\sigma(U)Z'$, and it holds that $F_Y(0)\geq 0$. Hence

$$\mathbb{E}YZ \leq \int_0^1 F_Y^{-1}(u)\sigma(u)F_{Z'}^{-1}(u)\mathrm{d}u.$$

One may apply Hölder's inequality now to arrive at

$$\begin{aligned}\mathbb{E}YZ &\leq \int_0^1 F_Y^{-1}(u)\sigma(u)^{\frac{1}{p}}\cdot\sigma(u)^{\frac{1}{q}}F_{Z'}^{-1}(u)\mathrm{d}u\\ &\leq \left(\mathbb{E}\sigma(U)Y^p\right)^{1/p}\cdot\left(\mathbb{E}\sigma(U)Z'^q\right)^{1/q} = \|Y\|_{\sigma,p}\cdot\|Z'\|_{\sigma,q}.\end{aligned}$$

By passing to the infimum it follows that

$$\mathbb{E}YZ \leq \|Y\|_{\sigma,p}\cdot\|Z\|_{\sigma,q}^*,$$

the assertion. □

5.2 The Hahn–Banach Functional for L_σ^p

To understand the dual space of L_σ^p for $p>1$ it is helpful to consider the random variable $Y\in L_\sigma^p$ together with

$$Z := \sigma(U)Y^{p-1} \text{ and } Z' := Y^{p-1} \tag{20}$$

first, where $Y\geq 0$ and U are coupled in a comonotone way. It is evident that $Z'{}_\sigma\!\succcurlyeq Z$ such that $\|Z\|_{\sigma,q}^*\leq\|Z'\|_{\sigma,q}$. However, $\|Z'\|_{\sigma,q}^q=\mathbb{E}\sigma(U)Y^{(p-1)q}=\mathbb{E}\sigma(U)Y^p=\|Y\|_{\sigma,p}^p$, and

$$\begin{aligned}\mathbb{E}YZ &= \mathbb{E}\sigma(U)Y^p = \|Y\|_{\sigma,p}^p = \|Y\|_{\sigma,p}^{p/q}\cdot\|Y\|_{\sigma,p}\\ &= \|Z'\|_{\sigma,q}\cdot\|Y\|_{\sigma,p}\geq\|Z\|_{\sigma,q}^*\cdot\|Y\|_{\sigma,p}.\end{aligned}$$

The constant $\|Z\|_{\sigma,q}^*$ thus cannot be improved in (19) and it follows that

$$\left\|\sigma(U)Y^{p-1}\right\|_{\sigma,q}^* = \|Z\|_{\sigma,q}^* = \|Z'\|_{\sigma,q} = \|Y\|_{\sigma,p}^{p/q} = \|Y\|_{\sigma,p}^{p-1}, \tag{21}$$

as Y and U are coupled in a comonotone way.

As well this demonstrates that

$$\|Y\|_{\sigma,p} = \max_{Z\neq 0}\frac{\mathbb{E}YZ}{\|Z\|_{\sigma,q}^*}, \tag{22}$$

as the supremum is attained for $Z:=\operatorname{sign}(Y)\,\sigma(U)\,|Y|^{p-1}$.

Example 31. As an example one deduces that

$$\|\sigma(U)\|_{\sigma,q}^{*}=1$$

(choose $Y\equiv 1$ in (21)).

Example 32 (The norm of simple functions)**.** Simple functions $Z=\mathbb{1}_E$ (E is a measurable set with probability $P(E)$) do not have a decomposition as in (20). To compute the norm $\|\mathbb{1}_E\|_{\sigma,q}^{*}$ consider

$$Z':=\frac{1}{\frac{1}{P(E)}\int_{1-P(E)}^{1}\sigma(u)\mathrm{d}u}\mathbb{1}_E$$

with norm $\|Z'\|_{\sigma,q}=P(E)\Big/\left(\int_{1-P(E)}^{1}\sigma(u)\mathrm{d}u\right)^{1-\frac{1}{q}}$.

Observe that

$$\begin{aligned}(1-\alpha)\mathsf{AV@R}_\alpha\left(\sigma Z'\right)\ge(1-\alpha)\mathsf{AV@R}_\alpha\left(Z\right)&=1-\alpha &&\text{if } \alpha\ge 1-P(E)\text{ and} \qquad (23)\\ \mathsf{AV@R}_\alpha\left(\sigma Z'\right)=\mathsf{AV@R}_\alpha\left(Z\right)&=\frac{P(E)}{1-\alpha} &&\text{if } \alpha\le 1-P(E),\end{aligned}$$

where (23) holds by concavity of the function $\alpha\mapsto(1-\alpha)\mathsf{AV@R}_\alpha(Z)$. It follows that $Z'\,{}_{\sigma}\!\succcurlyeq Z$. For $Y=\mathbb{1}_E$ it is thus evident that

$$\begin{aligned}\mathbb{E}ZY &= P(E)=\left(\int_{1-P(E)}^{1}\sigma(u)\mathrm{d}u\right)^{\frac{1}{p}}\cdot P(E)\left(\int_{1-P(E)}^{1}\sigma(u)\mathrm{d}u\right)^{\frac{1}{q}-1}\\ &= \|Y\|_{\sigma,p}\cdot\|Z'\|_{\sigma,q},\end{aligned}$$

from which we conclude that $\|Z\|_{\sigma,q}^{*}=\|Z'\|_{\sigma,q}$, or

$$\|\mathbb{1}_E\|_{\sigma,q}^{*}=\frac{1}{\frac{1}{P(E)}\left(\int_{1-P(E)}^{1}\sigma(u)\mathrm{d}u\right)^{1-\frac{1}{q}}}.$$

The useful bounds

$$P(E)\le\|\mathbb{1}_E\|_{\sigma,q}^{*}\le P(E)^{1/q}$$

are immediate from $P(E)\le\int_{1-P(E)}^{1}\sigma(u)\mathrm{d}u\le 1$ (cf. Example 8).

5.3 The dual of L_σ^{q*}

In order to establish duality for L_σ^{q*} we aim at finding a random variable $Y\in L_\sigma^{p}$ such that the supremum in

$$\|Z\|_{\sigma,q}^{*}=\sup_{Y\neq 0}\frac{\mathbb{E}YZ}{\|Y\|_{\sigma,p}} \qquad (24)$$

is attained (cf. (22)). A simple decomposition as in (20) is not always available, not even for simple functions (as outlined in Example 32).

A more involved construction for Y is required. This is explained in what follows.

The Hahn–Banach Functional for L_σ^{q*}. To observe that the inequality in (19) cannot be improved consider the concave functions

$$G(\alpha) := \int_\alpha^1 F_Z^{-1}(u)\mathrm{d}u \text{ and } S(\alpha) = \int_\alpha^1 \sigma(u)\mathrm{d}u$$

and define[4]

$$G_\sigma(\alpha) := \inf_{y\geq 0}\, y\cdot S(\alpha) - G_\sigma^*(y), \text{ where } G_\sigma^*(y) := \inf_{\alpha'\in[0,1]} y\cdot S(\alpha') - G(\alpha'). \tag{25}$$

It is evident that

$$G_\sigma(\alpha) = \inf_{y\geq 0}\, y\cdot S(\alpha) - G_\sigma^*(y) \geq G(\alpha).$$

Lemma 33. *G_σ is concave and nonincreasing, and there is an nonnegative, nondecreasing function $H(\cdot)$ such that $G_\sigma(\alpha) = \int_\alpha^1 H(u)\sigma(u)\mathrm{d}u$.*

Proof. $G_\sigma(\cdot)$ is concave, as $S(\cdot)$ is concave and as the infimum of concave functions in (25) is concave.

Then it holds that G_σ is bounded, as

$$0 \leq G(\alpha) \leq G_\sigma(\alpha) \leq 0\cdot S(\alpha) - G_\sigma^*(0) = \sup_\alpha G(\alpha) = G(0) = \mathbb{E}|Z|.$$

It follows from concavity of G_σ that there is a subderivative g_σ such that $G_\sigma(\alpha) = a + \int_\alpha^1 g_\sigma(v)\mathrm{d}v$ for some $a \in \mathbb{R}$.

Note, that $\tilde G_\sigma(\alpha') := \inf_{y\geq 0} y\cdot\alpha' - G_\sigma^*(y)$ is concave as well, that is

$$\tilde G_\sigma(x) \geq \frac{b-x}{b-a}\tilde G_\sigma(a) + \frac{x-a}{b-a}\tilde G_\sigma(b),$$

or equivalently

$$\frac{\tilde G_\sigma(u) - \tilde G_\sigma(a)}{u-a} \geq \frac{\tilde G_\sigma(b) - \tilde G_\sigma(u)}{b-u}$$

whenever $a < u < b$. As $G_\sigma = \tilde G_\sigma \circ S$ it follows that

$$\frac{G_\sigma(u) - G_\sigma(a)}{S(u)-S(a)} \leq \frac{G_\sigma(b) - G_\sigma(u)}{S(b)-S(u)},$$

as S is nonincreasing. Hence,

$$H(u) := \lim_{h\searrow 0} \frac{\int_u^{u+h} g_\sigma(v)\mathrm{d}v}{\int_u^{u+h}\sigma(v)\mathrm{d}v} = \frac{g_\sigma(u)}{\sigma(u)}$$

exists and is nondecreasing. ☐

Proposition 34. *For $1 \leq q < \infty$ the infimum in (18) to compute norm $\|Z\|_{\sigma,q}^*$ is attained and there exists $Y \neq 0$ for which the supremum in (24) is attained.*

[4] If S is invertible, then $G_\sigma = \left(G\circ S^{-1}\right)^{**}\circ S$, where f^{**} is f's upper semi-continuous, concave envelope.

Proof. Let U be coupled in a comonotone way with Z and let the function $H(\cdot)$ be defined as in Lemma 33. We shall show that the random variable

$$Y := H(U)^{q-1}$$

satisfies (19) with equality.

To compute $\mathbb{E}YZ$ we distinguish the following two situations first:

(i) Suppose that $G(u) < G_\sigma(u)$. By concavity of $\tilde{G}_\sigma$ there is y and an interval $(u_1, u_2) \ni u$ such that

$$\tilde{G}_\sigma(u_1) = y \cdot u_1 - G_\sigma^*(y),\ \tilde{G}_\sigma(u_2) = y \cdot u_2 - G_\sigma^*(y)$$

and

$$G(u) < G_\sigma(u) \text{ for all } u_1 < u < u_2.$$

In this interval it holds that $G_\sigma(u) = \tilde{G}_\sigma(S(u)) = y \cdot S(u) - G_\sigma^*(y)$ and hence $G'_\sigma(u) = y \cdot \sigma(u)$, such that $H(u) = y = \textit{constant}$. It follows that

$$\begin{aligned}\int_{u_1}^{u_2} H(u)^q \sigma(u)\mathrm{d}u &= y^{q-1}\int_{u_1}^{u_2} y\sigma(u)\mathrm{d}u = y^{q-1}\int_{u_1}^{u_2}\sigma(u)\mathrm{d}u = y^{q-1}\left(G_\sigma(u_2) - G_\sigma(u_1)\right)\\ &= y^{q-1}\left(G(u_2) - G(u_1)\right) = \int_{u_1}^{u_2} H(u)^{q-1} F_Z^{-1}(u)\mathrm{d}u. \qquad (26)\end{aligned}$$

(ii) Suppose further that $G(u) = G_\sigma(u)$, then there is a maximal interval with $G(u) = G_\sigma(u)$. The derivatives coincide in the interior, that is $F_Z^{-1}(u) = H(u)\sigma(u)$. It follows that

$$\int_{u_1}^{u_2} H(u)^q \sigma(u)\mathrm{d}u = \int_{u_1}^{u_2} H(u)^{q-1} F_Z^{-1}(u)\mathrm{d}u.$$

Combined with (26) it holds that

$$\mathbb{E}YZ = \int_0^1 H(u)^{q-1}F_Z^{-1}(u)\mathrm{d}u = \int_0^1 H(u)^q\sigma(u)\mathrm{d}u = \mathbb{E}\sigma(U)Y^p = \|Y\|_{\sigma,p}^p,$$

as $p = \frac{q}{q-1}$.

It follows along the same lines as (26) (choose $q = 1$) that

$$\begin{aligned}\int_{u_1}^{u_2} H(u)\sigma(u)\mathrm{d}u &= \int_{u_1}^{u_2} y\sigma(u)\mathrm{d}u = G_\sigma(u_2) - G_\sigma(u_1)\\ &= G(u_2) - G(u_1) = \int_{u_1}^{u_2} F_Z^{-1}(u)\mathrm{d}u,\end{aligned}$$

such that

$$Z' := H(U) \,{}_\sigma\!\succcurlyeq Z.$$

Hence

$$\|Z\|_{\sigma,q}^{*q} \le \|Z'\|_{\sigma,q}^q = \int H(u)^q\sigma(u)\mathrm{d}u = \|Y\|_{\sigma,p}^p$$

and consequently

$$\mathbb{E}YZ = \|Y\|_{\sigma,p}^p = \|Y\|_{\sigma,p}\cdot\|Y\|_{\sigma,p}^{p-1} \ge \|Y\|_{\sigma,p}\cdot\|Z\|_{\sigma,q}^*.$$

Thus, equality in (19) is established, the infimum in (18) is attained for $Z' = H(U)$ and the supremum in (24) for $Y = H(U)^{q-1}$. □

Corollary 35 (Comparison of norms)**.** *It holds that*

$$\|Z\|_1 \le \|Z\|_{\sigma,q}^* \le \|Z\|_q \le \|Z\|_{\sigma,q}\,,$$

and thus $L_1 \supseteq L_\sigma^{q*} \supseteq L^q$.

Proof.

$$\|Z\|_{\sigma,q}^* = \sup_{Y\neq 0} \frac{\mathbb{E}YZ}{\|Y\|_{\sigma,p}} \le \sup_{Y\neq 0} \frac{\|Y\|_p \cdot \|Z\|_q}{\|Y\|_{\sigma,p}} \le \|Z\|_q\,,$$

because $\|Y\|_{\sigma,p} \ge \|Y\|_p$ by (9).

The remaining inequalities follow from Proposition 26 and (9). □

Theorem 36. *The dual of the space* $\left(L_\sigma^p, \|\cdot\|_{\sigma,p}\right)$ *is isometric to* $\left(L_\sigma^{q*}, \|\cdot\|_{\sigma,q}^*\right)$*, where* $p > 1$ *and the exponents are conjugate,* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $\ell \in \left(L_\sigma^p, \|\cdot\|_{\sigma,p}\right)$ be fixed. The measure $\mu(E) := \ell\left(\mathbb{1}_E\right)$ is absolutely continuous with respect to P, as $P(E) = 0$ implies that

$$\left|\mu(E)\right| = \left|\ell(\mathbb{1}_E)\right| \le \|\ell\|\, \|\mathbb{1}_E\|_{\sigma,p} = \|\ell\| \cdot \int_{1-P(E)}^1 \sigma(u)\mathrm{d}u = 0.$$

μ is further σ-additive, as for pairwise disjoint sets E_i and $E := \bigcup_{i=1}^\infty E_i$

$$\left|\mu\left(E\right) - \sum_{i=1}^n \mu\left(E_i\right)\right| \le \|\ell\| \cdot \int_{1-P\left(E\setminus \bigcup_{i=1}^n E_i\right)}^1 \sigma(u)\mathrm{d}u \to 0,$$

whenever $n \to \infty$. Hence, by the Radon–Nikodým theorem there is a density Z and $\ell\left(\mathbb{1}_E\right) = \int_E Z\mathrm{d}P = \int \mathbb{1}_E Z\mathrm{d}P = \mathbb{E}\mathbb{1}_E Z$. By linearity and continuity it follows that $\ell\left(Y\right) = \mathbb{E}YZ$. Further it holds that $\|\ell\| = \sup_{Y\neq 0} \frac{\ell(Y)}{\|Y\|_{\sigma,p}} = \sup_{Y\neq 0} \frac{\mathbb{E}YZ}{\|Y\|_{\sigma,p}} = \|Z\|_{\sigma,q}^*$ by Proposition 30 and Proposition 34, which establishes the isometry. □

Theorem 37. *The spaces* $\left(L_\sigma^{q*}, \|\cdot\|_{\sigma,q}^*\right)$ *and* $\left(L_\sigma^p, \|\cdot\|_{\sigma,p}\right)$ *are reflexive, whenever* $q > 1$ *and* $p > 1$.

Proof. The proof is along the same lines as Theorem 36. Instead of Proposition 34, Section 5.2 has to be employed. □

6 Summary

This paper introduces Banach spaces, which naturally carry risk measures for vector-valued returns. Risk measures are continuous on these spaces. The spaces are built based on duality, and in this sense are natural for risk measures involving vector-valued returns. A consistent characterization of the topological dual is possible, if the state space enjoys the Radon–Nikodým property.

It turns out that the corresponding space is not reflexive. A reflexive space is obtained by adapting modifications, which follow L^p spaces..

It is a key property that the corresponding risk measure is continuous (in fact, Lipschitz continuous) with respect to any of the norms introduced, such that they all qualify as a domain space for the risk measure.

As regards further extensions, which should be addressed in future, we mention the situation with various distortion functions being present simultaneously. The corresponding duality theory probably involves Köthe-Toeplitz spaces.

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